MATH 147A PRACTICE PROBLEMS FOR THE FINAL

The final will 8 to 10 problems, some computational, some conceptual. Feel free to bring a reasonable sized paper for notes and formulas.

(1) Show that $EG - F^2 \geq 0$ for any parametrization of a surface. E, G, F are the coefficients of the first fundamental form.

(2) Let $\sigma(u, v)$ be a surface patch with standard unit normal $\mathbf{N}$. Show that

$$\mathbf{N} \times \sigma_u = \frac{E\sigma_v - F\sigma_u}{\sqrt{EG - F^2}}, \quad \mathbf{N} \times \sigma_v = \frac{F\sigma_v - G\sigma_u}{\sqrt{EG - F^2}}$$

(3) Sketch the surface given by

$$\sigma(u, v) = (b + r\cos u)\cos v, (b + r\cos u)\sin v, r\sin u)$$

with $b > r$ and $0 < u, v < 2\pi$. Then compute its first fundamental form, second fundamental form, surface area, and Gauss curvature.

(4) Gives examples of isometries, conformal maps, and equiareal maps. Then give relations and counter examples of each combination, i.e. isometries are conformal, stereographic projection is conformal but not an isometry etc. Make sure to give justification.

(5) Show that the Gaussian curvature of the surface $z = f(x, y)$, where f is a smooth function is

$$K = \frac{f_{xx}f_{yy} - f_{xy}^2}{(1 + f_x^2 + f_y^2)^2}.$$

(6) Compute the Gauss curvature of a sphere of radius R.

(7) What regular surface has a Gauss map that is constant?

(8) Parametrize a surface obtained by rotating the graph of $y = \frac{1}{x}$ from $x = 1$ to $x = \infty$ about the x-axis. Compute its volume and surface area. Can you give a physical interpretation of your answer?

(9) Show that a ruled surface has Gaussian curvature $K \leq 0$.

(10) Prove or give a counterexample: If S is a surface with Gaussian curvature $K > 0$, then the curvature of any curve $\gamma \subset S$ is everywhere strictly positive.

Solutions

(1) From the dot product formula,

$$F^2 = (\sigma_u \cdot \sigma_v)^2 = \|\sigma_u\|^2\|\sigma_v\|^2\cos^2\theta \leq \|\sigma_u\|^2\|\sigma_v\|^2 = EG.$$

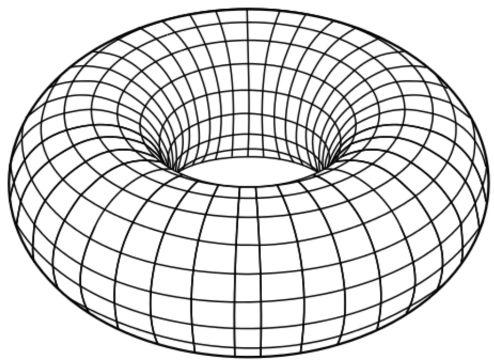
(2) We have $\mathbf{N} = \dfrac{\sigma_u \times \sigma_v}{\sqrt{EG - F^2}}$. Using the triple product identity, $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = -(\mathbf{b} \times \mathbf{c}) \times \mathbf{a} = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$, we have

$$\begin{aligned}
\mathbf{N} \times \sigma_u &= \frac{1}{\sqrt{EG - F^2}}((\sigma_u \times \sigma_v) \times \sigma_u) \\
&= \frac{1}{\sqrt{EG - F^2}}(\sigma_v(\sigma_u \cdot \sigma_u) - \sigma_u(\sigma_v \cdot \sigma_u)) \\
&= \frac{1}{\sqrt{EG - F^2}}(E\sigma_v - F\sigma_u).
\end{aligned}$$

and

$$\begin{aligned}\mathbf{N} \times \sigma_v &= \frac{1}{\sqrt{EG - F^2}}((\sigma_u \times \sigma_v) \times \sigma_v) \\ &= \frac{1}{\sqrt{EG - F^2}}(\sigma_v(\sigma_u \cdot \sigma_v) - \sigma_u(\sigma_v \cdot \sigma_v)) \\ &= \frac{1}{\sqrt{EG - F^2}}(F\sigma_v - G\sigma_u).\end{aligned}$$

(3) The surface is a torus:



The tangent vectors are given by

$$\begin{aligned}\sigma_u &= (-r\sin(u)\cos(v), -r\sin(u)\sin(v), r\cos(u)) \\ \sigma_v &= (-(b + r\cos(u)\sin(v), (b + r\cos(u))\cos(v), 0).\end{aligned}$$

The coefficients of the first fundamental form are given by

$$\begin{aligned}E &= \sigma_u \cdot \sigma_u = r^2 \\ F &= \sigma_u \cdot \sigma_v = 0 \\ G &= \sigma_v \cdot \sigma_v = (b + r\cos(u))^2\end{aligned}$$

We also compute the normal vector

$$\sigma_u \times \sigma_v = (-r(b + r\cos(u))\cos(u)\cos(v), -r(b + r\cos(u))\cos(u)\sin(v), -r(b + r\cos(u))\sin(u))$$

which has magnitude

$$\sqrt{EG - F^2} = r(b + r\cos(u))$$

and the second derivative terms

$$\begin{aligned}\sigma_{uu} &= (-r\cos(u)\cos(v), -r\cos(u)\sin(v), -r\sin(u)) \\ \sigma_{uv} &= (r\sin(u)\sin(v), -r\sin(u)\cos(v), 0) \\ \sigma_{vv} &= (-(b + r\cos(u))\cos(v), -(b + r\cos(u))\sin(v), 0).\end{aligned}$$

The coefficients of the second fundamental form is

$$\begin{aligned}L &= \sigma_{uu} \cdot \mathbf{N} = r\cos^2(u)\cos^2(v) + r\cos^2(u)\sin^2(v) + r\sin^2(u) = r \\ M &= \sigma_{uv} \cdot \mathbf{N} = 0 \\ N &= \sigma_{vv} \cdot \mathbf{N} = (b + r\cos(u))\cos^2(v)\cos(u) + (b + r\cos(u))\sin^2(v)\cos(u) = (b + r\cos(u))\cos(u).\end{aligned}$$

Hence its Gauss curvature is given by

$$\frac{LN - M^2}{EG - F^2} = \frac{r(b + r\cos(u))\cos(u)}{r^2(b + r\cos(u))^2} = \frac{\cos(u)}{r(b + r\cos(u))}.$$

Its surface area is given by

$$\int_R \sqrt{EG - F^2} dA$$

where R is the region which will cover the torus. We can take it to be $(0, 2\pi) \times (0, 2\pi)$ since we are only missing a one-dimensional circle, which has zero area. Thus

$$r \int_0^{2\pi} \int_0^{2\pi} (b + r\cos(u)) du dv = rb4\pi^2$$

(4) An example of an isometry is translation by some fixed constant, any isometry is a conformal map. A conformal map that is not an isometry is given by the stereographic projection. Any isometry is also an equiareal map. An equiareal map that is not an isometry or a conformal map is given by the Archimedes map. They can be analyzed by looking at the first fundamental forms.

(5) See text exercise 8.1.1

(6) See text example 8.1.4, note that it must be scaled by R, so the Gauss curvature is $1/R^2$.

(7) A regular surface that has constant Gauss map is a plane.

(8) A parametrization can be given by

$$\sigma(u, v) = \left(u, \frac{1}{u}\cos(v), \frac{1}{u}\sin(v)\right).$$

with $1 < u < \infty$ and $0 < v < 2\pi$ (Note that another patch would be necessary to cover the entire surface.) To compute the surface area, we have

$$\sigma_u = \left(1, -\frac{\cos(v)}{u^2}, -\frac{\sin(v)}{u^2}\right),$$

$$\sigma_v = \left(0, -\frac{\sin(v)}{u}, \frac{\cos(v)}{u}\right),$$

$$\sigma_u \times \sigma_v = \left(-\frac{1}{u^3}, -\frac{\cos(v)}{u}, -\frac{\sin(v)}{u}\right).$$

Hence the surface area is given by

$$\int_0^{2\pi} \int_1^{\infty} \|\sigma_u \times \sigma_v\| dA = 2\pi \int_1^{\infty} \frac{\sqrt{u^4 + 1}}{u^3} du \geq \int_1^{\infty} \frac{1}{u} du = \infty.$$

So the surface area is infinite. However, the volume is given by (from Calculus)

$$\pi \int_1^{\infty} \frac{1}{x^2} dx = \pi < \infty.$$

Hence this is a surface (called Gabriel's Horn) with finite volume but infinite surface area. This is a surface which cannot hold enough paint to paint itself.

(9) See text example 8.1.5.

(10) This is true. Suppose not, let S be a surface with Gaussian curvature $K > 0$. Let γ be a curve such that $\kappa = 0$ at a point $p \in S$. Let κ_n and κ_g be its normal and geodesic curvature, respectively. Then $\kappa^2 = \kappa_n^2 + \kappa_g^2$. Hence at the point p, $\kappa_n = 0$. By direct computation, or Proposition 7.3.3 in the text, at this point p,

$$0 = \kappa_n = II(\dot{\gamma}, \dot{\gamma}).$$

Now the Gauss curvature is given by

$$K = \frac{LN - M^2}{EG - F^2}$$

and from the first problem, $EG - F^2 \geq 0$. Assume that $EG - F^2 > 0$, then $LN - M^2 > 0$. Since $LN - M^2$ is the determinant of the matrix represented by the quadratic form II, we have a contradiction.