Math 5C – Midterm Review Outline

The exam will cover sections 10.1-10.8, 10.10, 11.1-11.3. The outline below summarizes the main topics. Note cards and calculators will not be permitted during the exam.

(1) Sequences
   (a) An infinite sequence of real numbers is a function from $\mathbb{N}$ to $\mathbb{R}$, written $a_k$, $k = 1, 2, \ldots$, or \( \{a_k\}_{k=1}^{\infty} \).
   (b) “\( \lim_{k \to \infty} a_k = L \)” means that the terms of sequence are eventually in every interval containing $L$.
   (c) The convergence or divergence of a sequence is not affected by changing a finite number of its terms.

(2) Infinite series
   (a) An infinite series is an expression of the form \( \sum_{k=1}^{\infty} a_k \).
   (b) The partial sums of an infinite series \( \sum_{k=1}^{\infty} a_k \) is the sequence \( s_n = \sum_{k=1}^{n} a_k \).
   (c) “\( \sum_{k=1}^{\infty} a_k \) converges to a sum $S$” means \( \lim_{n \to \infty} s_n = S \).
   (d) “\( \sum_{k=1}^{\infty} a_k \) converges absolutely” means \( \sum_{k=1}^{\infty} |a_k| \) converges. (Absolute convergence implies convergence.)
   (e) If \( \sum_{k=1}^{\infty} a_k \) converges, then \( \lim_{k \to \infty} a_k = 0 \). Equivalently, if \( \lim_{k \to \infty} a_k \neq 0 \), then \( \sum_{k=1}^{\infty} a_k \) diverges.
   (f) Geometric series
      (i) A geometric series is a series \( \sum_{k=1}^{\infty} a_k \) whose terms are in constant ratio, i.e. \( \frac{a_{k+1}}{a_k} = r \), for all $k$.
      (ii) If $r < 1$ then the series converges absolutely, and the sum is given by the formula \( \frac{a_1}{1-r} \), i.e. the first term divided by $1-r$.
      (iii) If $r \geq 1$, then the geometric series diverges.
   (g) Convergence tests
      (i) Ratio test
      (ii) Integral test
      (iii) Comparison test
      (iv) Ratio comparison test

(3) Taylor series
   (a) Given a function \( f(x) \) with derivatives of every order on an interval \( |x-x_0| < A \), its Taylor polynomial of degree $n$ about $x = x_0$ is
      \[ p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x-x_0)^k, \quad n = 0, 1, 2, \ldots. \]
   (b) Taylor’s theorem gives a formula for the error
      \[ f(x) - p_n(x) = R_n(x, x_0). \]
   (c) If \( \lim_{n \to \infty} |R_n(x, x_0)| = 0 \), for \( |x-x_0| < A \), then
      \[ f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} (x-x_0)^k, \quad |x-x_0| < A. \]
This is called the Taylor series for \( f(x) \) at \( x = x_0 \).

(4) Power series

(a) A power series is an infinite series of the form \( \sum_{k=0}^{\infty} a_k (x - x_0)^k \).

(b) A power series converges absolutely on some interval \( |x - x_0| < R \) and diverges for \( |x - x_0| > R \). \( R \) is called the radius of convergence. (\( R = \infty \) means the series converges for all \( x \in \mathbb{R} \).)

(c) Within its interval of convergence, a power series defines a function \( f(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k \) which has derivatives of all orders. In this case, the coefficients \( a_k \) are the Taylor coefficients of the function \( f(x) \), i.e. \( a_k = \frac{f^{(k)}(x_0)}{k!} \).

(d) New power series from old

(i) A power series may be integrated or differentiated term-by-term, within its interval of convergence.

(ii) Substitution.

(5) Fourier series

(a) A function \( f(x) \) is \( p \)-periodic if \( f(x + p) = f(x) \) for all \( x \in \mathbb{R} \).

(b) A \( 2L \)-periodic function which is piecewise continuous and has left and right derivatives at every point can be represented by a convergent Fourier series

\[
f(x) = a_0 + \sum_{n=1}^{\infty} [a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}],
\]

in which

\[
a_0 = \frac{1}{2L} \int_{-L}^{L} f(x) dx
\]

\[
a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} dx, \quad n = 1, 2, \ldots
\]

\[
b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} dx, \quad n = 1, 2, \ldots
\]

(c) If \( f(x) \) is even, then \( b_n = 0, \quad n = 1, 2, \ldots \).

(d) If \( f(x) \) is odd, then \( a_n = 0, \quad n = 0, 1, 2, \ldots \).

(e) Half range expansions for a function \( f(x) \) defined for \( 0 < x < L \).

(i) The Fourier cosine series

\[
a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L},
\]

with

\[
a_0 = \frac{1}{L} \int_{0}^{L} f(x) dx, \quad a_n = \frac{2}{L} \int_{0}^{L} f(x) \cos \frac{n\pi x}{L} dx, \quad n = 1, 2, \ldots
\]

converges to the even \( 2L \)-periodic extension of \( f(x) \).
(ii) The Fourier sine series

\[ \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}, \]

with

\[ b_n = \frac{2}{L} \int_{0}^{L} f(x) \sin \frac{n\pi x}{L} \, dx, \quad n = 1, 2, \ldots \]

converges to the odd $2L$-periodic extension of $f(x)$.

**Review Questions**

(1) What is the difference between an infinite sequence and an infinite series?
(2) What is meant by the sum of an infinite series?
(3) What is a geometric series? Give some examples of convergent and divergent geometric series.
(4) What is a $p$-series? Give some examples of convergent and divergent $p$-series.
(5) (a) What does it mean to say that an infinite series converges absolutely?
   (b) What relationship exists between convergence and absolute convergence of an infinite series?
(6) If a power series in $x - x_0$ has radius of convergence $R$, what can you say about the set of $x$-values at which the series converges? Diverges?
(7) True or false?
   (a) If $\sum_{k=1}^{\infty} a_k$ converges, then $\lim_{k \to \infty} a_k = 0$.
   (b) If $\lim_{k \to \infty} a_k = 0$, then $\sum_{k=1}^{\infty} a_k$ converges.
   (c) If an infinite series converges, then it converges absolutely.
   (d) If $0 \leq a_k \leq b_k$ and $\sum_{k=1}^{\infty} a_k$ converges, then $\sum_{k=1}^{\infty} b_k$ converges.
   (e) The function $f(x) = x^{1/3}$ has a Taylor expansion at $x = 0$.
   (f) $1 + 1/2 - 1/2 + 1/3 - 1/3 + 1/4 - 1/4 + \cdots = 1$
   (g) $1 + 1/2 - 1/2 + 1/2 - 1/2 + \cdots = 1$
(8) (a) Give an example of a bounded sequence that diverges.
   (b) Give an example of a monotone sequence that diverges.
(9) Determine whether the series diverge or converge.
   (a) $\sum_{k=1}^{\infty} \frac{1}{3^k}$
   (b) $\sum_{k=1}^{\infty} \frac{1}{5k+1}$
   (c) $\sum_{k=1}^{\infty} \frac{k+1}{k^2+k}$
   (d) $\sum_{k=1}^{\infty} \frac{k+2}{3k-1}$
   (e) $\sum_{k=1}^{\infty} \left( \frac{k+2}{3k-1} \right)^k$
   (f) $\sum_{k=1}^{\infty} \frac{\cos k}{k^2}$
(10) Suppose that $\sum_{k=1}^{n} a_k = 2 - \frac{1}{n}$.
    (a) Find $\sum_{k=1}^{\infty} a_k$.
    (b) Find $\lim_{k \to \infty} a_k$.
(11) Find the first 5 Taylor polynomials at $x = 0$ of the function $p(x) = 1 - 7x + 5x^2 + 4x^3$. 
(12) Use the known Taylor series \( \sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} \) at \( x = 0 \) to find the Taylor series at \( x = 0 \) for the following functions. Find the radius of convergence.

(a) \( \frac{1}{2} \sin 2x \)
(b) \( \sin x^2 \)

(13) Find the Taylor series expansion for \( \frac{1}{(1-x)^2} \) at \( x = 0 \) using the fact that \( \frac{d}{dx} \frac{1}{1-x} = \frac{1}{(1-x)^2} \).

(14) Let \( A > 0 \) be a fixed constant. What is the smallest positive period of the function \( f(x) = \sin Ax \)?

(15) What is a Fourier series? A Fourier sine series? A Fourier cosine series?

(16) Can a discontinuous function have a Fourier series? A Taylor series?

(17) Each of the periodic functions below are defined over one period. Find the period, decide whether the function is even, odd or neither even nor odd, and give the form of the Fourier series. (No calculations are necessary.)

(a) \( f(x) = \begin{cases} -3, & -1 < x < 0 \\ 3, & 0 < x < 1 \end{cases} \)
(b) \( f(x) = \begin{cases} 0, & -\pi/2 < x < \pi/2 \\ 3, & \pi/2 < x < 3\pi/2 \end{cases} \)
(c) \( f(x) = x, \quad -2\pi < x < 2\pi \)
(d) \( f(x) = |x|, \quad -2 < x < 2 \)

(18) Find the half range cosine and sine series expansions for the function \( f(x) = 1 - x, \quad 0 < x < 1 \).

**Solutions to Review Questions**

(1) See 1a and 2a in the outline above.
(2) See 2c in the outline.
(3) See 2(f)i in the outline.
(4) A \( p \)-series is an infinite series of the form \( \sum_{k=1}^{\infty} \frac{1}{k^p} \) for some \( p > 0 \). The \( p \)-series converges for \( p > 1 \) and diverges for \( 0 < p \leq 1 \), by the integral test.
(5) See 2d in the outline.
(6) See 4b in the outline.
(7) (a) True. See 2e in the outline.
   (b) False. Consider the harmonic series.
   (c) False. The converse is true, see 2d in the outline.
   (d) False. The comparison test says that if \( \sum_{k=1}^{\infty} b_k \) converges then \( \sum_{k=1}^{\infty} a_k \) converges, and if \( \sum_{k=1}^{\infty} a_k \) diverges then \( \sum_{k=1}^{\infty} b_k \) diverges.
   (e) False. This function is not differentiable at \( x = 0 \).
   (f) True. The partial sums are \( 1, 3/2, 1, 4/3, 1, 5/4, \ldots \) which converges to 1.
(g) False. The partial sums are 1, 3/2, 1, 3/2, 1, 3/2, ... which has no limit.

(8) (a) $a_k = (-1)^k, k = 1, 2, \ldots$
(b) $a_k = k, k = 1, 2, \ldots$

(9) (a) Convergent geometric series, $r = 1/5$.
(b) Convergent, compare with the previous problem, $\frac{1}{5^{k+1}} < \frac{1}{5^k}, k = 1, 2, \ldots$
(c) Divergent, compare with the harmonic series, $\frac{k+1}{k^2+k} > \frac{1}{k}, k = 1, 2, \ldots$
(d) Divergent, $\lim_{k \to \infty} \frac{k+2}{3k-1} = \frac{1}{3} \neq 0$.
(e) Convergent. $\frac{3k-1}{k+2} \leq \frac{1}{2}, k = 3, 4, \ldots$, compare with the geometric series $\sum_{k=1}^{\infty} \frac{1}{2^k}$.
(f) Absolutely convergent, and therefore convergent, since $\frac{|\cos k|}{k^2} \leq \frac{1}{k^2}$.

(10) (a) $\sum_{k=1}^{\infty} a_k = \lim_{n \to \infty} \sum_{k=1}^{n} a_k = \lim_{n \to \infty} 2 - \frac{1}{n} = 2$.
(b) Since $\sum_{k=1}^{\infty} a_k$ converges, $\lim_{k \to \infty} a_k = 0$.

(11) $p_0(x) = 1$, $p_1(x) = 1 - 7x$, $p_2(x) = 1 - 7x + 5x^2$, $p_3(x) = 1 - 7x + 5x^2 + 4x^3$, $p_4(x) = p_5(x) = \ldots$

(12) (a) $\frac{1}{2} \sin 2x = \frac{1}{2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} (2x)^{2k+1} = \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k} x^{2k+1}}{(2k+1)!}, R = \infty$.
(b) $\sin x^2 = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{4k+2}}{(2k+1)!}, R = \infty$.

(13) $\frac{1}{(1-x)^2} = \frac{d}{dx} \frac{1}{1-x} = \frac{d}{dx} \sum_{k=0}^{\infty} x^k = \sum_{k=0}^{\infty} \frac{d}{dx} x^k = \sum_{k=0}^{\infty} k x^{k-1}$, for $|x| < 1$.

(14) $\frac{2\pi}{A}$

(15) A Fourier series of period $2L$ is an infinite series of the form

$$a_0 + \sum_{n=1}^{\infty} \left[ a_n \cos \frac{n \pi x}{L} + b_n \sin \frac{n \pi x}{L} \right].$$

A Fourier cosine series has $b_n = 0$, $n = 1, 2, \ldots$ A Fourier sine series has $a_n = 0$, $n = 0, 1, \ldots$.

(16) Yes, see 5b in the outline. No, a function must be infinitely differentiable on an interval where it has a convergent Taylor series, and a discontinuous function is not differentiable.

(17) (a) Period 2, odd, Fourier sine series with $L = 1$.
(b) Period $2\pi$, even, Fourier cosine series with $L = \pi$.
(c) Period $4\pi$, odd, Fourier sine series with $L = 2\pi$.
(d) Period 4, even, Fourier cosine series with $L = 2$.

(18) Cosine series

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos n \pi x,$$

$$a_0 = \int_0^1 (1-x)dx, \quad a_n = 2 \int_0^1 (1-x) \cos n \pi x dx, \quad n = 1, 2, \ldots$$

Sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin n \pi x, \quad b_n = 2 \int_0^1 (1-x) \sin n \pi x dx, \quad n = 1, 2, \ldots$$