Ordinary Differential Equations
and
Dynamical Systems

Thomas C. Sideris

Department of Mathematics, University of California,
Santa Barbara, CA 93106
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## Contents

Chapter 1. Introduction 1

Chapter 2. Linear Systems 5
  2.1. Definition of a Linear System 5
  2.2. Exponential of a Linear Transformation 5
  2.3. Solution of the Initial Value Problem for Linear Homogeneous Systems 8
  2.4. Computation of the Exponential of a Matrix 8
  2.5. Asymptotic Behavior of Linear Systems 11

Chapter 3. Existence Theory 17
  3.1. The Initial Value Problem 17
  3.2. The Cauchy-Peano Existence Theorem 17
  3.3. The Picard Existence Theorem 18
  3.4. Extension of Solutions 22
  3.5. Continuous Dependence on Initial Conditions 23
  3.6. Flow of Nonautonomous Systems 27
  3.7. Flow of Autonomous Systems 29
  3.8. Global Solutions 32
  3.9. Stability 34
  3.10. Liapunov Stability 38
CHAPTER 1

Introduction

The most general $n^{\text{th}}$ order ordinary differential equation (ODE) has the form

$$F(t, y, y', \ldots, y^{(n)}) = 0,$$

where $F$ is a continuous function from some open set $\Omega \subset \mathbb{R}^{n+2}$ into $\mathbb{R}$. A real-valued function $y(t)$ is a solution on an interval $I$ if

$$F(t, y(t), y'(t), \ldots, y^{(n)}(t)) = 0, \quad t \in I.$$

A necessary condition for existence of a solution is the existence of points $p = (t, y_1, \ldots, y_{n+1}) \in \mathbb{R}^{n+2}$ such that $F(p) = 0$. For example, the equation

$$(y')^2 + y^2 + 1 = 0$$

has no (real) solutions, because $F(p) = y_2^2 + y_1^2 + 1 = 0$ has no real solutions.

If $F(p) = 0$ and $\frac{\partial F}{\partial y_{n+1}}(p) \neq 0$, then locally we can solve for $y_{n+1}$ in terms of the other variables by the implicit function theorem

$$y_{n+1} = G(t, y_1, \ldots, y_n),$$

and so we can our ODE as

$$y^{(n)} = G(t, y, y', \ldots, y^{(n-1)}).$$

This equation can, in turn, be written as a first order system by introducing additional unknowns. Setting

$$x_1 = y, \ x_2 = y', \ldots, \ x_n = y^{(n-1)},$$

we have that

$$x_1' = x_2, \ x_2' = x_3, \ldots, \ x_{n-1}' = x_n, \ x_n' = G(t, x_1, \ldots, x_n),$$

Therefore, if we define $n$-vectors

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}, \quad f(t, x) = \begin{bmatrix} x_2 \\ \vdots \\ x_n \\ G(t, x_1, \ldots, x_{n-1}, x_n) \end{bmatrix}.$$
we obtain the equivalent first order system
\[ x' = f(t, x). \]

The point of this discussion is that there is no loss of generality in studying the first order system above, where \( f(t, x) \) is a continuous function (at least) defined on some open region in \( \mathbb{R}^{n+1} \).

A fundamental question that we will address in this course is the existence and uniqueness of solutions to the initial value problem
\[ x' = f(t, x), \quad x(t_0) = x_0, \]
for points \((t_0, x_0)\) in the domain of \( f(t, x) \). We will then proceed to study the qualitative behavior of such solutions, including periodicity, asymptotic behavior, invariant structures, etc.

In the case where \( f(t, x) = f(x) \) is independent of \( t \), the system is called autonomous. Every first order system can be rewritten as an autonomous one by introducing an extra unknown. If
\[ z_1 = t, \quad z_2 = x_1, \ldots, z_{n+1} = x_n, \]
then we obtain the equivalent autonomous system
\[ z' = g(z), \quad g(z) = \begin{bmatrix} 1 \\ f(z) \end{bmatrix}. \]

Suppose that \( f(x) \) is a continuous map from an open set \( U \subset \mathbb{R}^n \) into \( \mathbb{R}^n \). We can regard a solution \( x(t) \) of an autonomous system
(1.0.1) \[ x' = f(x), \]
as a curve in \( \mathbb{R}^n \). This gives us a geometric interpretation of (1.0.1). If the vector \( x'(t) \neq 0 \), then it is tangent to the solution curve at \( x(t) \). The equation (1.0.1) tells us what the value of this tangent vector must be, namely, \( f(x(t)) \). So if there is one and only one solution through each point of \( U \), we know just from the equation (1.0.1) its tangent direction at every point of \( U \). For this reason, \( f(x) \) is called a vector filed or direction field on \( U \).

The collection of all solution curves in \( U \) is called the phase diagram of \( f(x) \). If \( f \neq 0 \) in \( U \), then locally, the curves are parallel. Near a point \( x_0 \in U \) where \( f(x_0) = 0 \), the picture becomes more interesting.

A point \( x_0 \in U \) such that \( f(x_0) = 0 \) is called, interchangibly, a critical point, a stationary point, or an equilibrium point of \( f \). If \( x_0 \in U \) is a critical point of \( f \), then by direct substitution, \( x(t) = x_0 \) is a solution of (1.0.1). Such solutions are referred to as equilibrium or stationary solutions.

To understand the phase diagram near a critical point we are going to attempt to approximate solutions of (1.0.1) by solutions of an
associated linearized system. Suppose that \( x_0 \) is a critical point of \( f \). If \( f \in C^1(U) \), then the mean value theorem says that

\[
f(x) \approx Df(x_0)(x - x_0),
\]

when \( x - x_0 \) is small. The linearized system near \( x_0 \) is

\[
y' = Ay, \quad A = Df(x_0).
\]

An important goal is to understand when \( y \) is a good approximation to \( x - x_0 \). Linear systems are simple, and this is the benefit of replacing a nonlinear system by a linearized system near a critical point. For this reason, our first topic will be the study of linear systems.
CHAPTER 2

Linear Systems

2.1. Definition of a Linear System

Let \( f(t,x) \) be a continuous map from an open set in \( \mathbb{R}^{n+1} \) to \( \mathbb{R}^n \). A first order system
\[
x' = f(t,x)
\]
will be called linear when
\[
f(t,x) = A(t)x + g(t).
\]
Here \( A(t) \) is a continuous \( n \times n \)-matrix valued function and \( g(t) \) is a continuous \( \mathbb{R}^n \)-valued function, both defined for \( t \) belonging to some interval in \( \mathbb{R} \).

A linear system is homogeneous when \( g(t) = 0 \). A linear system is said to have constant coefficients if \( A(t) = A \) is constant.

In this chapter, we shall study linear, homogeneous systems with constant coefficients, i.e. systems of the form
\[
x' = Ax,
\]
where \( A \) is an \( n \times n \) matrix (with real entries).

2.2. Exponential of a Linear Transformation

Let \( V \) be a finite dimensional normed vector space over \( \mathbb{R} \) or \( \mathbb{C} \). \( L(V) \) will denote the set of linear transformations from \( V \) into \( V \).

**Definition 2.2.1.** Let \( A \in L(V) \). Define the operator norm
\[
\|A\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|} = \sup_{\|x\|=1} \|Ax\|.
\]

Properties:

- \( \|A\| < \infty \), for every \( A \in L(V) \).
- \( L(V) \) with the operator norm is a finite dimensional normed vector space.
- Given \( A \in L(V) \), \( \|Ax\| \leq \|A\|\|x\| \), for every \( x \in V \), and \( \|A\| \) is the smallest number with this property.
- \( \|AB\| \leq \|A\|\|B\| \), for every \( A,B \in L(V) \).
2. LINEAR SYSTEMS

Definition 2.2.2. A sequence $\{A_n\}$ in $L(V)$ converges to $A$ if and only if

$$\lim_{n \to \infty} \|A_n - A\| = 0.$$ 

With this notion of convergence, $L(V)$ is complete.

All norms on a finite dimensional space are equivalent, so $A_n \to A$ in the operator norm implies componentwise convergence in any coordinate system.

Definition 2.2.3. Given $A \in L(V)$, define $\exp A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k$.

The exponential is well-defined in the sense that the sequence of partial sums

$$S_n = \sum_{k=0}^{n} \frac{1}{k!} A^k$$

has a limit. This can be seen by showing that $S_n$ is a Cauchy sequence. Let $m < n$. Then,

$$\|S_n - S_m\| = \| \sum_{k=m+1}^{n} \frac{1}{k!} A^k \|$$

$$\leq \sum_{k=m+1}^{n} \frac{1}{k!} \|A^k\|$$

$$\leq \sum_{k=m+1}^{n} \frac{1}{k!} \|A\|^k$$

$$= \frac{1}{(m+1)!} \|A\|^{m+1} \sum_{k=0}^{n-m-1} \frac{(m+1)!}{(k+m+1)!} \|A\|^k$$

$$\leq \frac{1}{(m+1)!} \|A\|^{m+1} \sum_{k=0}^{\infty} \frac{1}{k!} \|A\|^k$$

$$= \frac{1}{(m+1)!} \|A\|^{m+1} \exp \|A\|.$$ 

From this, we see that $S_n$ is Cauchy.

It also follows that $\|\exp A\| \leq \exp \|A\|$. 

Lemma 2.2.1. Given $A, B \in L(V)$, we have the following properties:

1. $\exp At$ exists for all $t \in \mathbb{R}$.
2. $\exp (A(t + s)) = \exp At \exp As = \exp As \exp At$, for all $t, s \in \mathbb{R}$.
3. $\exp(A + B) = \exp A \exp B = \exp B \exp A$, provided $AB = BA$. 

2.2. EXPONENTIAL OF A LINEAR TRANSFORMATION

(4) \( \exp At \) is invertible for every \( t \in \mathbb{R} \), and \( (\exp At)^{-1} = \exp(-At) \).

(5) \( \frac{d}{dt} \exp At = A \exp At = \exp At A \).

**Proof.** (1) was shown in the preceding paragraph.
(2) is a consequence of (3).
To prove (3), we first note that when \( AB = BA \) the binomial expansion is valid:

\[
(A + B)^k = \sum_{j=0}^{k} \binom{k}{j} A^j B^{k-j}.
\]

Thus, by definition

\[
\exp(A + B) = \sum_{k=0}^{\infty} \frac{1}{k!}(A + B)^k
\]

\[
= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} A^j B^{k-j}
\]

\[
= \sum_{j=0}^{\infty} \frac{1}{j!} A^j \sum_{k=j}^{\infty} \frac{1}{(k-j)!} B^{k-j}
\]

\[
= \sum_{j=0}^{\infty} \frac{1}{j!} A^j \sum_{\ell=0}^{\infty} \frac{1}{\ell!} B^\ell
\]

\[
= \exp A \exp B.
\]

The rearrangements are justified by the absolute convergence of all series.
(4) is an immediate consequence of (2).
(5) is proven as follows. We have

\[
\| (\Delta t)^{-1} [\exp A(t + \Delta t) \exp At] - \exp At A \|
\]

\[
= \| \exp At \{ (\Delta t)^{-1} [\exp A\Delta t - I] - A \} \|
\]

\[
= \left\| \exp At \sum_{k=2}^{\infty} \frac{(\Delta t)^{k-1}}{k!} A^k \right\|
\]

\[
\leq \| \exp At \| \left\| A^2 \Delta t \sum_{k=2}^{\infty} \frac{(\Delta t)^{k-2}}{k!} A^{k-2} \right\|
\]

\[
\leq |\Delta t| \|A\|^2 \exp \|A\|(|t| + |\Delta t|).
\]
This last expression tends to 0 as $\Delta t \to 0$. Thus, we have shown that $\frac{d}{dt} \exp At = \exp At A$. This also equals $A\exp At$ because $A$ commutes with the partial sums for $\exp At$ and hence with $\exp At$ itself. \hfill \Box

### 2.3. Solution of the Initial Value Problem for Linear Homogeneous Systems

**Theorem 2.3.1.** Let $A$ be an $n \times n$ matrix over $\mathbb{R}$, and let $x_0 \in \mathbb{R}^n$. The initial value problem

\begin{equation}
(2.3.1) \quad x'(t) = Ax(t), \quad x(t_0) = x_0
\end{equation}

has a unique solution defined for all $t \in \mathbb{R}$ given by

\begin{equation}
(2.3.2) \quad x(t) = \exp A(t - t_0) x_0.
\end{equation}

**Proof.** We use the method of the integrating factor. Multiplying the system (2.3.1) by $\exp(-At)$ and using Lemma 2.2.1, we see that $x(t)$ is a solution of the IVP if and only if

$$\frac{d}{dt}[\exp(-At)x(t)] = 0, \quad x(t_0) = x_0.$$ Integration of this identity yields the equivalent statement

$$\exp(-At)x(t) - \exp(-At_0)x_0 = 0,$$

which in turn is equivalent to (2.3.2). This establishes existence, and uniqueness. \hfill \Box

### 2.4. Computation of the Exponential of a Matrix

The main computational tool will be reduction to an elementary case by similarity transformation.

**Lemma 2.4.1.** Let $A, S \in L(V)$ with $S$ invertible. Then

$$\exp(SAS^{-1}) = S(\exp A)S^{-1}.$$  

**Proof.** This follows immediately from the definition of the exponential together with the fact that $(SAS^{-1})^k = SA^kS^{-1}$, for every $k \in \mathbb{N}$. \hfill \Box

The simplest case is that of a diagonal matrix $D = \text{diag} [\lambda_1, \ldots, \lambda_n]$. Since $D^k = \text{diag} [\lambda_1^k, \ldots, \lambda_n^k]$, we immediately obtain

$$\exp Dt = \text{diag} [\exp \lambda_1 t, \ldots, \exp \lambda_n t].$$

Now if $A$ is diagonalizable, i.e. $A = SDS^{-1}$, then we can use Lemma 2.4.1 to compute

$$\exp At = S \exp Dt \ S^{-1}.$$
An $n \times n$ matrix $A$ is diagonalizable if and only if there is a basis of eigenvectors $\{v_j\}_{j=1}^n$. If such a basis exists, let $\{\lambda_j\}_{j=1}^n$ be the corresponding set of eigenvalues. Then

$$A = SDS^{-1},$$

where $D = \text{diag} \{\lambda_1, \ldots, \lambda_n\}$ and $S = [v_1 \ldots v_n]$ is the matrix whose columns are formed by the eigenvectors. Even if $A$ has real entries, it can have complex eigenvalues, in which case the matrices $D$ and $S$ will have complex entries. However, if $A$ is real, complex eigenvectors and eigenvalues occur in conjugate pairs.

In the diagonalizable case, the solution of the initial value problem (2.3.1) is

$$x(t) = \exp At \ x_0 = S \exp Dt \ S^{-1}x_0 = \sum_{j=1}^n c_j \exp \lambda_j t \ v_j,$$

where the coefficients $c_j$ are the coordinates of the vector $c = S^{-1}x_0$. Thus, the solution space is spanned by the elementary solutions $\exp \lambda_j t \ v_j$.

There are two important situations where an $n \times n$ matrix can be diagonalized.

- $A$ is real and symmetric, i.e. $A = A^T$. Then $A$ has real eigenvalues and there exists an orthonormal basis of real eigenvectors. Using this basis yields an orthogonal diagonalizing matrix $S$, i.e. $S^T = S^{-1}$.

- $A$ has distinct eigenvalues. For each eigenvalue there is always at least one eigenvector, and eigenvectors corresponding to distinct eigenvalues are independent. Thus, there is a basis of eigenvectors.

An $n \times n$ matrix over $\mathbb{C}$ may not be diagonalizable, but it can always be reduced to Jordan canonical (or normal) form. A matrix $J$ is in Jordan canonical form if it is block diagonal

$$J = \begin{bmatrix} B_1 & & \\ & \ddots & \\ & & B_p \end{bmatrix},$$

and each Jordan block has the form

$$B = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ & & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & \cdots & 0 & \lambda \end{bmatrix}.$$
Since $B$ is upper triangular, it has the single eigenvalue $\lambda$ with multiplicity equal to the size of the block $b$.

Computing the exponential of a Jordan block is easy. Write

$$B = \lambda I + N,$$

where $N$ has 1’s along the superdiagonal and 0’s everywhere else. The matrix $N$ is nilpotent. If the block size is $d \times d$, then $N^d = 0$. We also clearly have that $\lambda I$ and $N$ commute. Therefore,

$$\exp Bt = \exp (\lambda I + N) t = \exp \lambda t \exp N t = \exp (\lambda t) \sum_{j=1}^{d-1} \frac{t^j}{j!} N^j.$$

The entries of $\exp N t$ are polynomials in $t$ of degree at most $d - 1$.

Again using the definition of the exponential, we have that the exponential of a matrix in Jordan canonical form is the block diagonal matrix

$$\exp J t = \begin{bmatrix} \exp B_1 t & & \\ & \ddots & \\ & & \exp B_p t \end{bmatrix}.$$

The following central theorem in linear algebra will enable us to understand the form of $\exp At$ for a general matrix $A$.

**Theorem 2.4.1.** Let $A$ be an $n \times n$ matrix over $\mathbb{C}$. There exists a basis $\{v_j\}_{j=1}^n$ for $\mathbb{C}^n$ which reduces $A$ to Jordan normal form $J$. That is, if $S = [v_1 \cdots v_n]$ is the matrix whose columns are formed from the basis vectors, then

$$A = S J S^{-1}.$$

The Jordan normal form of $A$ is unique up to the permutation of its blocks.

When $A$ is diagonalizable, the basis $\{v_j\}_{j=1}^n$ consists of eigenvectors of $A$. In this case, the Jordan blocks are $1 \times 1$. Thus, each vector $v_j$ lies in the kernel of $A - \lambda_j I$ for the corresponding eigenvalue $\lambda_j$.

In the general case, the basis $\{v_j\}_{j=1}^n$ consists of appropriately chosen generalized eigenvectors of $A$. A vector $v$ is a generalized eigenvector of $A$ corresponding to an eigenvalue $\lambda_j$ if it lies in the kernel of $(A - \lambda_j I)^k$ for some $k \in \mathbb{N}$. The set of generalized eigenvectors of $A$ corresponding to a given eigenvalue $\lambda_j$ is a subspace, $E(\lambda_j)$, of $\mathbb{C}^n$, called the generalized eigenspace of $\lambda_j$. These subspaces are invariant under $A$. If $\{\lambda_j\}_{j=1}^d$ are the distinct eigenvalues of $A$, then

$$\mathbb{C}^n = E(\lambda_1) \oplus \cdots \oplus E(\lambda_d),$$

is a direct sum.
We arrive at the following algorithm for computing $\exp At$. Given an $n \times n$ matrix $A$, reduce it to Jordan canonical form $A = SJS^{-1}$, and then write
\[
\exp At = S \exp Jt S^{-1}.
\]
Even if $A$ (and hence also $\exp At$) has real entries, the matrices $J$ and $S$ may have complex entries. However, if $A$ is real, then any complex eigenvalues and generalized eigenvectors occur in conjugate pairs. It follows that the entries of $\exp At$ are linear combinations of terms of the form $t^k e^{\mu t} \cos \nu t$ and $t^k e^{\mu t} \sin \nu t$, where $\lambda = \mu \pm i\nu$ is an eigenvalue of $A$ and $k = 0, 1, \ldots, p$, with $p + 1$ being the size of the largest Jordan block for $\lambda$.

2.5. Asymptotic Behavior of Linear Systems

**Definition 2.5.1.** Let $A$ be an $n \times n$ matrix over $\mathbb{R}$. Define the complex stable, unstable, and center subspaces of $A$, denoted $E^C_s$, $E^C_u$, and $E^C_c$, respectively, to be the linear span over $\mathbb{C}$ of the generalized eigenvectors of $A$ corresponding to eigenvalues with negative, positive, and zero real parts, respectively.

Arrange the eigenvalues of $A$ so that $\text{Re} \lambda_1 \leq \ldots \leq \text{Re} \lambda_n$. Partition the set $\{1, \ldots, n\} = I_s \cup I_c \cup I_u$ so that
\[
\text{Re} \lambda_j < 0, \quad j \in I_s
\]
\[
\text{Re} \lambda_j = 0, \quad j \in I_c
\]
\[
\text{Re} \lambda_j > 0, \quad j \in I_u.
\]

Let $\{v_j\}_{j=1}^n$ be a basis of generalized eigenvectors corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_n$. Then
\[
\text{span} \{v_j : j \in I_s\} = E^C_s
\]
\[
\text{span} \{v_j : j \in I_c\} = E^C_c
\]
\[
\text{span} \{v_j : j \in I_u\} = E^C_u.
\]
In other words, we have
\[
E^C_s = \oplus_{j \in I_s} E(\lambda_j), \quad E^C_c = \oplus_{j \in I_c} E(\lambda_j), \quad E^C_u = \oplus_{j \in I_u} E(\lambda_j).
\]
It follows that $\mathbb{C}^n = E^C_s \oplus E^C_c \oplus E^C_u$ is a direct sum. Thus, any vector $x \in \mathbb{C}^n$ is uniquely represented as
\[
x = P_s x + P_c x + P_u x \in E^C_s \oplus E^C_c \oplus E^C_u.
\]
These subspaces are invariant under $A$. 
The maps $P_s, P_c, P_u$ are linear projections onto the complex stable, center, and unstable subspaces. Thus, we have

$$P_s^2 = P_s, \quad P_c^2 = P_c, \quad P_u^2 = P_u.$$ 

Since these subspaces are independent of each other, we have that

$$P_s P_c = P_c P_s = 0,$$

Since these subspaces are invariant under $A$, the projections commute with $A$, and thus also any function of $A$, including $\exp A t$.

If $A$ is real and $v \in \mathbb{C}^n$ is a generalized eigenvector with eigenvalue $\lambda \in \mathbb{C}$, then its complex conjugate $\bar{v}$ is a generalized eigenvector with eigenvalue $\bar{\lambda}$. Since $\text{Re} \lambda = \text{Re} \bar{\lambda}$, it follows that the subspaces $E_s^C, E_c^C,$ and $E_u^C$ are closed under complex conjugation. For any vector $x \in \mathbb{C}^n$, we have

$$\overline{P_s x + P_c x + P_u x} = \bar{x} = P_s \bar{x} + P_c \bar{x} + P_u \bar{x}.$$ 

This gives two representations of $\bar{x}$ in $E_s^C \oplus E_c^C \oplus E_u^C$. By uniqueness of representations, we must have

$$P_s \bar{x} = \overline{P_s x}, \quad P_c \bar{x} = \overline{P_c x}, \quad P_u \bar{x} = \overline{P_u x}.$$ 

So if $x \in \mathbb{R}^n$, we have that

$$P_s x = \overline{P_s x}, \quad P_c x = \overline{P_c x}, \quad P_u x = \overline{P_u x}.$$ 

Therefore, the projections leave $\mathbb{R}^n$ invariant:

$$P_s : \mathbb{R}^n \to \mathbb{R}^n, \quad P_c : \mathbb{R}^n \to \mathbb{R}^n, \quad P_u : \mathbb{R}^n \to \mathbb{R}^n.$$ 

**Definition 2.5.2.** Let $A$ be an $n \times n$ matrix over $\mathbb{R}$. Define the real stable, unstable, and center subspaces of $A$, denoted $E_s, E_u, E_c$, to be the images of $\mathbb{R}^n$ under the corresponding projections:

$$E_s = P_s \mathbb{R}^n, \quad E_c = P_c \mathbb{R}^n, \quad E_u = P_u \mathbb{R}^n.$$ 

Equivalently, we could define $E_s = E_s^C \cap \mathbb{R}^n$, etc.

We have that $\mathbb{R}^n = E_s \oplus E_c \oplus E_u$ is a direct sum. When restricted to $\mathbb{R}^n$, the projections possess the same properties as they do on $\mathbb{C}^n$.

The real stable subspace can also be characterized as the linear span over $\mathbb{R}$ of the real and imaginary parts of all generalized eigenvectors of $A$ corresponding to an eigenvalue with negative real part. Similar statements hold for $E_c$ and $E_u$.

We are now ready for the first main result of this section, which estimates the norm of $\exp A t$ on the invariant subspaces. These estimates will be used many times.
2.5. ASYMPTOTIC BEHAVIOR OF LINEAR SYSTEMS

Theorem 2.5.1. Let $A$ an $n \times n$ matrix over $\mathbb{R}$. Define

$$-\lambda_s = \max \{ \text{Re} \lambda_j : j \in I_s \} \quad \text{and} \quad \lambda_u = \min \{ \text{Re} \lambda_j : j \in I_u \}.$$ 

There is a constant $C > 0$ and an integer $0 \leq p < n$, depending on $A$, such that for all $x \in \mathbb{C}^n$,

\begin{align*}
\| \exp At P_s x \| &\leq C(1 + t)^p e^{-\lambda_s t} \| P_s x \|, \quad t > 0 \\
\| \exp At P_c x \| &\leq C(1 + |t|)^p \| P_c x \|, \quad t \in \mathbb{R} \\
\| \exp At P_u x \| &\leq C(1 + |t|)^p e^{\lambda_u t} \| P_u x \|, \quad t < 0.
\end{align*}

Proof. We will prove the first of these inequalities. The other two are similar.

Let $\{v_j\}_{j=1}^n$ be a basis generalized eigenvectors with indices ordered as above. For any $x \in \mathbb{C}^n$, we have

$$x = \sum_{j=1}^n c_j v_j, \quad \text{and} \quad P_s x = \sum_{j \in I_s} c_j v_j.$$

Let $S$ be the matrix whose columns are the vectors $v_j$. Then $S$ reduces $A$ to Jordan canonical form: $A = S(D + N)S^{-1}$, where $D = \text{diag}(\lambda_1, \ldots, \lambda_n)$ and $N^{p+1} = 0$, for some $p < n$.

If $\{e_j\}_{j=1}^n$ is the standard basis, then $Se_j = v_j$, and so, $e_j = S^{-1}v_j$.

We may write

\begin{align*}
\exp At P_s x &= S \exp Nt \exp Dt S^{-1} P_s x \\
&= S \exp Nt \exp Dt \sum_{j \in I_s} c_j e_j \\
&= S \exp Nt \sum_{j \in I_s} c_j \exp(\lambda_j t) e_j \\
&\equiv S \exp Nt y.
\end{align*}

Taking the norm, we have

$$\| \exp At P_s x \| \leq \| S \| \| \exp Nt \| \| y \|.$$

Now, if $p + 1$ is the size of the largest Jordan block, then $N^{p+1} = 0$.

Thus, we have that

$$\exp Nt = \sum_{j=0}^p \frac{t^j}{k!} N^k,$$

and so,

$$\| \exp Nt \| \leq \sum_{j=0}^p \frac{|t|^j}{k!} \| N \|^k \leq C_1 (1 + |t|)^p.$$
Next, we have, for $t > 0$,

$$
\|y\|^2 = \left\| \sum_{j \in I_s} c_j \exp(\lambda_j t) e_j \right\|^2 \\
= \sum_{j \in I_s} |c_j|^2 \exp(2 \Re \lambda_j t) \\
\leq \exp(-2\lambda_s t) \sum_{j \in I_s} |c_j|^2 \\
= \exp(-2\lambda_s t) \|S^{-1} P_s x\|^2 \\
\leq \exp(-2\lambda_s t) \|S^{-1}\|^2 \|P_s x\|^2,
$$

and so $\|y\| \leq \|S^{-1}\| \exp(-\lambda_s t) \|P_s x\|$.

The result follows with $C = C_1 \|S\| \|S^{-1}\|$. \qed

**Remark.** Examination of the proof of Theorem 2.5.1 shows that the exponent $p$ in these inequalities has the property that $p + 1$ is the size of the largest Jordan block of the Jordan form of $A$. In fact, the value of this exponent can be made more precise by noting, for example, that $\exp At P_s = \exp AP_s t P_s$, and so in the first case, $p + 1$ may be taken to be the size of the largest Jordan block of $AP_s$, i.e. the size of the largest Jordan block of $A$ corresponding to eigenvalues with negative real part. Similar statements hold in the other two cases.

It will often be convenient to use the following slightly weaker version of Theorem 2.5.1.

**Corollary 2.5.1.** Let $A$ be an $n \times n$ matrix over $\mathbb{R}$. Define $\lambda_s$ and $\lambda_u$ as in Theorem 2.5.1. Assume that $0 < \alpha < \lambda_s$ and $0 < \beta < \lambda_u$.

There is a constant $C > 0$ depending on $A$, such that for all $x \in \mathbb{C}^n$,

$$
\| \exp At P_s x \| \leq C e^{-\alpha t} \|P_s x\|, \quad t > 0 \\
\| \exp At P_u x \| \leq C e^\beta t \|P_u x\|, \quad t < 0.
$$

**Proof.** Notice that for any $\varepsilon > 0$, the function $(1 + t)^p \exp(-\varepsilon t)$ is bounded on the interval $t > 0$. Thus, for any constant $0 < \alpha < \lambda_s$, we have that

$$(1 + t)^p \exp(-\lambda_s t) = (1 + t)^p \exp[-(\lambda_s - \alpha)t] \exp(-\alpha t) \leq C \exp(-\alpha t),$$

for $t > 0$. The first statement now follows from Theorem 2.5.1. The second statement is analogous. \qed

**Corollary 2.5.2.** Let $A$ be an $n \times n$ matrix over $\mathbb{R}$. 


There is a constant $C > 0$ and an integer $0 \leq p < n$, depending on $A$, such that for all $x \in \mathbb{C}^n$,

\[
\| \exp At P_s x \| \geq C(1 + |t|)^{-p} e^{\lambda_s(-t)} \| P_s x \|,
\]
\[
\| \exp At P_u x \| \geq C(1 + t)^{-p} e^{\lambda_u t} \| P_u x \|,
\]

$t < 0$

Proof. Write $P_s x = \exp(-At) \exp At P_s x$. Since the eigenvalues of $-A$ are the negatives of the eigenvalues of $A$ the stable and unstable subspaces are exchanged as well as the numbers $\lambda_s$ and $\lambda_u$. Since $\exp At P_s x$ belongs to the stable subspace of $A$ and hence to the unstable subspace of $-A$, we obtain from Theorem 2.5.1 that for $t < 0$

\[
\| P_s x \| \leq C(1 + |t|)^p e^{\lambda_u t} \| \exp At P_s x \|,
\]

which proves the first statement. The second statement is similarly proven. \qed

Lemma 2.5.1. Let $A$ be an $n \times n$ matrix over $\mathbb{R}$. If $x \in E_c$ and $x \neq 0$, then

\[
\liminf_{|t| \to \infty} \| \exp At x \| > 0.
\]

Proof. Express $A$ in Jordan normal form as above,

\[
A = S(D + N)S^{-1}.
\]

The columns $\{v_j\}$ of $S$ comprise a basis of generalized eigenvectors for $A$.

Suppose that $y \in E_c$. Then $y = \sum_{j \in I_c} c_j v_j$, and so, $S^{-1} y = \sum_{j \in I_c} c_j e_j$. It follows that $\exp Dt S^{-1} y = \sum_{j \in I_c} c_j e_j$, and since Re $\lambda_j = 0$ for $j \in I_c$, we have that $\| \exp Dt S^{-1} y \| = \| c \| = \| S^{-1} y \|$. The general fact that $\| S z \| \geq \| S^{-1} \|^{-1} \| z \|$, combines with the preceding to give

\[
\| S \exp Dt S^{-1} y \| \geq \| S^{-1} \|^{-1} \| \exp Dt S^{-1} y \| \geq \| S^{-1} \|^{-1} \| S^{-1} y \|,
\]

for all $y \in E_c$.

Using the Jordan Normal Form, we have

\[
\exp At = S \exp(D + N)t S^{-1} = S \exp Dt \exp Nt S^{-1} = S \exp Dt S^{-1} S \exp Nt S^{-1}.
\]

If $x \in E_c$, then $y = S \exp Nt S^{-1} x \in E_c$. Thus, from the preceding we have

\[
\| \exp At x \| = \| S \exp Dt S^{-1} y \| \geq \| S^{-1} \|^{-1} \| S^{-1} y \| = \| S^{-1} \|^{-1} \| \exp Nt S x \|.
\]
Now since $N$ is nilpotent,
\[
\exp Nt = \sum_{k=0}^{p} \frac{t^k}{k!} N^k.
\]
If $x \neq 0$, then $Sx \neq 0$ and there exists a largest integer $m$ between 0 and $p$ such that $N^m Sx \neq 0$. We have by the triangle inequality that
\[
\| \exp Nt Sx \| = \left\| \sum_{k=0}^{m} \frac{t^k}{k!} N^k Sx \right\| \geq \frac{|t|^m}{m!} \| N^m Sx \| - \sum_{k<m} \frac{|t|^k}{k!} \| N^k Sx \|.
\]
It follows that if $m > 0$, then $\lim \inf_{|t| \to \infty} \| \exp At x \|$ is equal to $+\infty$. When $m = 0$, it is bounded below by $\|S^{-1} S x\|$. □

The next result serves as a converse to Theorem 2.5.1.

**Theorem 2.5.2.** Let $A$ be an $n \times n$ matrix over $\mathbb{R}$. Let $x \in \mathbb{R}^n$.

If $\lim_{t \to \infty} \exp At x = 0$, then $x \in E_s$.

If $\lim_{t \to -\infty} \exp At x = 0$, then $x \in E_u$.

If $\lim_{|t| \to \infty} (1 + |t|)^{-p} \exp At x = 0$, where $p$ is the size of the largest Jordan block of $A$, then $x \in E_c$.

**Proof.** Again, we shall only prove the first statement. Let $x \in \mathbb{R}^n$. Then by Theorem 2.5.1, $\lim_{t \to \infty} \| \exp At P_s x \| = 0$. If $\lim_{t \to \infty} \exp At x = 0$, then we have that
\[
(2.5.1) \quad 0 = \lim_{t \to \infty} \exp At x = \lim_{t \to \infty} \exp At(P_s x + P_c x + P_u x) = \lim_{t \to \infty} \exp At(P_c x + P_u x).
\]
By the triangle inequality, Theorem 2.5.1, and Corollary 2.5.2 we have
\[
\| \exp At(P_u x + P_c x) \| \geq \| \exp At P_u x \| - \| \exp At P_c x \|
\geq C_1 (1 + t)^{-p} e^{\lambda_u t} \| P_u x \| - C_2 (1 + |t|)^p \| P_c x \|.
\]
The last expression grows exponentially when $P_u x \neq 0$, so (2.5.1) forces $P_u x = 0$. Therefore, we are left with $\lim_{t \to \infty} \| \exp At P_s x \| = 0$. By Lemma 2.5.1, we must also have $P_c x = 0$. Thus, $x = P_s x \in E_s$.

The proofs of the other two statements are similar. □

All of the results in this section hold for complex matrices $A$, except for the remarks concerning the projections on $\mathbb{R}^n$ and the ensuing definitions of real invariant subspaces. We will not need this, however.
CHAPTER 3

Existence Theory

3.1. The Initial Value Problem

Let \( \Omega \subset \mathbb{R}^{n+1} \) be an open connected set. We will denote points in \( \Omega \) by \((t,x)\) where \( t \in \mathbb{R} \) and \( x \in \mathbb{R}^n \). Let \( f: \Omega \to \mathbb{R}^n \) be a continuous map. In this context, \( f(t,x) \) is called a vector field on \( \Omega \). Given any initial point \((t_0,x_0)\) \( \in \Omega \), we wish to construct a unique solution to the initial value problem

\[
(3.1.1) \quad x'(t) = f(t,x(t)) \quad x(t_0) = x_0.
\]

In order for this to make sense, \( x(t) \) must be a \( C^1 \) function from some interval \( I \subset \mathbb{R} \) containing the initial time \( t_0 \) into \( \mathbb{R}^n \) such that the solution curve satisfies

\[
\{(t,x(t)) : t \in I\} \subset \Omega.
\]

Such a solution is referred to as a local solution when \( I \neq \mathbb{R} \). When \( I = \mathbb{R} \), the solution is called global.

3.2. The Cauchy-Peano Existence Theorem

Theorem 3.2.1 (Cauchy-Peano). If \( f: \Omega \to \mathbb{R}^n \) is continuous, then for every point \((t_0,x_0)\) \( \in \Omega \) the initial value problem (3.1.1) has local solution.

The problem with this theorem is that it does not guarantee uniqueness. We will skip the proof, except to mention that it is uses a compactness argument based on the Arzela-Ascoli Theorem.

Example. Here is a simple example that demonstrates that uniqueness can indeed fail. Let \( \Omega = \mathbb{R}^2 \) and consider the autonomous vector field \( f(t,x) = |x|^{1/2} \). When \((t_0,x_0) = (0,0)\), the initial value problem has infinitely many solutions. In addition to the zero solution \( x(t) = 0 \), for any \( \alpha, \beta \geq 0 \), the following is a family of solutions.

\[
x(t) = \begin{cases} 
-\frac{1}{4}(t + \alpha)^2, & t \leq -\alpha \\
0, & -\alpha \leq t \leq \beta \\
\frac{1}{4}(t - \beta)^2, & \beta \leq t. 
\end{cases}
\]
This can be verified by direct substitution.

3.3. The Picard Existence Theorem

The failure of uniqueness can be rectified by placing an additional restriction on the vector field. The next definitions introduce this key property.

**Definition 3.3.1.** Let $S \subset \mathbb{R}^m$. Suppose $x \mapsto f(x)$ is a function from $S$ to $\mathbb{R}^n$.

The function $f$ is said to be Lipschitz continuous on $S$ if there exists a constant $C > 0$ such that

$$\|f(x_1) - f(x_2)\|_{\mathbb{R}^n} \leq C\|x_1 - x_2\|_{\mathbb{R}^m},$$

for all $x_1, x_2 \in S$.

The function $f$ is said to be locally Lipschitz continuous on $S$ if for every compact subset $K \subset S$, there exists a constant $C_K > 0$ such that

$$\|f(x_1) - f(x_2)\|_{\mathbb{R}^n} \leq C_K\|x_1 - x_2\|_{\mathbb{R}^m},$$

for all $x_1, x_2 \in K$.

**Remark.** If a function is Lipschitz continuous on $S$, then it is continuous on $S$.

**Example.** The function $\|x\|^\alpha$ from $\mathbb{R}^m$ to $\mathbb{R}$ is Lipschitz continuous for $\alpha = 1$, locally Lipschitz continuous for $\alpha > 1$, and not Lipschitz continuous (on any neighborhood of 0) when $\alpha < 1$.

**Definition 3.3.2.** Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set. A continuous function $(t, x) \mapsto f(t, x)$ from $\Omega$ to $\mathbb{R}^n$ is said to be locally Lipschitz continuous in $x$ if for every compact set $K \subset \Omega$, there is a constant $C_K > 0$ such that

$$\|f(t, x_1) - f(t, x_2)\|_{\mathbb{R}^n} \leq C_K\|x_1 - x_2\|_{\mathbb{R}^m},$$

for every $(t, x_1), (t, x_2) \in K$. If there is a constant for which the inequality holds for all $(t, x_1), (t, x_2) \in \Omega$, then $f$ is said to be Lipschitz continuous in $x$.

**Lemma 3.3.1.** If $f : \Omega \to \mathbb{R}^n$ is $C^1$, then it is locally Lipschitz continuous in $x$.

**Theorem 3.3.1 (Picard).** Let $\Omega \subset \mathbb{R}^{n+1}$ be open. Assume that $f : \Omega \to \mathbb{R}^n$ is continuous and that $f(t, x)$ is locally Lipschitz continuous in $x$. Let $K \subset \Omega$ be any compact set. Then there is a $\delta > 0$ such that for every $(t_0, x_0) \in K$, the initial value problem (3.1.1) has a unique local solution defined on the interval $|t - t_0| < \delta$. 
Before proving this important theorem, it is convenient to have the following technical “Covering Lemma”.

First, some notation: Given a point \((t, x) \in \mathbb{R}^{n+1}\) and positive numbers \(r\) and \(a\), define the cylinder

\[ C(t, x) \equiv \{(t', x') \in \mathbb{R}^{n+1} : \|x - x'\| \leq r, |t - t'| \leq a\}. \]

**Lemma 3.3.2 (Covering Lemma).** Let \(K \subset \Omega \subset \mathbb{R}^n \times \mathbb{R}\) with \(\Omega\) an open set and \(K\) a compact set. There exists a compact set \(K'\) and positive numbers \(r\) and \(a\) such that \(K \subset K' \subset \Omega\) and \(C(t, x) \subset K'\), for all \((t, x) \in K\).

**Proof.** For every point \(p = (t, x) \in K\), choose positive numbers \(a(p)\) and \(r(p)\) such that

\[ D(p) = \{(t', x') \in \mathbb{R}^{n+1} : \|x - x'\| \leq 2r(p), |t - t'| \leq 2a(p)\} \subset \Omega. \]

This is possible because \(\Omega\) is open.

Define the cylinders

\[ C(p) = \{(t', x') \in \mathbb{R}^{n+1} : \|x - x'\| < r(p), |t - t'| < a(p)\}. \]

The collection of open sets \(\{C(p) : p \in K\}\) forms an open cover of the set \(K\). \(K\) is compact, therefore there is a finite number of cylinders \(C(p_1), \ldots, C(p_N)\) whose union contains \(K\). Set

\[ K' = \bigcup_{i=1}^{N} D(p_i). \]

Then \(K'\) is compact, and

\[ K \subset \bigcup_{i=1}^{N} C(p_i) \subset \bigcup_{i=1}^{N} D(p_i) = K' \subset \Omega. \]

Define

\[ a = \min\{a(p_i) : i = 1, \ldots, N\} \quad \text{and} \quad r = \min\{r(p_i) : i = 1, \ldots, N\}. \]

The claim is that, for this uniform choice of \(a\) and \(r\), \(C(t, x) \subset K'\), for all \((t, x) \in K\).

If \((t, x) \in K\), then \((t, x) \in C(p_i)\) for some \(i = 1, \ldots, N\). Let \((t', x') \in C(t, x)\). Then

\[ \|x' - x_i\| \leq \|x' - x\| + \|x - x_i\| \leq a + a(p_i) \leq 2a(p_i) \]

and

\[ |t' - t_i| \leq |t' - t| + |t - t_i| \leq r + r(p_i) \leq 2r(p_i). \]

This shows that \((t', x') \in D(p_i)\), from which follows the conclusion \(C(t, x) \subset D(p_i) \subset K'\). \qed
Proof of the Picard Theorem. The first step of the proof is to reformulate the problem. If \(x(t)\) is a \(C^1\) solution of the initial value problem (3.1.1) for \(|t - t_0| \leq \delta\), then by integration we find that
\[
(3.3.1) \quad x(t) = x_0 + \int_{t_0}^t f(s, x(s))ds,
\]
for \(|t - t_0| \leq \delta\). Conversely, if \(x(t)\) is a \(C^0\) solution of the integral equation, then it is \(C^1\) and it solves the initial value problem (3.1.1).

Given a compact subset \(K \subset \Omega\), choose \(a, r, K'\) as in the covering lemma.

Choose \((t_0, x_0) \in K\). Let \(\delta < a\) and set
\[
I_\delta = \{|t - t_0| \leq \delta\}, \quad B_r = \{\|x - x_0\| \leq r\}, \quad X_\delta = C^0(I_\delta; B_r).
\]
Note that \(X_\delta\) is a complete metric space with the sup norm metric.

By definition, if \(x \in X_\delta\), then \((s, x(s)) \in C(t_0, x_0) \subset K' \subset \Omega\), for \(s \in I_\delta\). Thus, the operator
\[
Tx(t) = x_0 + \int_{t_0}^t f(s, x(s))ds
\]
is well-defined on \(X_\delta\) and the function \(Tx(t)\) is continuous for \(t \in I_\delta\).

Define \(M_1 = \max_{K'} |f(t, x)|\). The claim is that if \(\delta\) is chosen small enough so that \(M_1 \delta \leq r\), then \(T : X_\delta \to X_\delta\). If \(x \in X_\delta\), we have from (3.3.1)
\[
\sup_{t \in I_\delta} \|Tx(t) - x_0\| \leq M_1 \delta \leq r,
\]
for \(t \in I_\delta\). Thus, \(Tx \in X_\delta\).

Next, let \(M_2\) be a Lipschitz constant for \(f(t, x)\) on \(K'\). If \(\delta\) is further restricted so that \(M_2 \delta < 1/2\), then we claim that \(T : X_\delta \to X_\delta\) is a contraction. Let \(x_1, x_2 \in X_\delta\). Then from (3.3.1), we have
\[
\sup_{t \in I_\delta} \|Tx_1(t) - Tx_2(t)\| \leq M_2 \delta \sup_{t \in I_\delta} \|x_1(t) - x_2(t)\| \\
\leq 1/2 \sup_{t \in I_\delta} \|x_1(t) - x_2(t)\|.
\]

So by the Contraction Mapping Principle, there exists a unique function \(x \in X_\delta\) such that \(Tx = x\). In other words, \(x\) solves (3.3.1). \(\square\)

Note that the final choice of \(\delta\) is \(\min\{a, r/M_1, 1/2M_2\}\) which depends only on the set \(K\) and on \(f\).

In the proof of the Picard Existence Theorem, we used the Contraction Mapping Principle, the proof of which is based on iteration:
choose an arbitrary element \( x_0 \) of the metric space and define the sequence \( x_k = Tx_{k-1} \), \( k = 1, 2, \ldots \). Then the fixed point \( x \) is obtained as a limit: \( x = \lim_{k \to \infty} x_k \). In the context of the existence theorem, the sequence elements \( x_k \) are known as the Picard iterates.

The alert reader will notice that the solution constructed above is unique within the metric space \( X_\delta \), but it is not necessarily unique in \( C^0(I_\delta, B_r) \). The next result fills in this gap.

**Theorem 3.3.2 (Uniqueness).** Suppose that \( f : \Omega \to \mathbb{R}^n \) satisfies the hypotheses of the Picard Theorem. For \( j = 1, 2 \), let \( x_j(t) \) be solutions of \( x'(t) = f(t, x(t)) \) on the interval \( I_j \). If there is a point \( t_0 \in I_1 \cap I_2 \) such that \( x_1(t_0) = x_2(t_0) \), then \( x_1(t) = x_2(t) \) on the interval \( I_1 \cap I_2 \). Moreover, the function

\[
x(t) = \begin{cases} 
  x_1(t), & t \in I_1 \\
  x_2(t), & t \in I_2 
\end{cases}
\]

defines a solution on the interval \( I_1 \cup I_2 \).

**Proof.** Let \( J \subset I_1 \cap I_2 \) be any closed interval with \( t_0 \in J \). Let \( M \) be a Lipschitz constant for \( f(t, x) \) on the compact set

\[
\{(t, x_1(t)) : t \in J \} \cup \{(t, x_2(t)) : t \in J \}.
\]

The solutions \( x_j(t) \), \( j = 1, 2 \), satisfy the integral equation (3.3.1) on the interval \( J \). Thus, estimating as before

\[
\|x_1(t) - x_2(t)\| \leq \left| \int_{t_0}^t M\|x_1(s) - x_2(s)\|ds \right|,
\]

for \( t \in J \). It follows from Gronwall’s Lemma (below) that

\[
\|x_1(t) - x_2(t)\| = 0
\]

for \( t \in J \). Since \( J \subset I_1 \cap I_2 \) was any closed interval containing \( t_0 \), we have that \( x_1 = x_2 \) on \( J \).

From this it follows that \( x(t) \) is well-defined, is \( C^1 \), and is a solution.

\( \square \)

**Lemma 3.3.3 (Gronwall).** Let \( f(t), \varphi(t) \) be nonnegative continuous functions on an open interval \( J = (\alpha, \beta) \) containing the point \( t_0 \). Let \( c_0 \geq 0 \). If

\[
f(t) \leq c_0 + \left| \int_{t_0}^t \varphi(s)f(s)ds \right|,
\]

for all \( t \in J \), then

\[
f(t) \leq c_0 \exp \left| \int_{t_0}^t \varphi(s)ds \right|,
\]
for \( t \in J \).

**Proof.** Suppose first that \( t \in [t_0, \beta) \). Define

\[
F(t) = c_0 + \int_{t_0}^{t} \varphi(s)f(s)ds.
\]

Then \( F \) is \( C^1 \) and

\[
F'(t) = \varphi(t)f(t) \leq \varphi(t)F(t),
\]

for \( t \in [t_0, \beta) \), since \( f(t) \leq F(t) \). This implies that

\[
\frac{d}{dt} \left[ \exp \left( -\int_{t_0}^{t} \varphi(s)ds \right) F(t) \right] \leq 0,
\]

for \( t \in [t_0, \beta) \). Integrate this over the interval \([t_0, \tau)\) to get

\[
f(\tau) \leq F(\tau) \leq c_0 \exp \int_{t_0}^{\tau} \varphi(s)ds,
\]

for \( \tau \in [t_0, \beta) \).

On the interval \((\alpha, t_0]\), perform the analogous argument to the function

\[
G(t) = c_0 + \int_{t}^{t_0} \varphi(s)f(s)ds.
\]

\[\square\]

### 3.4. Extension of Solutions

**Theorem 3.4.1.** For every \((t_0, x_0) \in \Omega\) the solution to the initial value problem (3.1.1) extends to a maximal existence interval \( I = (\alpha, \beta) \). Furthermore, if \( K \subset \Omega \) is any compact set containing the point \((t_0, x_0)\), then there exist times \( \alpha(K) > \alpha, \beta(K) < \beta \) such that \((t, x(t)) \in \Omega \setminus K\), for \( t \in (\alpha, \alpha(K)) \cup (\beta(K), \beta) \).

**Proof.** Define \( A \) to be the collection of intervals \( J \) containing the initial time \( t_0 \) on which there exists a solution \( x_J \) of the initial value problem (3.1.1). The existence theorem 3.3.1 guarantees that \( A \) is nonempty, so we may define \( I = \bigcup_{J \in A} J \). Then \( I \) is an interval containing \( t_0 \). Write \( I = (\alpha, \beta) \). Note that \( \alpha \) and/or \( \beta \) could be infinite – that’s ok.

Suppose \( \bar{t} \in I \). Then \( \bar{t} \in J \) for some \( J \in A \), and there is a solution \( x_J \) of (3.1.1) defined on \( J \). If \( J' \in A \) is any other interval containing the value \( \bar{t} \) with corresponding solution \( x_{J'} \), then by the uniqueness theorem 3.3.2, we have that \( x_J = x_{J'} \) on \( J \cap J' \). In particular, \( x_J(\bar{t}) = x_{J'}(\bar{t}) \). Therefore, the following function is well-defined on \( I \):

\[
x(t) = x_J(t), \quad t \in J, \quad J \in A.
\]
Moreover, it follows from this definition that \( x(t) \) is a solution of (3.1.1) on \( I \), and it is unique, again thanks to Theorem 3.3.2. Thus, \( I \in A \) is maximal.

Now let \( K \subset \Omega \) be compact, with \( (t_0, x_0) \in K \). Define
\[
G(K) = \{ t \in I : (t, x(t)) \in K \}.
\]

By the existence theorem, we know \( G(K) \) is nonempty. The set \( G(K) \) is bounded since \( K \) is compact, so \( \beta(K) \equiv \sup G(K) < \infty \). We must show that \( \beta(K) < \beta \).

If \( t_1 \in G(K) \), then \( (t_1, x_1) = (t_1, x(t_1)) \in K \). By the existence theorem, we can solve the initial value problem
\[
y'(t) = f(t, y(t)), \quad y(t_1) = x_1
\]
on the interval \( \{|t - t_1| < \delta\} \). By the uniqueness theorem, \( x(t) = y(t) \) on the interval \( I \cap \{|t - t_1| < \delta\} \). Thus, \( y(t) \) extends \( x(t) \) to the interval \( I \cup \{|t - t_1| < \delta\} \). But, by maximality, \( I \cup \{|t - t_1| < \delta\} \subset I \). Therefore, \( t_1 + \delta \leq \beta \), for all \( t_1 \in G(K) \). Take the supremum over all \( t_1 \in G(K) \). We conclude that \( \beta(K) + \delta \leq \beta \). Thus, \( \beta(K) < \beta \).

Similarly, \( \alpha(K) = \inf G(K) > \alpha \).

\[ \square \]

Example. Consider the IVP
\[
x' = x^2, \quad x(0) = x_0,
\]
the solution of which is
\[
x(t) = \frac{x_0}{1 - x_0 t}.
\]
We see that the maximal interval of existence depends on the initial value \( x_0 \):
\[
I = (\alpha, \beta) = \begin{cases} 
(-\infty, \infty), & \text{if } x_0 = 0 \\
(-\infty, 1/x_0), & \text{if } x_0 > 0 \\
(1/x_0, \infty), & \text{if } x_0 < 0.
\end{cases}
\]

3.5. Continuous Dependence on Initial Conditions

Definition 3.5.1. A function \( g \) from \( \mathbb{R}^m \) into \( \mathbb{R} \cup \{\infty\} \) is lower semi-continuous at a point \( y_0 \) provided \( \liminf_{y \to y_0} g(y) \geq g(y_0) \).

Equivalently, a function \( g \) into \( \mathbb{R} \cup \{\infty\} \) is lower semi-continuous at a point \( y_0 \) provided for every \( L < g(y_0) \) there is a neighborhood \( V \) of \( y_0 \) such that \( L \leq g(y) \) for \( y \in V \).
Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set. Let $f : \Omega \rightarrow \mathbb{R}^n$ satisfy the hypotheses of the Picard Theorem 3.3.1. Given $(t_0, x_0) \in \Omega$, let $x(t, t_0, x_0)$ denote the unique solution of the IVP

$$x' = f(t, x), \quad x(t_0) = x_0,$$

with maximal existence interval $I(t_0, x_0) = (\alpha(t_0, x_0), \beta(t_0, x_0))$.

**Theorem 3.5.1.** The domain of $x(t, t_0, x_0)$, namely

$$D = \{(t, t_0, x_0) : (t_0, x_0) \in \Omega, \ t \in I(t_0, x_0)\},$$

is an open set in $\mathbb{R}^{n+2}$.

The function $x(t, t_0, x_0)$ is continuous on $D$.

The function $\beta(t_0, x_0)$ is lower semi-continuous on $\Omega$, and the function $\alpha(t_0, x_0)$ is upper semi-continuous on $\Omega$.

**Example.** Suppose that $\Omega = \mathbb{R}^{2+1} \setminus \{0\}$ and $f(t, x) = 0$ on $\Omega$. Then $x(t, t_0, x_0) = x_0$ where, because the solution curve must remain in $\Omega$, the maximal existence interval $I(t_0, x_0)$ has the right endpoint

$$\beta(t_0, x_0) = \begin{cases} 0, & \text{if } x_0 = 0, \ t_0 < 0, \\ +\infty, & \text{otherwise}. \end{cases}$$

Thus, we can not improve upon the lower semi-continuity of the function $\beta$.

The proof of this theorem will be based on the following lemma.

**Lemma 3.5.1.** Choose any closed interval $J = [\bar{\alpha}, \bar{\beta}]$ such that $t_0 \in J \subset I(t_0, x_0)$. Given any $\varepsilon > 0$, there exists a neighborhood $V \subset \Omega$ containing the point $(t_0, x_0)$ such that for any $(t_1, x_1) \in V$, the solution $x(t, t_1, x_1)$ is defined for $t \in [\bar{\alpha}, \bar{\beta}]$, and

$$\|x(t, t_1, x_1) - x(t, t_0, x_0)\| < \varepsilon,$$

for $t \in [\bar{\alpha}, \bar{\beta}]$.

**Proof of Theorem 3.5.1.** Let’s assume that the Lemma 3.5.1 holds and use it to establish the theorem.

Fix $(t_0, x_0) \in \Omega$. To show that $x(t, t_0, x_0)$ is continuous, fix a point $(t', t_0, x_0) \in D$ and let $\varepsilon > 0$ be given. Choose $J = [\bar{\alpha}, \bar{\beta}] \subset I(t_0, x_0)$ such that $\bar{\alpha} < t' < \bar{\beta}$. By the Lemma (using $\varepsilon/2$ for $\varepsilon$), there is a neighborhood $V \subset \Omega$ of the point $(t_0, x_0)$, such that for any $(t_1, x_1) \in V$, the solution $x(t, t_1, x_1)$ is defined for $t \in J$, and

$$\|x(t, t_1, x_1) - x(t, t_0, x_0)\| < \varepsilon/2,$$

for $t \in J$. 
3.5. CONTINUOUS DEPENDENCE ON INITIAL CONDITIONS

Now \( x(t, t_0, x_0) \) is continuous as a function of \( t \) on \( J \subset I(t_0, x_0) \), since it’s a solution of the IVP. So there is a \( \delta > 0 \) with \( \{ |t - t'| < \delta \} \subset J \), such that
\[
\| x(t, t_0, x_0) - x(t', t_0, x_0) \| < \varepsilon / 2,
\]
provided \( |t - t'| < \delta \).

Thus, for any \((t, t_1, x_1)\) in the neighborhood \( \{ |t - t'| < \delta \} \times V \) of \((t', t_0, x_0)\), we have that
\[
\| x(t, t_1, x_1) - x(t', t_0, x_0) \| \leq \| x(t, t_1, x_1) - x(t, t_0, x_0) \| + \| x(t, t_0, x_0) - x(t', t_0, x_0) \|
\]
\[
\leq \varepsilon / 2 + \varepsilon / 2 = \varepsilon.
\]

This proves continuity.

Let \((t', t_0, x_0) \in D\). In the preceding, we have constructed the set \( \{ |t - t'| < \delta \} \times V \) which is a neighborhood of \((t', t_0, x_0)\) contained in \( D \). This shows that \( D \) is open.

Using the lemma again, we have that for any \( \beta < \beta(t_0, x_0) \), there is a neighborhood \( V \subset \Omega \) of \((t_0, x_0)\) such that \((t_1, x_1) \in V \) implies that \( x(t, t_1, x_1) \) is defined for \( t \in [t_1, \beta] \). This says that \( \beta(t_1, x_1) \geq \beta \), in other words, the function \( \beta \) is lower semi-continuous at \((t_0, x_0)\). The proof that \( \alpha \) is upper semi-continuous is similar.

It remains to prove Lemma 3.5.1.

**Proof of Lemma 3.5.1.** Fix \((t_0, x_0) \in \Omega\). For notational convenience, we will use the abbreviation \( x_0(t) = x(t, t_0, x_0) \). Let \( J = [\bar{\alpha}, \bar{\beta}] \subset I(t_0, x_0) \).

Consider the compact set \( K = \{(s, x_0(s)) : s \in [\bar{\alpha}, \bar{\beta}]\} \). By the covering lemma, there exist a compact set \( K' \) and numbers \( a, r > 0 \) such that \( K \subset K' \subset \Omega \) and
\[
C(s, x_0(s)) = \{(s', x') : |s' - s| < a, \| x' - x_0(s) \| < r \} \subset K',
\]
for all \( s \in [\bar{\alpha}, \bar{\beta}] \). Define \( M_1 = \max_{K'} \| f(t, x) \| \) and let \( M_2 \) be a Lipschitz constant for \( f \) on \( K' \).

Given \( 0 < \varepsilon < r \), we are going to produce a sufficiently small \( \delta > 0 \), such that the neighborhood \( V = \{(t, x) : |t - t_0| < \delta, \| x - x_0 \| < \delta \} \) satisfies the requirements of the lemma. In fact, choose \( \delta \) small enough so that \( \delta < a, \delta < r, \{ |t - t_0| < \delta \} \subset [\bar{\alpha}, \bar{\beta}] \), and
\[
\delta(M_1 + 1) \exp M_2(\bar{\beta} - \bar{\alpha}) < \varepsilon < r.
\]
Notice that \( V \subset C(t_0, x_0) \subset K' \).
Choose \((t_1, x_1) \in V\). Let \(x_1(t) = x(t, t_1, x_1)\). This solution is defined on the maximal interval \(I(t_1, x_1) = (\alpha(t_1, x_1), \beta(t_1, x_1))\). Let \(I^* = [\alpha^*, \beta^*]\) be the largest subinterval of \(I(t_1, x_1)\) such that
\[
\{(s, x_1(s)) : s \in I^*\} \subset K'.
\]
Note that by the existence theorem, \(I^* \neq \emptyset\) and that \(t_1 \in I^*\). Since \(|t_1 - t_0| < \delta\), we have that \(t_1 \in J\). Thus, \(J \cap I^* \neq \emptyset\). Since \((t, x_1(t))\) must eventually exit \(K'\), we have that \(I^*\) is properly contained in \(I(t_1, x_1)\).

Recall that our solutions satisfy the integral equation
\[
x_i(t) = x_i + \int_{t_i}^{t} f(s, x_i(s))ds,
\]
\(i = 0, 1\). So we have for all \(t \in J \cap I^*\),
\[
x_1(t) - x_0(t) = x_1 - x_0 + \int_{t_1}^{t} f(s, x_1(s))ds - \int_{t_0}^{t} f(s, x_0(s))ds
\]
\[
= x_1 - x_0 + \int_{t_0}^{t_1} f(s, x_0(s))ds
\]
\[
+ \int_{t_1}^{t} [f(s, x_1(s)) - f(s, x_0(s))]ds.
\]
Since the solution curves remain in \(K'\) on the interval \(J \cap I^*\), we now have the following estimate:
\[
\|x_1(t) - x_0(t)\| \leq \|x_1 - x_0\| + \left| \int_{t_0}^{t_1} \|f(s, x_0(s))\|ds \right|
\]
\[
+ \left| \int_{t_1}^{t} \|f(s, x_1(s)) - f(s, x_0(s))\|ds \right|
\]
\[
\leq \delta + M_1|t_1 - t_0| + \left| \int_{t_1}^{t} M_2\|x_1(s) - x_0(s)\|ds \right|
\]
\[
\leq \delta(1 + M_1) + \left| \int_{t_1}^{t} M_2\|x_1(s) - x_0(s)\|ds \right|.
\]
By Gronwall’s inequality and our choice of \(\delta\), we obtain
\[
\|x_1(t) - x_0(t)\| \leq \delta(1 + M_1) \exp M_2|t - t_1| \leq \varepsilon < r,
\]
for \(t \in J \cap I^*\). This estimate shows that throughout the time interval \(J \cap I^*\), \((t, x_1(t)) \in C(t, x_0(t))\) and so \((t, x(t))\) is contained in the interior of \(K'\). Thus, we have shown that \(J \subset J^*\), and therefore \(\beta \leq \beta^* < \beta(t_1, x_1)\) and \(x_1(t)\) remains within \(\varepsilon\) of \(x_0(t)\) on \(J\). This completes the proof of the lemma.

\(\square\)
3.6. Flow of Nonautonomous Systems

Let $f : \Omega \to \mathbb{R}^n$ be a vector field which satisfies the hypotheses of the Picard Theorem 3.3.1. Given $(t_0, x_0) \in \Omega$, let $x(t, t_0, x_0)$ be the corresponding solution of the initial value problem defined on the maximal existence interval $I(t_0, x_0) = (\alpha(t_0, x_0), \beta(t_0, x_0))$. The domain of $x(t, t_0, x_0)$ is

$$D = \{(t, t_0, x_0) \in \mathbb{R}^{n+2} : (t, x_0) \in \Omega, \ t \in I(t_0, x_0)\}.$$  

The domain $D$ is open, and $x(t, t_0, x_0)$ is continuous on $D$. The next result summarizes some key properties of the solution.

**Lemma 3.6.1.**

1. If $(s, t_0, x_0), (t, t_0, x_0) \in D$, then $(t, s, x(s, t_0, x_0)) \in D$, and
   $$x(t, t_0, x_0) = x(t, s, x(s, t_0, x_0)) \quad \text{for} \quad t \in I(t_0, x_0).$$
2. If $(s, t_0, x_0) \in D$, then $I(t_0, x_0) = I(s, x(s, t_0, x_0))$.
3. If $(t_0, t_0, x_0) \in D$, then $x(t_0, t_0, x_0) = x_0$.
4. If $(s, t_0, x_0) \in D$, then $(t_0, s, x(s, t_0, x_0)) \in D$ and
   $$x(t_0, s, x(s, t_0, x_0)) = x_0.$$

**Proof.** The first two statements are a consequence of the uniqueness theorem (3.3.2). The two solutions $x(t, t_0, x_0)$ and $x(t, s, x(s, t_0, x_0))$ pass through the point $(s, x(s, t_0, x_0))$, and so they share the same maximal existence interval and they are equal on that interval.

The third statement follows from the definition of $x$.

Finally, if $(s, t_0, x_0) \in D$, then by (2),

$$t_0 \in I(t_0, x_0) = I(s, x(s, t_0, x_0)).$$

Thus, $(t_0, s, x(s, t_0, x_0)) \in D$, and we may substitute $t_0$ for $t$ in (1) to get the result:

$$x_0 = x(t_0, t_0, x_0) = x(t_0, s, x(s, t_0, x_0)).$$

$\square$

**Example.** Let $A$ be an $n \times n$ matrix over $\mathbb{R}$. The solution of the initial value problem

$$x' = Ax, \quad x(t_0) = x_0$$

is $x(t, t_0, x_0) = \exp A(t-t_0)x_0$. Here, we have $\Omega = \mathbb{R}^{n+1}$, $I(t_0, x_0) = \mathbb{R}$, for every $(t_0, x_0) \in \mathbb{R}^{n+1}$, and $D = \mathbb{R}^{n+2}$. In this case, Lemma 3.6.1 says

1. $\exp A(t-t_0)x_0 = \exp A(t-s) \exp A(s-t_0)x_0$,
2. $\exp A(t_0-t_0)x_0 = x_0$. 


3. EXISTENCE THEORY

(4) \exp A(t_0 - s) \exp A(s - t_0) x_0 = x_0,

which follow from Lemma 2.2.1.

DEFINITION 3.6.1. Let \( t, s \in \mathbb{R} \). The flow of the vector field \( f \) from time \( s \) to time \( t \) is the map \( y \mapsto \Phi_{t,s}(y) \equiv x(t, s, y) \). The domain of the flow map is therefore the set

\[ U(t, s) \equiv \{ y \in \mathbb{R}^n : (s, y) \in \Omega, t \in I(s, y) \} = \{ y \in \mathbb{R}^n : (t, s, y) \in D \} . \]

Notice that \( U(t, s) \subset \mathbb{R}^n \) is open because the domain \( D \) is open. It is possible that \( U(t, s) \) is empty for some pairs \( t, s \).

EXAMPLE. Continuing the previous example of linear systems, we have \( \Phi_{t,t_0} = \exp A(t - t_0) \).

EXAMPLE. Consider the scalar initial value problem

\[ x' = x^2, \quad x(t_0) = x_0. \]

Then \( \Phi_{t,t_0}(x_0) = x(t, t_0, x_0) = x_0/\[1 - x_0(t - t_0)] \) on \( D = \mathbb{R}^3 \). If \( t > t_0 \), then \( U(t, t_0) = (-\infty, (t - t_0)^{-1}) \) and \( U(t_0, t) = ((t_0 - t)^{-1}, \infty) \).

LEMMA 3.6.2.

1. If \( t_0 \leq s \leq t \) or if \( t \leq s \leq t_0 \), then \( U(t, t_0) \subset U(s, t_0) \).
2. Transitivity property:

\[ \Phi_{s,t_0} : U(t, t_0) \cap U(s, t_0) \to U(t, s) \]

and

\[ \Phi_{t,t_0} = \Phi_{t,s} \circ \Phi_{s,t_0}, \quad \text{on} \quad U(t, t_0) \cap U(s, t_0) . \]

3. If \( x_0 \in U(t_0, t_0) \), then \( \Phi_{t_0,t_0}(x_0) = x_0 \).
4. Inverse property:

\[ \Phi_{t,t_0} : U(t, t_0) \to U(t_0, t) \]

and

\[ \Phi_{t_0,t} \circ \Phi_{t,t_0}(x_0) = x_0, \quad x_0 \in U(t, t_0) . \]

5. \( \Phi_{t,t_0} \) is a homeomorphism from \( U(t, t_0) \) onto \( U(t_0, t) \).

PROOF. Suppose that \( t_0 \leq s \leq t \). If \( x_0 \in U(t, t_0) \), then \( t \in I(t_0, x_0) \). Since \( I(t_0, x_0) \) is an interval containing \( t_0 \), we must have \([t_0, t] \subset I(t_0, x_0)\), and so \([t_0, s] \subset [t_0, t] \subset I(t_0, x_0)\). Thus, \( x_0 \in U(s, t_0) \). The other case is identical.

The remaining statements are simply a restatement of Lemma 3.6.1 using our new notation.

Lemma 3.6.1, (1), says that if \( x_0 \in U(s, t_0) \cap U(t, t_0) \), then \( \Phi_{s,t_0}(x_0) \in U(t, s) \) and \( \Phi_{t,t_0}(x_0) = \Phi_{t,s} \circ \Phi_{s,t_0}(x_0) \), which is statement (2) above.
Statements (3) and (4) are equivalent to (3) and (4) of Lemma 3.6.1.

The continuity of \( x(t,t_0,x_0) \) on \( D \) implies that \( \Phi_{t,t_0} \) is continuous on \( U(t,t_0) \). It is one-to-one and onto by (4).

\[ \square \]

**Remark.** If \( \Phi_{t,t_0} \) is a map which satisfies the properties in Lemma 3.6.2 and which is \( C^1 \) in the \( t \)-variable, then it is the flow of the vector field

\[ f(t,x) = \frac{d}{ds} \Phi_{s,t}(x) \bigg|_{s=t}, \]

with domain \( \Omega = \cup_t U(t,t). \)

### 3.7. Flow of Autonomous Systems

Suppose now that the vector field \( f(t,x) = f(x) \) is autonomous. Then we may assume that its domain has the form \( \Omega = \mathbb{R} \times \mathcal{O} \) for an open set \( \mathcal{O} \subset \mathbb{R}^n \). As usual, given \( x_0 \in \mathcal{O}, x(t,t_0,x_0) \) denotes the solution of the (autonomous) IVP

\[ x' = f(x), \quad x(t_0) = x_0, \]

with maximal existence interval \( I(t_0,x_0) \).

**Lemma 3.7.1.** Let \( x_0 \in \mathcal{O}, t, \tau \in \mathbb{R} \). Then

\[ t + \tau \in I(t_0,x_0) \quad \text{if and only if} \quad t \in I(t_0-\tau,x_0), \]

and

\[ x(t+\tau,t_0,x_0) = x(t,t_0-\tau,x_0), \quad \text{for} \quad t \in I(t_0-\tau,x_0). \]

**Proof.** Since \( x(t,t_0,x_0) \) is a solution on \( I(t_0,x_0) \), we have

\[ x'(t,t_0,x_0) = f(x(t,t_0,x_0)), \quad \text{for all} \quad t \in I(t_0,x_0). \]

Fix \( \tau \in \mathbb{R} \) and define \( J = \{ t : t + \tau \in I(t_0,x_0) \} \). Substituting \( t + \tau \) for \( t \), we see that

\[ x'(t+\tau,t_0,x_0) = f(x(t+\tau,t_0,x_0)), \quad \text{for all} \quad t \in J. \]

Let \( y(t) = x(t+\tau,t_0,x_0) \) be a translate of the solution. Here is the key point. *Since the system is autonomous*, we claim that \( y(t) \) solves the equation on \( J \). Using the chain rule and (3.7.1), we have

\[ y'(t) = \frac{d}{dt} [x(t+\tau,t_0,x_0)] = x'(t+\tau,t_0,x_0) = f(x(t+\tau,t_0,x_0)) = f(y(t)), \]

on the interval \( J \). Since \( y(t_0-\tau) = x_0 \), it follow by the uniqueness theorem 3.3.2 that

\[ x(t+\tau,t_0,x_0) = y(t) = x(t,t_0-\tau,x_0), \]

and \( I(t_0-\tau,x_0) = J \).
Using the flow notation, this can be restated as:

**Lemma 3.7.2.** For \( t, t_0 \in \mathbb{R} \), we have

\[
U(t + \tau, t_0) = U(t, t_0 - \tau) \quad \text{and} \quad \Phi_{t + \tau, t_0} = \Phi_{t, t_0 - \tau}.
\]

**Corollary 3.7.1.** For \( t, s \in \mathbb{R} \), we have that

\[
\Phi_{t + s, 0} = \Phi_{t, 0} \circ \Phi_{s, 0}, \quad \text{on the domain} \quad U(t + s, 0) \cap U(s, 0).
\]

**Proof.** By the general result, Lemma 3.6.2, we have

\[
\Phi_{t + s, 0} = \Phi_{t, 0} \circ \Phi_{s, 0}, \quad \text{on the domain} \quad U(t + s, 0) \cap U(s, 0).
\]

By Lemma 3.7.2, \( \Phi_{t + s, 0} = \Phi_{t, 0} \). \( \square \)

**Remark.** Because of Lemma 3.7.2, valid only in the autonomous case, there is no loss of generality in using \( \Phi_{t, 0} \) since \( \Phi_{t - s, 0} = \Phi_{t, s} \). As is commonly done, we shall use simply \( \Phi_t \) to denote the flow \( \Phi_{t, 0} \).

**Example.** Suppose that \( f(x) = Ax \) with \( A \) an \( n \times n \) matrix over \( \mathbb{R} \). Then \( \Phi_t = \Phi_{t, 0} = \exp At \), and the corollary is the familiar property that \( \exp A(t + s) = \exp At \exp As \).

**Example.** Suppose that \( \Phi_{t,s}(x_0) \) is the flow associated to an \( n \)-dimensional nonautonomous initial value problem

\[
x'(t) = f(t, x(t)), \quad x(s) = x_0.
\]

We saw in Chapter 1 that a nonautonomous system can be reformulated as an autonomous system by treating the time variable as a dependent variable. If \( y(t) = (u(t), x(t)) \in \mathbb{R}^{n+1} \), then the equivalent autonomous system is

\[
y'(t) = g(y(t)), \quad y(s) = y_0,
\]

with

\[
g(y) = g(u, x) = (1, f(u, x)) \quad \text{and} \quad y_0 = (u_0, x_0).
\]

By direct substitution, the flow of this system is

\[
\Psi_{t,s}(y_0) = \Psi_{t,s}(u_0, x_0) = (t - s + u_0, \Phi_{t-s+u_0, u_0}(x_0)).
\]

It is immediate that \( \Psi_{t,s} \) satisfies the property of Lemma 3.7.2, namely

\[
\Psi_{t + \tau, s}(y_0) = \Psi_{t, s - \tau}(y_0).
\]

**Definition 3.7.1.** Given \( x_0 \in \mathcal{O} \), define the orbit of \( x_0 \) to be the curve

\[
\gamma(x_0) = \{ x(t, 0, x_0) : t \in I(0, x_0) \} = \{ \Phi_t(x_0) : t \in I(0, x_0) \}.
\]
Notice that the orbit is a curve in the phase space $\mathcal{O} \subset \mathbb{R}^n$, as opposed to the solution trajectory \( \{(t, x(t, t_0, x_0)) : t \in I(t_0, x_0)\} \) which is a curve in the space-time domain $\Omega = \mathbb{R} \times \mathcal{O} \subset \mathbb{R}^{n+1}$.

For autonomous flow, we have the following strengthening of the Uniqueness Theorem 3.3.2.

**Theorem 3.7.1.** If $z \in \gamma(x_0)$, then $\gamma(x_0) = \gamma(z)$. Thus, if two orbits intersect, then they are identical.

**Proof.** Suppose that $z \in \gamma(x_0)$. Then $z = x(\tau, 0, x_0)$, for some $\tau \in I(0, x_0)$. By Lemma 3.6.1, we have that $I(0, x_0) = I(\tau, z)$ and $x(t, 0, x_0) = x(t, \tau, z)$ on this interval. Thus, we may write

$$
\gamma(x_0) = \{x(t, 0, x_0) : t \in I(0, x_0)\} \\
= \{x(t, \tau, z) : t \in I(\tau, z)\} \\
= \{x(t + \tau, \tau, z) : t + \tau \in I(\tau, z)\}.
$$

On the other hand, Lemma 3.7.1, says that

$$
t + \tau \in I(\tau, z) \quad \text{if and only if} \quad t \in I(0, z),
$$

and

$$
x(t + \tau, \tau, z) = x(t, 0, z), \quad \text{for} \quad t \in I(0, z).
$$

Thus, we see that

$$
\gamma(z) = \{x(t, 0, z) : t \in I(0, z)\} \\
= \{x(t + \tau, \tau, z) : t \in I(\tau, z)\}.
$$

This shows that $\gamma(x_0) = \gamma(z)$.

From the existence and uniqueness theory for general systems, we have that the domain $\Omega$ is foliated by the solution trajectories

$$
\{(t, x(t, t_0, x_0)) : t \in I(t_0, x_0)\}.
$$

That is, every point $(t_0, x_0) \in \Omega$ has a unique trajectory passing through it. Theorem 3.7.1 says that, for autonomous systems, the phase space $\mathcal{O}$ is foliated by the orbits. Each point of the phase space $\mathcal{O}$ has a unique orbit passing through it. For this reason, phase diagrams are meaningful in the autonomous case.

Since $x' = f(x)$, the orbits are curves in $\mathcal{O}$ everywhere tangent to the vector field $f(x)$. They are sometimes also referred to as integral curves. They can be obtained by solving the system

$$
\frac{dx_1}{f_1(x)} = \cdots = \frac{dx_n}{f_n(x)}.
$$
Example. Consider the harmonic oscillator
\[ x_1' = x_2, \quad x_2' = -x_1. \]
The system for the integral curves is
\[ \frac{dx_1}{x_2} = \frac{dx_2}{-x_1}. \]
Solutions satisfy
\[ x_1^2 + x_2^2 = c, \]
and so we confirm that the orbits are concentric circles centered at the origin.

3.8. Global Solutions

As usual, we assume that \( f : \Omega \to \mathbb{R}^n \) satisfies the hypotheses of the Picard Existence Theorem 3.3.1.

Recall that a global solution of the initial value problem is one whose maximal interval of existence is \( \mathbb{R} \). For this to be possible, it is necessary for the domain \( \Omega \) to be unbounded in the time direction. So let’s assume that \( \Omega = \mathbb{R} \times O \), where \( O \subset \mathbb{R}^n \) is open.

**Theorem 3.8.1.** Let \( I = (\alpha, \beta) \) be the maximal interval of existence of some solution \( x(t) \) of the initial value problem. Then either \( \beta = +\infty \) or for every compact set \( K \subset O \), there exists a time \( \beta(K) \) such that \( x(t) \notin K \) for all \( t \in (\beta(K), \beta) \).

**Proof.** Assume that \( \beta < +\infty \). Suppose that \( K \subset O \) is compact. Then \( K' = [t_0, \beta] \times K \subset \Omega \) is compact. By Theorem 3.4.1, there exists a time \( \beta(K') < \beta \) such that \( (t, x(t)) \notin K' \) for \( t \in (\beta(K'), \beta) \). Since \( t < \beta \), this implies that \( x(t) \notin K \) for \( t \in (\beta(K'), \beta) \). \( \square \)

Of course, the analogous statement holds for \( \alpha \), the left endpoint of \( I \).

**Theorem 3.8.2.** If \( \Omega = \mathbb{R}^{n+1} \), then either \( \beta = +\infty \) or
\[ \lim_{t \to \beta^-} \|x(t)\| = +\infty. \]

**Proof.** If \( \beta < +\infty \), then by the previous result, for every \( R > 0 \), there is a time \( \beta(R) \) such that \( x(t) \notin \{x : \|x\| \leq R\} \) for \( t \in (\beta(R), \beta) \). In other words, \( \|x(t)\| > R \) for \( t \in (\beta(R), \beta) \), i.e. the desired conclusion holds. \( \square \)

**Corollary 3.8.1.** Assume that \( \Omega = \mathbb{R}^{n+1} \). If there exists a non-negative continuous function \( \psi(t) \) defined for all \( t \in \mathbb{R} \) such that
\[ \|x(t)\| \leq \psi(t) \quad \text{for all} \quad t \in [t_0, \beta), \]

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3.8. GLOBAL SOLUTIONS

then $\beta = +\infty$.

**Proof.** The assumed estimate implies that $\|x(t)\|$ remains bounded on any bounded time interval. By the previous result, we must have that $\beta = +\infty$. $\square$

**Theorem 3.8.3.** Let $\Omega = \mathbb{R}^{n+1}$. Suppose that there exist continuous nonnegative functions $c_1(t), c_2(t)$ defined on $\mathbb{R}$ such that the vector field $f$ satisfies

$$\|f(t, x)\| \leq c_1(t)\|x\| + c_2(t), \quad \text{for all } (t, x) \in \mathbb{R}^{n+1}.$$ 

Then the solution to the initial value problem is global, for every $(t_0, x_0) \in \mathbb{R}^{n+1}$.

**Proof.** We know that a solution of the initial value problem also solves the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds,$$

for all $t \in (\alpha, \beta)$. Applying the estimate for $\|f(t, x)\|$, we find that

$$\|x(t)\| \leq \|x_0\| + \left| \int_{t_0}^{t} [c_1(s)\|x(s)\| + c_2(s)]ds \right|$$

$$\leq \|x_0\| + \left| \int_{t_0}^{t} c_2(s)ds \right| + \left| \int_{t_0}^{t} c_1(s)\|x(s)\|ds \right|$$

$$= C(t) + \left| \int_{t_0}^{t} c_1(s)\|x(s)\|ds \right|,$$

with

$$C(t) = \|x_0\| + \int_{t_0}^{t} c_2(s)ds.$$

We now adapt our version of Gronwall’s inequality to this slightly more general situation. Notice that $C(t)$ is nondecreasing for $t \in [t_0, \beta)$. Fix $T \in [t_0, \beta)$. Then for $t \in [t_0, T)$ we have

$$\|x(t)\| \leq C(T) + \int_{t_0}^{t} c_1(s)\|x(s)\|ds.$$

Then Gronwall’s Lemma 3.3.3 implies that

$$\|x(t)\| \leq C(T) \exp \int_{t_0}^{t} c_1(s)ds,$$

for $t \in [t_0, T]$. In particular, this holds for $t = T$. Thus, we obtain

$$\|x(T)\| \leq C(T) \exp \int_{t_0}^{T} c_1(s)ds,$$
for all \( T \in [t_0, \beta) \). Therefore, by Corollary 3.8.1, we conclude that \( \beta = +\infty \).

In the same way, we have that
\[
\|x(T)\| \leq C(T) \exp \int_{T_0}^{T} c_1(s)\,ds,
\]
for all \( T \in (\alpha, t_0] \), and hence \( \alpha = -\infty \). □

**Corollary 3.8.2.** Let \( A(t) \) be an \( n \times n \) matrix and let \( F(t) \) be an \( n \)-vector which depend continuously on \( t \in \mathbb{R} \). Then the linear initial value problem
\[
x'(t) = A(t)x(t) + F(t), \quad x(t_0) = x_0
\]
have a unique global solution for every \((t_0, x_0) \in \mathbb{R}^{n+1}\).

**Proof.** This is an immediate corollary of the preceding result, since the vector field
\[
f(t, x) = A(t)x + F(t)
\]
satisfies
\[
\|f(t, x)\| \leq \|A(t)\|\|x\| + \|F(t)\|.
\]
□

### 3.9. Stability

Let \( \Omega = \mathbb{R} \times \mathcal{O} \) for some open set \( \mathcal{O} \subset \mathbb{R}^n \), and suppose that \( f : \Omega \to \mathbb{R}^n \) satisfies the hypotheses of the Picard Theorem.

**Definition 3.9.1.** A point \( \bar{x} \in \mathcal{O} \) is called an equilibrium point (singular point, critical point) if \( f(t, \bar{x}) = 0 \), for all \( t \in \mathbb{R} \).

**Definition 3.9.2.** An equilibrium point \( \bar{x} \) is stable if given any \( \varepsilon > 0 \), there exists a \( \delta > 0 \) such that for all \( \|x_0 - \bar{x}\| < \delta \), the solution of the initial value problem \( x(t, 0, x_0) \) exists for all \( t \geq 0 \) and
\[
\|x(t, 0, x_0) - \bar{x}\| < \varepsilon, \quad t \geq 0.
\]

An equilibrium point \( \bar{x} \) is asymptotically stable if it is stable and there exists a \( b > 0 \) such that if \( \|x_0 - \bar{x}\| < b \), then
\[
\lim_{t \to \infty} \|x(t, 0, x_0) - \bar{x}\| = 0.
\]

An equilibrium point \( \bar{x} \) is unstable if it is not stable.

**Example.**
- A center in \( \mathbb{R}^2 \) is stable, but not asymptotically stable.
- A sink or a spiral sink is asymptotically stable.
- A saddle is unstable.
Remark. If \( \bar{x} \) is a stable equilibrium (or asymptotically stable equilibrium), then the conclusion of Definition 3.9.2 holds for any initial time \( t_0 \in \mathbb{R} \), by Lemma 3.5.1.

Theorem 3.9.1. Let \( A \) be an \( n \times n \) matrix over \( \mathbb{R} \), and define the linear vector field \( f(x) = Ax \).

The equilibrium \( \bar{x} = 0 \) is asymptotically stable if and only if \( \text{Re} \lambda < 0 \) for all eigenvalues of \( A \).

The equilibrium \( \bar{x} = 0 \) is stable if and only if \( \text{Re} \lambda \leq 0 \) for all eigenvalues of \( A \) and \( A \) has no generalized eigenvectors corresponding to eigenvalues with \( \text{Re} \lambda = 0 \).

Proof. Recall that the solution of the initial value problem is \( x(t, 0, x_0) = \exp At x_0 \).

If \( \text{Re} \lambda < 0 \) for all eigenvalues, then \( E_s = \mathbb{R}^n \) and by Corollary 2.5.1, there is a constant \( C, \alpha > 0 \) such that

\[
\|x(t, 0, x_0)\| \leq C \exp[-\alpha t]\|x_0\|,
\]

for all \( t \geq 0 \). Asymptotic stability follows from this estimate.

If \( \bar{x} = 0 \) is asymptotically stable, then by linearity,

\[
\lim_{t \to \infty} x(t, 0, x_0) = \lim_{t \to \infty} \exp At x_0 = 0,
\]

for all \( x_0 \in \mathbb{R}^n \). By Theorem 2.5.2, we have that \( E_s = \mathbb{R}^n \). Thus, \( \text{Re} \lambda < 0 \) for all eigenvalues of \( A \).

If \( \text{Re} \lambda \leq 0 \) for all eigenvalues, then \( E_u + E_c = \mathbb{R}^n \) and by Corollary 2.5.1 and the remark following Theorem 2.5.1,

\[
\|x(t, 0, x_0)\| = \|\exp At (P_s + P_u)x_0\| \leq C \exp[-\alpha t]\|P_s x_0\| + C(1 + (t - t_0)^p)\|P_c x_0\|,
\]

for all \( t > t_0 \), where \( p + 1 \) is the size of the Jordan block corresponding to eigenvalues with zero real part. Now if \( A \) has no generalized eigenvectors corresponding to eigenvalues with zero real part, then we may take \( p = 0 \), above. Thus, the right-hand side is bounded by \( C\|x_0\| \). Stability follows from this estimate.

If \( A \) has an eigenvalue with positive real part, then \( E_u \neq \{0\} \). By Corollary 2.5.2,

\[
\lim_{t \to \infty} \|x(t, 0, x_0)\| = \lim_{t \to \infty} \|\exp At x_0\| = +\infty,
\]

for all \( 0 \neq x_0 \in E_u \). This implies that the origin is unstable.

Finally, if \( A \) has an eigenvalue \( \lambda \) with \( \text{Re} \lambda = 0 \) and a generalized eigenvector, then there exists \( 0 \neq x_0 \in N(A - \lambda I)^2 \setminus N(A - \lambda I) \). Thus,
$y = (A - \lambda I)x_0 \neq 0$ and $(A - \lambda I)^k x_0 = 0, \ k = 2, 3, \ldots$. From the definition of the exponential, we have that
\[
\exp At x_0 = \exp \lambda t \exp (A - \lambda I)^t x_0 \\
= \exp \lambda t \cdot [I + t(A - \lambda I)] x_0 \\
= \exp \lambda t (x_0 + ty).
\]

Since $Re \lambda = 0$, this gives the estimate
\[
\|x(t, 0, x_0)\| = \|\exp At x_0\| = \|x_0 + ty\| \geq t\|y\| - \|x_0\|, \ t \geq 0,
\]
which implies that the origin is unstable.

\[\square\]

**Theorem 3.9.2.** Let $O \subset \mathbb{R}^n$ be an open set, and let $f : O \to \mathbb{R}^n$ be $C^1$. Suppose that $\bar{x} \in O$ is an equilibrium point of $f$ and that the eigenvalues of $A = Df(\bar{x})$ all satisfy $Re \lambda < 0$. Then $\bar{x}$ is asymptotically stable.

**Proof.** Let $g(y) = f(\bar{x} + y) - Ay$ for $y \in O' = \{y \in \mathbb{R}^n : x_0 + y \in O\}$. Then $g \in C^1(O')$, $g(0) = 0$, and $Dg(0) = 0$.

Asymptotic stability of the equilibrium $x = \bar{x}$ for the system $x' = f(x)$ is equivalent to asymptotic stability of the equilibrium $y = 0$ for the system $y' = Ay + g(y)$.

Since $Re \lambda < 0$ for all eigenvalues of $A$, know that from Corollary 2.5.1 that there exists $0 < \alpha < \lambda_\gamma$ and a constant $C_1 > 0$ such that
\[
\|\exp A(t - s)\| \leq C_1 \exp[-\alpha(t - s)], \ t \geq s.
\]

Since $Dg(y)$ is continuous and $Dg(0) = 0$, given $\rho < \alpha/C_1$, there is a $\mu > 0$ such that
\[
\|Dg(y)\| \leq \rho, \text{ for } \|y\| \leq \mu.
\]

Using the elementary formula
\[
g(y) = \int_0^1 \frac{d}{ds} [g(sy)]ds = \int_0^1 Dg(sy)yds,
\]
we see that for $\|y\| \leq \mu$,
\[
\|g(y)\| \leq \int_0^1 \|Dg(sy)\|\|y\|ds \leq \rho\|y\|.
\]

Choose a $\delta < \mu$, and assume that $\|y_0\| < \delta$. Let $y(t) = y(t, 0, y_0)$ be the solution of the initial value problem
\[
y'(t) = Ay(t) + g(y(t)), \quad y(0) = y_0,
\]
defined for $t \in (\alpha, \beta)$. Define
\[
T = \sup\{t : \|y(s)\| < \mu, \text{ for } 0 \leq s < t\}.
\]
Then $T > 0$, by continuity, and of course, $T \leq \beta$.

If we treat $g(y(t))$ as an inhomogeneous term, then by the variation of parameters formula, see Corollary ??,

$$y(t) = \exp At \cdot y_0 + \int_0^t \exp A(t-s)g(y(s))ds.$$ 

Thus, since $\|y(t)\| < \mu$ on the interval $[0, T)$, we have

$$\|y(t)\| \leq C_1 e^{-\alpha t} \|y_0\| + \int_0^t e^{-\alpha(t-s)} \rho \|y(s)\| ds.$$ 

Set $z(t) = e^{\alpha t}\|y(t)\|$. Then

$$z(t) \leq C_1 \|y_0\| + \int_0^t \rho z(s) ds.$$ 

By the Gronwall inequality 3.3.3, we obtain

$$z(t) \leq C_1 \|y_0\| e^{C_1 \rho t}, \quad 0 \leq t < T.$$ 

In other words, we have

$$\|y(t)\| \leq C_1 \|y_0\| e^{(C_1 \rho - \alpha) t} \leq C_1 \delta e^{(C_1 \rho - \alpha) t}, \quad 0 \leq t < T.$$ 

Let $\epsilon < \mu/2$ be given. Recall that $C_1 \rho - \alpha < 0$. We may choose $\delta$ small enough so that in addition to $\delta < \mu$, as above, we also have $C_1 \delta < \epsilon < \mu/2$. Now if $T < \beta$, then $y(T)$ is defined and the estimate would show that $\|y(T)\| < \epsilon < \mu/2$, contradicting the maximality of the interval $[0, T)$. Thus, we must have $T = \beta$. But then, we conclude that $\|y(t)\| < \epsilon$ throughout its interval of existence, and thus, by Corollary 3.8.1, we have that $T = \beta = +\infty$. Since $\epsilon$ was arbitrary, we have, moreover, that the origin is a stable equilibrium. The estimate also shows that $\|y(t)\| \to 0$, as $t \to \infty$, which establishes asymptotic stability of the origin.

\[\square\]

**Remark.** Recall from Chapter 1 that the equation $y' = Ay$, with $A = Df(\bar{x})$, is the linearized equation for the nonlinear system $x' = f(x)$, near an equilibrium $\bar{x}$. The previous two results say that $\bar{x}$ is asymptotically stable for the nonlinear system if the origin is asymptotically stable for the linearized system.

**Example.** The scalar problem $x' = -x + x^2$ has equilibria at $x = 0, 1$. The origin is asymptotically stable for the linearized problem $y' = -y$, and so it is also asymptotically stable for the nonlinear problem. The linearized problem at $x = 1$ is $y' = y$, for which the origin is
unstable, and thus, \( x = 1 \) is an unstable equilibrium. In this simple example, we can write down the explicit solution

\[
x(t, 0, x_0) = \frac{x_0}{x_0 + (1 - x_0)e^t}.
\]

Note that \( x(t, 0, x_0) \to 0 \) exponentially, as \( t \to \infty \), provided \( x_0 < 1 \). This demonstrates both the asymptotic stability of \( x = 0 \) as well as the instability of \( x = 1 \). Of course, the using the phase diagram easiest way to analyze stability in this example.

### 3.10. Liapunov Stability

Let \( f(x) \) be a locally Lipschitz continuous vector field on an open set \( \mathcal{O} \subset \mathbb{R}^n \). Assume that \( f \) has an equilibrium point at \( \bar{x} \in \mathcal{O} \).

**Definition 3.10.1.** Let \( U \subset \mathcal{O} \) be a neighborhood of \( \bar{x} \). A Liapunov function for an equilibrium point \( \bar{x} \) of a vector field \( f \) is a function \( E : U \to \mathbb{R} \) such that

(i) \( E \in C(U) \cap C^1(U \setminus \{\bar{x}\}) \),

(ii) \( E(x) > 0 \) for \( x \in U \setminus \{\bar{x}\} \) and \( E(\bar{x}) = 0 \),

(iii) \( DE(x) f(x) \leq 0 \) for \( x \in U \setminus \{\bar{x}\} \).

If strict inequality holds in (iii), then \( E \) is called a strict Liapunov function.

**Theorem 3.10.1.** If an equilibrium point \( \bar{x} \) of \( f \) has a Liapunov function, then it is stable.

If \( \bar{x} \) has a strict Liapunov function, then it is asymptotically stable.

**Proof.** Suppose that \( E \) is a Liapunov function for \( \bar{x} \).

Choose any \( \varepsilon > 0 \) such that \( \overline{B}_\varepsilon(\bar{x}) \subset U \). Define

\[
m = \min\{E(x) : \|x - \bar{x}\| = \varepsilon\} \quad \text{and} \quad U_\varepsilon = \{x \in U : E(x) < m\} \cap B_\varepsilon(\bar{x}).
\]

Notice that \( U_\varepsilon \subset U \) is a neighborhood of \( \bar{x} \).

The claim is that for any \( x_0 \in U_\varepsilon \), the solution \( x(t) = x(t, 0, x_0) \) of the IVP \( x' = f(x), x(0) = x_0 \) is defined for all \( t \geq 0 \) and remains in \( U_\varepsilon \).

By the local existence theorem and continuity of \( x(t) \), we have that \( x(t) \in U_\varepsilon \) on some nonempty interval of the form \([0, \tau)\). Let \([0, T)\) be the maximal such interval. The claim amounts to showing that \( T = \infty \).

On the interval \([0, T)\), we have that \( x(t) \in U_\varepsilon \subset U \) and since \( E \) is a Liapunov function,

\[
\frac{d}{dt} E(x(t)) = DE(x(t)) \cdot x'(t) = DE(x(t)) \cdot f(x(t)) \leq 0.
\]

From this it follows that

\[
E(x(t)) \leq E(x(0)) = E(x_0) < m,
\]
on $[0, T)$. So, if $T < \beta$, we would have $E(x(T)) \leq E(x(0)) < m$, and so, by definition of $m$, $x(T)$ cannot belong to the set $\|x - \bar{x}\|=\varepsilon$. Thus, we would have that $x(T) \in U_\varepsilon$. But this contradicts the maximality of the interval $[0, T)$. It follows that $T = \beta$. Since $x(t)$ remains in $U_\varepsilon$ on $[0, T) = [0, \beta)$, it remains bounded. So by Theorem 3.8.1, we have that $\beta = T = +\infty$.

We now use the claim to establish stability. Let $\varepsilon > 0$ be given. Without loss of generality, we may assume that $B_\varepsilon(\bar{x}) \subset U$. Choose $\delta > 0$ so that $B_\delta(\bar{x}) \subset U_\varepsilon$. Then for every $x_0 \in B_\delta(\bar{x})$, we have that $x(t) \in U_\varepsilon \subset B_\varepsilon(\bar{x})$, for all $t > 0$.

Suppose now that $E$ is a strict Liapunov function, and let us prove asymptotic stability.

The equilibrium $\bar{x}$ is stable, so given $\varepsilon > 0$ with $B_\varepsilon(\bar{x}) \subset U$, there is a $\delta > 0$ so that $x_0 \in B_\delta(\bar{x})$ implies $x(t) \in B_\varepsilon(\bar{x})$, for all $t > 0$.

Let $x_0 \in B_\delta(\bar{x})$. We must show that $x(t) = x(t, 0, x_0)$ satisfies $\lim_{t \to \infty} x(t) = \bar{x}$. We may assume that $x_0 \neq \bar{x}$, so that, by uniqueness, $x(t) \neq \bar{x}$, on $[0, \infty)$.

Since $E$ is strict and $x(t) \neq \bar{x}$, we have that

$$\frac{d}{dt} E(x(t)) = DE(x(t)) \cdot x'(t) = DE(x(t)) \cdot f(x(t)) < 0.$$ 

Thus, $E(x(t))$ is a monotonically decreasing function bounded below by 0. Set $E^* = \inf\{E(x(t)) : t > 0\}$. Then $E(x(t)) \downarrow E^*$.

Since the solution $x(t)$ remains in the bounded set $U_\varepsilon$, it has a limit point. That is, there exist a point $z \in \overline{U_\varepsilon} \subset U$ and a sequence of times $t_k \to \infty$ such that $x(t_k) \to z$. We have, moreover, that $E^* = \lim_{k \to \infty} E(x(t_k)) = E(z)$.

Let $s > 0$. By the properties of autonomous flow, Lemma 3.7.1, we have that

$$x(s + t_k) = x(s + t_k, 0, x_0) = x(s, 0, x(t_k, 0, x_0)) = x(s, 0, x(t_k)).$$

By continuous dependence on initial conditions, Theorem 3.5.1, we have that

$$\lim_{k \to \infty} x(s + t_k) = \lim_{k \to \infty} x(s, 0, x(t_k)) = x(s, 0, z).$$

From this and the fact that $E(x(s, 0, z))$ is nonincreasing, it follows that

$$E^* \leq \lim_{k \to \infty} E(x(s + t_k)) = E(x(s, 0, z)) \leq E(x(0, 0, z)) = E^*.$$
Thus, \( x(s, 0, z) \) is a solution along which \( E \) is constant. But then
\[
0 = \frac{d}{dt} E(x(t, 0, z)) = DE(x(t, 0, z)) f(x(t, 0, z)).
\]
By assumption, this forces \( x(t, 0, z) = \bar{x} \) for all \( t \geq 0 \), and thus, \( z = \bar{x} \).
We have shown that the unique limit point of \( x(t) \) is \( \bar{x} \), which equivalent to \( \lim_{t \to \infty} x(t) = \bar{x} \). □

**Example (Hamiltonian Systems).** Let \( H : \mathbb{R}^2 \to \mathbb{R} \) be a \( C^2 \) function. A system of the form
\[
\dot{p} = H_q(p, q) \\
\dot{q} = -H_p(p, q)
\]
is referred to as Hamiltonian, the function \( H \) being called the Hamiltonian. Suppose that \( H(0, 0) = 0 \) and \( H(p, q) > 0 \), for \( (p, q) \neq (0, 0) \). Then since the origin is a minimum for \( H \), we have that \( H_p(0, 0) = H_q(0, 0) = 0 \). In other words, the origin is an equilibrium for the system. Since
\[
DHf = (H_p, H_q)(H_q, -H_p)^T = 0,
\]
we see that \( H \) is a Liapunov function. Therefore, the origin is stable.
Moreover, notice that for any solution \( (p(t), q(t)) \) we have
\[
\frac{d}{dt} H(p(t), q(t)) = 0.
\]
Thus, \( H(p(t), q(t)) \) is constant in time. This says that each orbit lies on a level set of the Hamiltonian.

More generally, we may take \( H : \mathbb{R}^{2n} \to \mathbb{R} \) in \( C^1 \). Again with \( H(0, 0) = 0 \) and \( H(p, q) > 0 \), for \( (p, q) \neq (0, 0) \). The origin is stable for the system
\[
\dot{p} = D_q H(p, q) \\
\dot{q} = -D_p H(p, q).
\]

**Example (Newton’s equation).** Let \( G : \mathbb{R} \to \mathbb{R} \) be \( C^2 \), with \( G(0) = 0 \) and \( G(u) > 0 \), for \( u \neq 0 \). Consider the second order equation
\[
\ddot{u} + G'(u) = 0.
\]
With \( x_1 = u \) and \( x_2 = \dot{u} \), this is equivalent to the first order system
\[
\dot{x}_1 = x_2, \quad \dot{x}_2 = -G'(x_1).
\]
Notice that the origin is an equilibrium, since \( G'(0) = 0 \). The system is, in fact, Hamiltonian with \( H(x_1, x_2) = \frac{1}{2}x_2^2 + G(x_1) \). Since \( H \) is positive away from the equilibrium, we have that the origin is stable.

The nonlinear pendulum arises when \( G(u) = 1 - \cos u \).
Example (Liénard’s equation). Let $\Phi : \mathbb{R} \to \mathbb{R}$ be $C^1$, with $\Phi(0) = 0$. Liénard’s equation is

$$\ddot{u} + \Phi'(u)\dot{u} + u = 0.$$ 

It models certain electrical circuits, among other things. We rewrite this as a first order system in a nonstandard way. If we let

$$x_1 = u, \quad x_2 = \dot{u} + \Phi(u),$$

then

$$\dot{x}_1 = x_2 - \Phi(x_1), \quad \dot{x}_2 = -x_1.$$ 

Notice that the origin is an equilibrium.

Suppose that and $\Phi'(u) \geq 0$. The function $E(x) = \frac{1}{2}[x_1^2 + x_2^2]$ serves as a Liapunov function at the origin, since

$$DE(x)f(x) = -x_1\Phi(x_1) = -x_1\int_0^{x_1} \Phi'(u)du \leq 0,$$

by our assumptions on $\Phi$. So the origin is a stable equilibrium. If $\Phi'(u) > 0$, for $u \neq 0$, then $E$ is a strict Liapunov function, and the origin is asymptotically stable.