(1) Let $f : \mathbb{R}^{n+1} \to \mathbb{R}^n$ be the vector field

$$f(t, x) = \frac{(1 + t^2) x}{(1 + t^2 + \|x\|^2)^{1/2}}.$$  

Prove that for every $(t_0, x_0) \in \mathbb{R}^{n+1}$, the initial value problem

$$x' = f(t, x), \quad x(t_0) = x_0$$

has a unique solution defined on the interval $-\infty < t < \infty$.

(2) Let $\mathcal{O} \subset \mathbb{R}^n$ be an open set, and set $\Omega = \mathbb{R} \times \mathcal{O}$. Suppose that $f : \Omega \to \mathbb{R}^n$ satisfies the hypotheses of the Picard existence and uniqueness theorem. Suppose also that $\bar{x} \in \mathcal{O}$ is a stable equilibrium for $f$, according to Definition 3.9.2 of the notes.

Proof that given any $\varepsilon > 0$ and $t_0 \in \mathbb{R}$, there exists a $\delta > 0$ such that if $\|x_0 - \bar{x}\| < \delta$, then the solution of the initial value problem $x(t, t_0, x_0)$ exists for all $t \geq t_0$ and

$$\|x(t, t_0, x_0) - \bar{x}\| < \varepsilon, \quad \text{for} \quad t \geq t_0.$$  

(Thus, the notion of stability does not depend upon the choice of the initial time.)

(3) Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ by $f(x) = Ax$, where $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is a nonzero matrix with real entries. Let

$$\text{tr} \ A = a_{11} + a_{22} \quad \text{and} \quad \text{det} \ A = a_{11}a_{22} - a_{12}a_{21}.$$  

Prove the following statements.

(a) If $\text{tr} \ A < 0$ and $\text{det} \ A > 0$, then the origin is asymptotically stable for $f$.

(b) If $\text{tr} \ A < 0$ and $\text{det} \ A = 0$, then the origin is stable for $f$.

(c) If $\text{tr} \ A = 0$ and $\text{det} \ A > 0$, then the origin is stable for $f$.

(d) In all other cases, the origin is unstable for $f$.

(4) Let $A_k$ be $2 \times 2$ matrices over $\mathbb{R}$, and let $f_k(x) = A_kx$, $k = 1, 2$.

(a) Assume that the origin is asymptotically stable for $f_1$.

Prove that there exists a $\delta > 0$ (depending on $A_1$) such that if $\|A_1 - A_2\| < \delta$, then the origin is asymptotically stable for $f_2$.

(b) Is the preceding statement true if “asymptotically stable” is replaced by “stable”? Explain.