Math 243C – Assignment 1
Due 04/13/10

(1) Let $h : \mathbb{R}^+ \to \mathbb{R}^+$ be $C^1$ and strictly increasing. For $x = (x_1, x_2) \in \mathbb{R}^2$, let $r = \|x\|$. Show that all solutions of the system

\[
x'_1 = h(r^2)x_2, \quad x'_2 = -h(r^2)x_1.
\]

are periodic and orbitally stable, but not stable.

(2) Consider the system

\[
x' = y + \varepsilon(x^2 \sin 2t - \sin 2t), \quad y' = -4x.
\]

(a) Verify that this is a critical perturbation.
(b) Compute a Lagrange standard form for this system.
(c) Find the corresponding averaged equations.
(d) What does the averaged system say about the existence and stability of periodic solutions?

(3) Show that Mathieu’s equation

\[
u'' + (m^2 - \varepsilon \beta + \varepsilon \cos 2t)u = 0
\]

has $\pi$-periodic solutions for $m = 1, 2, 3, \ldots$, when $\varepsilon$ is small enough.

(4) Find the averaged vector field and frequency response function for the damped Duffing’s equation

\[
u'' + \varepsilon \alpha u' + (1 + \varepsilon \beta)u + \varepsilon \gamma u^3 = F \cos t,
\]

with $\alpha, \gamma, F > 0$. 