From prime ideals to physics, via total nonnegativity

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Preliminary report

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Quantum matrices

k an infinite field of arbitrary characteristic, $q \in k^\times$ not a root of unity.

Quantized coordinate ring of 2×2 matrices

$$\mathcal{O}_q(M_2) := k\langle a, b, c, d \rangle \Big/ \begin{pmatrix} ab - qba, & bd - qdb, \\ ac - qca, & cd - qdc, \\ & bc - cb, \\ & ad - da - (q - q^{-1})bc \end{pmatrix}$$

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Scale up to $\mathcal{O}_q(M_{m,n})$: every "square" of variables $\begin{matrix} X_{ij} & X_{il} \\ X_{kj} & X_{kl} \end{matrix}$ satisfies a copy of the 2×2 relations.

$\mathcal{H}$-primes in quantum matrices

$\mathcal{O}_q(M_{m,n})$ is $\mathbb{Z}^{m+n}$-graded, with X_{ij} in degree $e_i + e_{m+j}$.

Understanding the graded primes, or $\mathcal{H}$-**primes**, in $\mathcal{O}_q(M_{m,n})$ is a key step towards understanding the entire prime spectrum.

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- Finitely many $\mathcal{H}$-primes (Cauchon, Goodearl-Letzter).
- Completely prime, i.e. prime in the commutative sense (Goodearl-Letzter).
- Generated by the quantum minors they contain (Casteels).
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Cauchon diagrams appeared independently as **Le diagrams** in Postnikov's 2006 preprint on total nonnegativity in the Grassmannian.

Totally nonnegative matrices

Definition

A **totally nonnegative (TNN) real matrix** is an $m \times n$ real matrix for which every minor is nonnegative.

More generally, a point in the real Grassmannian $Gr(r, d)$ is totally nonnegative if it can be represented by a matrix whose Plücker coordinates ($r \times r$ minors) are all nonnegative.

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Can partition the set of TNN matrices (or the TNN Grassmannian) into cells by considering which minors are zero or positive.

Total nonnegativity and $\mathcal{H}$-primes

Theorem (Goodearl-Launois-Lenagan 2011)

1. *The $\mathcal{H}$-primes in $\mathcal{O}_q(M_{m,n})$ are in bijection with the non-empty TNN cells of real $m \times n$ matrices*
2. *A list of minors defines the zeros of a non-empty TNN cell if and only if there is a $\mathcal{H}$-prime containing those minors and no others.*

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For example:

If $\begin{pmatrix} a & b \\ c & 0 \end{pmatrix}$ is a TNN matrix, then at least one of b and c must also be zero, i.e. **there is no TNN matrix for which d is the only zero minor**.

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In $\mathcal{O}_q(M_2)$, **the ideal $\langle d \rangle$ is not a prime ideal**, thanks to the relation

$$ad - da = (q - q^{-1})bc.$$

Recap

- $\mathcal{O}_q(M_{m,n})$ is a nice noncommutative deformation of the classical coordinate ring $\mathcal{O}(M_{m,n})$.
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- A totally nonnegative matrix is a matrix for which every minor is nonnegative.
- Study TNN matrices in terms of which minors are zero / positive.
- Studying which minors belong to a $\mathcal{H}$-prime corresponds to identifying vanishing patterns of minors in TNN cells.

A very informal introduction to scattering amplitudes

$N = 4$ supersymmetric Yang Mills ($N = 4$ SYM) is a simplified model of quantum field theory currently of great interest to physicists.

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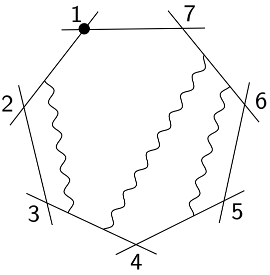
Two modern approaches:

- BCFW recursion / on-shell diagrams
- MHV diagrams / **Wilson loop diagrams**.

A Wilson loop diagram is a convex polygon with d vertices and r **propagators** (wavy internal lines) connecting pairs of edges.



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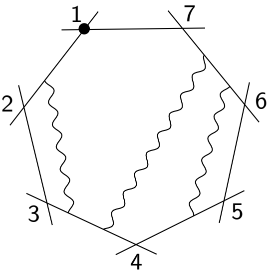
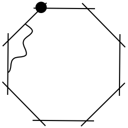


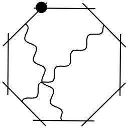
Each Wilson loop diagram defines a family of $r \times d$ matrices of rank r, i.e. a family of points in $Gr(r, d)$.

Admissible Wilson loop diagrams

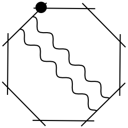
If we impose some fairly mild conditions on the propagators:



Can't connect adjacent edges



No crossing propagators



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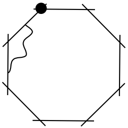

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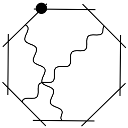

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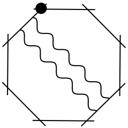

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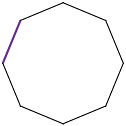
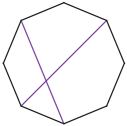
This looks more familiar if we dualise the boundary of the diagram:



No boundary edges



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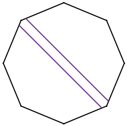

No doubled edges

Results

Question 1

When do two different Wilson loop diagrams define the same TNN cell?

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Theorem (Agarwala-Fryer)

Two Wilson loop diagrams define the same TNN cell if and only if they are flip-equivalent.

flip move

Results

Question 2

Can we read off the TNN cell directly from the Wilson loop diagram?

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Theorem (Agarwala-Fryer)

There is an algorithm for reading the Grassmann necklace of the cell directly from the Wilson loop diagram.

Future work

1. Compute some small examples, e.g. $Gr(2,6)$ or $Gr(3,7)$.
2. Classify which TNN cells are *not* hit by Wilson loop diagrams.
3. Use ideas from $\mathcal{H}$-primes to study boundaries (lower-dimensional cells) of Wilson loop diagrams.