On Almost Global Existence for Nonrelativistic Wave Equations in 3D

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Abstract

Almost global solutions are constructed to three-dimensional, quadratically nonlinear wave equations. The proof relies on generalized energy estimates and a new decay estimate. The method applies to equations that are only classically invariant, such as the nonlinear system of hyperelasticity. © 1996 John Wiley & Sons, Inc.

0. Introduction

This article establishes the almost global existence of solutions for three-dimensional, quadratically nonlinear wave equations, with the use of only the classical invariance of the equations under translations, rotations, and changes of scale. Previous proofs utilized, in addition, either Lorentz invariance [6] or direct estimation of the fundamental solution of the linear wave equation [5]. The approach used here has the advantage of applying to classical equations, such as the system of homogeneous, isotropic hyperelasticity. We also give a simplified proof of John's almost global existence result for this system [4], which bypasses estimation of the fundamental solution of the linear operator. The development for the scalar and vector cases will be presented in parallel.

In gross generality, the plan is as follows. Consider a linear hyperbolic partial differential operator of the form $Pu = \partial_t^2 u - Au$, for certain second-order linear elliptic operators A. The crux of the matter is contained in Lemma 2.3, where it is shown that, simply by manipulating differential operators, $|Au|$ and $|\nabla\partial_t u|$ can be controlled *pointwise* by a decaying factor times derivatives of u with respect to the generators of the invariants plus a term involving Pu. From this follows a weighted estimate for second derivatives of u in L^2, by a simple Gårding-type inequality in Lemma 3.1. The idea of using elliptic methods in a similar manner appeared in the work of Christodoulou and Klainerman on the Einstein equations [1]. Specially adapted Sobolev-type inequalities (see Lemma 4.2) then lead to a decay estimate for the *sup norm* of the second derivatives of u in terms of an *energy norm* plus derivatives of Pu in L^2, given in Theorem 5.1. As a consequence of

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the decay estimate we can prove almost global existence in both the scalar case, Theorem 6.1, and the vector case, Theorem 6.2, provided that the nonlinearities can be written in spatial divergence form. This is not much of a restriction in the scalar case, and it is the physically natural form for the equations in the case of elasticity.

1. Notation and Preliminaries

Partial derivatives will be denoted by $\partial_0 = \partial_t = \partial/\partial t$ and $\partial_i = \partial/\partial x^i$, $i = 1,2,3$, with $\nabla = (\partial_1, \partial_2, \partial_3)$. The angular-momentum operators are given by

$$\Omega = (\Omega_1, \Omega_2, \Omega_3) = x \wedge \nabla; \tag{1.1}$$

$\wedge$ is the vector cross product. The scaling operator is defined by

$$S = t\,\partial_t + r\,\partial_r, \quad \text{where} \quad r = |x|, \quad \text{and} \quad \partial_r = \frac{x}{r} \cdot \nabla. \tag{1.2}$$

Note too that

$$\nabla = \frac{x}{r}\partial_r - \frac{x}{r^2} \wedge \Omega. \tag{1.3}$$

The eight vector fields will be written as $\Gamma = (\Gamma_0, \ldots, \Gamma_7) = (\partial, \Omega, S)$. We will write $\Gamma^\alpha = \Gamma_{\alpha_1} \cdots \Gamma_{\alpha_k}$, for any ordered product of order $|\alpha| = k$.

The d'Alembertian is

$$\Box = \partial_t^2 - \Delta.$$

We will also consider the operator from homogeneous isotropic (infinitesimal) hyperelasticity, which acts on vector functions $v(t,x) = (v^1, v^2, v^3)$ by

$$Lv \equiv \partial_t^2 v - c_2^2 \Delta v - (c_1^2 - c_2^2)\nabla(\nabla \cdot v),$$

with speeds $0 < c_2 < c_1$; see [4].

The operators ∂ and Ω commute with $\Box$, and $[S, \Box] = -2\Box$. For L, one introduces the generators of simultaneous rotations,

$$\widetilde{\Omega} = \Omega I + U, \tag{1.4}$$

where

$$U_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad U_2 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad U_3 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{1.5}$$

Then L commutes with $\widetilde{\Omega}$. Moreover, $[S, L] = -2L$.

For technical reasons, we need to use the operators $|D|$ and $|D|^{-1}$, whose symbols are $|\xi|$ and $|\xi|^{-1}$, respectively. Using the Fourier transform, it is easy to verify that $|D|$ and $|D|^{-1}$ commute with ∂, Ω, and $\widetilde{\Omega}$. We also have that $[S, |D|] = |D|$ and $[S, |D|^{-1}] = -|D|^{-1}$.

We will be using the standard energy norm

$$E_1(u(t)) = \int_{\mathbb{R}^3} \left[|\partial_t u(t,x)|^2 + |\nabla u(t,x)|^2 \right] dx = \|\partial u(t)\|^2,$$

with $\|\cdot\|$ the L^2 norm on $\mathbb{R}^3$, as well as the so-called generalized energy norms

$$E_k(u(t)) = \sum_{|\alpha|\leqq k-1} E_1(\Gamma^\alpha u(t)), \quad k = 2,3,\ldots.$$

It is also convenient to introduce the nonlocal energy,

$$\mathscr{E}_0(u(t)) = E_1(|D|^{-1}u(t)),$$

and its higher-order versions,

$$\mathscr{E}_k(u(t)) = \sum_{|\alpha|\leqq k} \mathscr{E}_0(\Gamma^\alpha u(t)).$$

The Sobolev norms will be

$$\|u(t)\|_k^2 = \sum_{|\alpha|\leqq k} \|\Gamma^\alpha u(t)\|^2, \quad k = 1,2,\ldots,$$

and the sup norm will be written as

$$|u(t)|_k = \sum_{|\alpha|\leqq k} \sup_{x\in\mathbb{R}^3} |\Gamma^\alpha u(t,x)|.$$

The solutions to be constructed will have $E_k(u(t))$ and $\mathscr{E}_{k-1}(u(t))$ finite on some interval $[0,T)$ for some $k \geqq 7$. To describe the solution space, we introduce the time-independent analogue of the vector fields. Set

$$\Lambda = (\Lambda_1,\ldots,\Lambda_7) = (\nabla,\Omega,r\partial_r).$$

The Λ's have the same commutation relations as the Γ. By H_Λ^k we will mean the set of functions u on $\mathbb{R}^3$ such that $\sum_{|\alpha|\leqq k} \|\Lambda^\alpha u\| < \infty$. (Note that $|D|H_\Lambda^k \subset H_\Lambda^{k-1}$.) In the scalar case, solutions will lie in the space

$$X^k(T) = \left\{ u(t,x) : \partial u, \partial |D|^{-1} u \in \bigcap_{j=0}^{k-1} C^j\left([0,T), H_\Lambda^{k-1-j}\right) \right\}.$$

In the case of the elasticity system, it is more convenient to replace Ω by $\widetilde{\Omega}$, because of the commutation properties of L. We note that for vector functions, all norms with $\widetilde{\Omega}$ in place of Ω are equivalent. Therefore, we make the obvious modifications in all of the preceding definitions, basing them instead on $\widetilde{\Omega}$. Accordingly, we will write $\widetilde{\Lambda}$, $\widetilde{\Gamma}$, and $\widetilde{X}^k(T)$.

2. Identities for Differential Operators

The first result separates the dominant terms of the elliptic part of the operator.

LEMMA 2.1. *Let* $u \in C^2(\mathbb{R}^3)$ *and* $v \in C^2(\mathbb{R}^3)^3$. *Then*

$$|\Delta u(x) - \partial_r^2 u(x)| \leqq \frac{C}{r} \sum_{|\alpha| \leqq 1} |\nabla \Omega^\alpha u(x)|, \tag{2.1}$$

$$\left|\frac{x}{r} \cdot [\nabla(\nabla \cdot v(x)) - \partial_r^2 v(x)]\right| \leqq \frac{C}{r} \sum_{|\alpha| \leqq 1} |\nabla \Omega^\alpha v(x)|, \tag{2.2}$$

$$\left|\frac{x}{r} \wedge \nabla(\nabla \cdot v(x))\right| \leqq \frac{C}{r} \sum_{|\alpha| \leqq 1} |\nabla \Omega^\alpha v(x)|. \tag{2.3}$$

The following immediate corollary of Lemma 2.1 plays a crucial role in the analysis of the operator L.

COROLLARY 2.2. *Let* $v \in C^2(\mathbb{R}^3)^3$. *Set*

$$\mathcal{W} v(t,x) = c_2^2 \Delta v(t,x) + (c_1^2 - c_2^2)\nabla(\nabla \cdot v(t,x)). \tag{2.4}$$

Then

$$\left|\frac{x}{r} \cdot \left[\mathcal{W} v - c_1^2 \partial_r^2 v\right]\right| \leqq \frac{C}{r} \sum_{|\alpha| \leqq 1} |\nabla \Omega^\alpha v(x)|, \tag{2.5}$$

and

$$\left|\frac{x}{r} \wedge \left[\mathcal{W} v - c_2^2 \partial_r^2 v\right]\right| \leqq \frac{C}{r} \sum_{|\alpha| \leqq 1} |\nabla \Omega^\alpha v(x)|. \tag{2.6}$$

Proof of Lemma 2.1: If we expand the Laplacian into its radial and angular parts,

$$\Delta = \partial_r^2 + \frac{2}{r}\partial_r + \frac{1}{r^2}\Omega \cdot \Omega,$$

we see that inequality (2.1) follows from the fact that by (1.2) and (1.1),

$$|\partial_r(\cdot)| \leqq |\nabla(\cdot)| \quad \text{and} \quad \left|\frac{1}{r}\Omega(\cdot)\right| \leqq C|\nabla(\cdot)|. \tag{2.7}$$

By (1.3), we may write

$$\begin{aligned} \frac{x}{r} \cdot \nabla(\nabla \cdot v(x)) &= \partial_r \left[\frac{x}{r} \cdot \partial_r v(x) - \left(\frac{x}{r^2} \wedge \Omega\right) \cdot v(x)\right] \\ &= \frac{x}{r} \cdot \partial_r^2 v(x) + \left(\frac{x}{r^3} \wedge \Omega\right) \cdot v(x) - \left(\frac{x}{r^2} \wedge \partial_r \Omega\right) \cdot v(x). \end{aligned}$$

If we notice (2.7), we see that (2.2) holds.

The estimate (2.3) can be seen immediately by writing

$$\frac{x}{r} \wedge \nabla(\nabla \cdot v(x)) = \frac{1}{r}\Omega(\nabla \cdot v(x)),$$

and noting that the commutator of Ω and ∇ involves only ∇.

Using this, we now derive weighted pointwise bounds for certain combinations of second derivatives.

LEMMA 2.3. *Let $u \in C^2(\mathbb{R}^+ \times \mathbb{R}^3)$ and $v \in C^2(\mathbb{R}^+ \times \mathbb{R}^3)^3$. Set*

$$\sigma(t,r) = \left[1 + (t-r)^2\right]^{1/2}, \quad \tau(t,r) = \frac{\sigma(c_1 t, r)\sigma(c_2 t, r)}{\sigma(c_1 t, r) + \sigma(c_2 t, r)}. \tag{2.8}$$

Then

$$\sigma(t,r)|\Delta u(t,x)| \leqq C\left[\sum_{|\alpha|\leqq 1} |\partial\Gamma^\alpha u(t,x)| + t|\Box u(t,x)|\right], \tag{2.9}$$

$$\sigma(t,r)|\partial_t^2 u(t,x)| \leqq C\left[\sum_{|\alpha|\leqq 1} |\partial\Gamma^\alpha u(t,x)| + r|\Box u(t,x)|\right], \tag{2.10}$$

$$\sigma(t,r)|\nabla\partial_t u(t,x)| \leqq C\left[\sum_{|\alpha|\leqq 1} |\partial\Gamma^\alpha u(t,x)| + t|\Box u(t,x)|\right]. \tag{2.11}$$

Also, with the use of the notation of (2.4),

$$\tau(t,r)\,|\mathscr{W} v(t,x)| \leqq C\left[\sum_{|\alpha|\leqq 1} |\partial\Gamma^\alpha v(t,x)| + t|Lv(t,x)|\right], \tag{2.12}$$

$$\tau(t,r)|\partial_t^2 v(t,x)| \leqq C\left[\sum_{|\alpha|\leqq 1} |\partial\Gamma^\alpha v(t,x)| + r|Lv(t,x)|\right], \tag{2.13}$$

$$\tau(t,r)|\nabla\partial_t v(t,x)| \leqq C\left[\sum_{|\alpha|\leqq 1} |\partial\Gamma^\alpha v(t,x)| + t|Lv(t,x)|\right]. \tag{2.14}$$

Proof: This lemma is based on the simple observation that

$$\partial_t Su - \partial_t u = t\partial_t^2 u + r\partial_t\partial_r u, \tag{2.15}$$

$$\partial_r Su - \partial_r u = r\partial_r^2 u + t\partial_r\partial_t u. \tag{2.16}$$

Hence,

$$t(\partial_t Su - \partial_t u) - r(\partial_r Su - \partial_r u) = t^2\partial_t^2 u - r^2\partial_r^2 u. \tag{2.17}$$

In order to get (2.9), add and subtract the term $(t^2 - r^2)\Delta u$:

$$t^2\partial_t^2 u - r^2\partial_r^2 u = t^2\Box u + (t^2 - r^2)\Delta u + r^2(\Delta u - \partial_r^2 u).$$

Thus, upon rearrangement we have

$$(t - r)\Delta u = \frac{1}{t + r}\left[(t^2\partial_t^2 u - r^2\partial_r^2 u) - r^2(\Delta u - \partial_r^2 u) - t^2\Box u\right].$$

Inequality (2.9) now follows from (2.17) and (2.1).

If we go back to (2.17), we can also group the terms as follows:

$$t^2\partial_t^2 u - r^2\partial_r^2 u = (t^2 - r^2)\partial_t^2 u + r^2\Box u + r^2(\Delta u - \partial_r^2 u).$$

If we use (2.17) and (2.1), we obtain (2.10).

To check (2.11), we subtract (2.15) from (2.16) to get

$$(t - r)\partial_r\partial_t u = -\partial_t Su + \partial_t u + \partial_r Su - \partial_r u - t\Box u + (t - r)\Delta u + r(\Delta u - \partial_r^2 u).$$

Thus we refer to (2.1) and (2.9) to obtain

$$\begin{aligned} |t - r||\partial_r\partial_t u| &\leqq C[|\partial_t Su| + |\partial_t u| + |\partial_r Su| + |\partial_r u| \\ &\quad + t|\Box u| + |t - r||\Delta u| + r|\Delta u - \partial_r^2 u|] \\ &\leqq C\left[\sum_{|\alpha|\leqq 1} |\partial\Gamma^\alpha u(t,x)| + t|\Box u(t,x)|\right]. \end{aligned} \tag{2.18}$$

This leaves only the angular component of the gradient to be estimated. Once again, we exploit an algebraic relationship:

$$\begin{aligned} (t - r)\frac{x}{r^2} \wedge \Omega\partial_t u &= \frac{x}{r^2} \wedge \Omega(Su - r\partial_r u) - \frac{x}{r} \wedge \Omega\partial_t u \\ &= \frac{x}{r^2} \wedge \Omega Su - \frac{x}{r} \wedge \partial_r\Omega u - \frac{x}{r} \wedge \partial_t\Omega u, \end{aligned}$$

from which it follows that

$$|t - r|\left|\frac{x}{r^2} \wedge \Omega\partial_t u\right| \leqq C[|\nabla Su| + |\nabla\Omega u| + |\partial_t\Omega u|], \tag{2.19}$$

with the use of (2.7). If we combine (2.18) and (2.19) and recall (1.3), we have proven (2.11).

Of course, (2.17) holds for vectors v as well. By analogy with the above, write

$$t^2\partial_t^2 v - r^2\partial_r^2 v = t^2 Lv + \frac{(c_i^2 t^2 - r^2)}{c_i^2}\mathscr{W}v + \frac{r^2}{c_i^2}\left[\mathscr{W}v - c_i^2\partial_r^2 v\right], \quad i = 1, 2. \tag{2.20}$$

Take the dot product of (2.20) with x/r, set $i = 1$, and use (2.17) and (2.5) to get

$$\sigma(c_1 t, r)\left|\frac{x}{r} \cdot \mathscr{W}v\right| \leqq C\left[\sum_{|\alpha|\leqq 1} |\partial\Gamma^\alpha v(t,x)| + t|Lv(t,x)|\right]. \tag{2.21}$$

We cross (2.20) with x/r, set $i = 2$, and appeal to (2.17) and (2.6), to obtain

$$\sigma(c_2 t, r)\left|\frac{x}{r} \wedge \mathcal{W} v\right| \leqq C\left[\sum_{|\alpha| \leqq 1}|\partial \Gamma^\alpha v(t, x)|+t|L v(t, x)|\right]. \tag{2.22}$$

Now because $\tau(t,r) \leqq \min(\sigma(c_1 t, r), \sigma(c_2 t, r))$, and because $|\mathcal{W} v|^2 = |x/r \cdot \mathcal{W} v|^2 + |x/r \wedge \mathcal{W} v|^2$, the estimate (2.12) is seen to be a consequence of (2.21) and (2.22). The estimates (2.21) and (2.22) capture the two speeds of the operator L.

In order to derive (2.13), we write

$$t^2\partial_t^2 v - r^2\partial_r^2 v = \frac{(c_i^2 t^2 - r^2)}{c_i^2}\partial_t^2 v + \frac{r^2}{c_i^2} L v + \frac{r^2}{c_i^2}\left[\mathcal{W} v - c_i^2 \partial_r^2 v\right], \quad i = 1,2.$$

We get the estimate (2.13) by an argument similar to the one above.

We turn to the proof of (2.14), and use (2.15) and (2.16) again to write, for $i = 1,2$,

$$\begin{aligned}
(c_i t - r)\partial_r\partial_t v &= -(\partial_t S v - \partial_t v) + c_i(\partial_r S v - \partial_r v) + t\partial_t^2 v - c_i r \partial_r^2 v \\
&= -(\partial_t S v - \partial_t v) + c_i(\partial_r S v - \partial_r v) + tLv + t\mathcal{W} v - c_i r \partial_r^2 v \\
&= -(\partial_t S v - \partial_t v) + c_i(\partial_r S v - \partial_r v) + tLv \\
&\quad + \frac{1}{c_i}(c_i t - r)\mathcal{W} v + \frac{r}{c_i}(\mathcal{W} v - c_i^2 \partial_r^2 v).
\end{aligned}$$

By (2.5) and (2.6), we have

$$\begin{aligned}
|c_1 t - r|\left|\frac{x}{r} \cdot \partial_r\partial_t v\right| &\leqq C\left[\sum_{|\alpha| \leqq 1}|\partial \Gamma^\alpha v(t, x)|+t|L v(t, x)|\right], \\
|c_2 t - r|\left|\frac{x}{r} \wedge \partial_r\partial_t v\right| &\leqq C\left[\sum_{|\alpha| \leqq 1}|\partial \Gamma^\alpha v(t, x)|+t|L v(t, x)|\right].
\end{aligned}$$

Hence, if we proceed exactly as before,

$$\tau(t,r)|\partial_r\partial_t v| \leqq C\left[\sum_{|\alpha| \leqq 1}|\partial \Gamma^\alpha v(t, x)|+t|L v(t, x)|\right].$$

The angular portion of the gradient is handled as in the scalar case. The proof is now complete.

3. Weighted L^2 Estimates

The previous pointwise bounds can be translated into L^2 estimates, essentially with the use of the argument of Gårding's inequality.

LEMMA 3.1. *Let σ and τ be as in (2.8). Then for all $u \in X^2(T)$ and $v \in \widetilde{X}^2(T)$,*

$$\|\sigma(t,\cdot)\nabla\partial u(t,\cdot)\|^2 \leqq C\left[E_2(u(t)) + t^2\|\Box u(t,\cdot)\|^2\right], \tag{3.1}$$

$$\|\tau(t,\cdot)\nabla\partial v(t,\cdot)\|^2 \leqq C\left[E_2(v(t)) + t^2\|Lv(t,\cdot)\|^2\right]. \tag{3.2}$$

Proof: If we square equations (2.9) and (2.11) of Lemma 2.3 and integrate over space (and suppress the fixed time variable), we see that the quantities $\|\sigma\Delta u\|^2$ and $\|\sigma\nabla\partial_t u\|^2$ have the desired bound. However, integration by parts twice gains control of all of the second-order spatial derivatives:

$$\begin{aligned}\sum_{|\alpha|=2}\|\sigma\nabla^\alpha u\|^2 &= \sum_{i,j}\int \sigma^2(\partial_i\partial_j u)(\partial_i\partial_j u)dx\\ &= \int \sigma^2(\Delta u)^2 dx - \sum_{i,j}\int(\partial_i\sigma^2)(\partial_j u)(\partial_i\partial_j u)dx\\ &\quad + \sum_{i,j}\int(\partial_j\sigma^2)(\partial_j u)(\partial_i^2 u)dx.\end{aligned} \tag{3.3}$$

The derivatives of σ are uniformly bounded, and so a combination of the Cauchy-Schwarz inequality and Young's inequality shows that the last two integrals are each bounded by

$$\frac{1}{4}\sum_{|\alpha|=2}\|\sigma\nabla^\alpha u\|^2 + C\|\nabla u\|^2.$$

The second derivatives are absorbed on the left-hand side of (3.3), and the first derivatives are bounded by the energy. This proves (3.1).

With the use of the notation (2.4), we will show that

$$\|\tau\Delta v\| \leqq C\left[\|\tau\mathscr{W}v\|^2 + \|\nabla v\|^2\right]. \tag{3.4}$$

We expand the inner product and we find that

$$\begin{aligned}\|\tau\mathscr{W}v\|^2 &= c_2^4\int\tau^2|\Delta v|^2 dx + 2c_2^2(c_1^2 - c_2^2)\int\tau^2\Delta v\cdot\nabla(\nabla\cdot v)dx\\ &\quad + (c_1^2 - c_2^2)^2\int\tau^2|\nabla(\nabla\cdot v)|^2\,dx\end{aligned} \tag{3.5}$$

The middle term can be integrated by parts to give

$$\begin{aligned}\int\tau^2\Delta v\cdot\nabla(\nabla\cdot v)dx &= \int\tau^2|\nabla(\nabla\cdot v)|^2 dx - \int(\nabla\tau^2\cdot\Delta v)(\nabla\cdot v)dx\\ &\quad + \int(\nabla\tau^2\cdot\nabla(\nabla\cdot v))(\nabla\cdot v)dx\\ &\geqq -\frac{1}{2}\int\tau^2(\Delta v)^2 dx - C\int|\nabla v|^2\,dx.\end{aligned}$$

This proves (3.4), and thus by (2.12), we obtain from (3.5) the correct bound for $\|\tau\Delta v\|^2$. The rest is as in (3.3).

4. Sobolev Inequalities

The following inequalities are a mild generalization, in the flat Minkowski space-time, of some similar results proved in [1]. They are designed to exploit the degenerate weight factors σ, τ that arose in Lemma 3.1.

LEMMA 4.1. *Let $\sigma(r) \geqq 1$ be a smooth radial function whose derivatives are uniformly bounded. Then for all sufficiently regular functions ϕ on $\mathbb{R}^3$ and any $\lambda \geqq 0$,*

$$\left(r\sigma^{\lambda+1}(r)\right)^{1/2}\left(\int_{S_2}|\phi(r\omega)|^4 r^2 d\omega\right)^{1/4} \leqq C\left[\sum_{|\alpha|\leqq 1}\|\sigma^\lambda \Omega^\alpha\phi\| + \|\sigma\partial_r\phi\|\right].$$

(Here $S_2 = \{\omega \in \mathbb{R}^3 : |\omega| = 1\}$ is the unit sphere, and $d\omega$ is surface measure on S_2.)

Proof: We have the following straightforward calculation:

$$\begin{aligned}
&r^2\sigma^{2(\lambda+1)}(r)\int_{S_2}|\phi(r\omega)|^4 r^2 d\omega \\
&= -r^4\int_r^\infty \partial_\rho\left(\sigma^{2(\lambda+1)}(\rho)\int_{S_2}|\phi(\rho\omega)|^4 d\omega\right)d\rho \\
&\leqq r^4\int_r^\infty\int_{S_2}\sigma^{2(\lambda+1)}(\rho)\, 4|\phi(\rho\omega)|^3|\partial_r\phi(\rho\omega)|\, d\omega\, d\rho \\
&\quad + r^4\int_r^\infty\int_{S_2} 2(\lambda+1)\sigma^{2\lambda+1}\sigma'(\rho)|\phi(\rho\omega)|^4\, d\omega\, d\rho \qquad (4.1)\\
&\leqq Cr^2\int_{|x|\geqq r}\left[\sigma^{2(\lambda+1)}|\phi|^3|\partial_r\phi| + \sigma^{2\lambda+1}|\phi|^4\right]dx \\
&\leqq C\left(\int_{|x|\geqq r}|x|^4\sigma^{2(2\lambda+1)}|\phi|^6\, dx\right)^{1/2}\left(\int_{|x|\geqq r}[\sigma^2|\partial_r\phi|^2 + |\phi|^2]\, dx\right)^{1/2} \\
&\leqq C\left(\int_{|x|\geqq r}|x|^4\sigma^{2(2\lambda+1)}|\phi|^6\, dx\right)^{1/2}\left(\int_{|x|\geqq r}[\sigma^2|\partial_r\phi|^2 + \sigma^{2\lambda}|\phi|^2]\, dx\right)^{1/2},
\end{aligned}$$

because $\sigma \geqq 1$.

Next, by the isoperimetric-Sobolev inequality (see [1], p. 32), we have

$$\left(\int_{S_2} |\phi(\rho\omega)|^6\,d\omega\right)^{\frac{1}{6}}$$
$$\leqq C\left(\int_{S_2} |\phi(\rho\omega)|^4\,d\omega\right)^{\frac{1}{4}\cdot\frac{2}{3}}\left(\int_{S_2}\left[|\Omega\phi(\rho\omega)|^2+|\phi(\rho\omega)|^2\right]d\omega\right)^{\frac{1}{2}\cdot\frac{1}{3}}.$$

We use this to obtain

$$\rho^4\sigma^{2(2\lambda+1)}(\rho)\int_{S_2}|\phi(\rho\omega)|^6\rho^2\,d\omega$$
$$\leqq C\left(\int_{S_2}\rho^2\sigma^{2(\lambda+1)}(\rho)|\phi(\rho\omega)|^4\rho^2\,d\omega\right)\left(\sum_{|\alpha|\leqq 1}\int_{S_2}\sigma^{2\lambda}(\rho)|\Omega^\alpha\phi(\rho\omega)|^2\rho^2\,d\omega\right).$$

An integration with respect to ρ yields

$$\begin{aligned}&\int_{|x|\geqq r}|x|^4\sigma^{2(2\lambda+1)}|\phi|^6\,dx\\ &\leqq C\sup_{\rho\geqq r}\left(\rho^2\sigma^{2(\lambda+1)}(\rho)\int_{S_2}|\phi(\rho\omega)|^4\rho^2\,d\omega\right)\left(\sum_{|\alpha|\leqq 1}\int_{|x|\geqq r}\sigma^{2\lambda}|\Omega^\alpha\phi|^2\,dx\right).\end{aligned}\tag{4.2}$$

The result now follows from (4.1) and (4.3).

LEMMA 4.2. *Let σ be as in Lemma 4.1. For all sufficiently regular functions ϕ on $\mathbb{R}^3$,*

$$\sup_{S_2} r\sigma^{(\lambda+1)/2}(r)|\phi(r\omega)| \leqq C\left[\sum_{|\alpha|\leqq 2}\|\sigma^\lambda\Omega^\alpha\phi\| + \sum_{|\alpha|\leqq 1}\|\sigma\partial_r\Omega^\alpha\phi\|\right].$$

Proof: Again with the use of a standard Sobolev inequality on S_2, we have

$$\sup_{S_2}|\phi(r\omega)| \leqq C\left(\sum_{|\alpha|\leqq 1}\int_{S_2}|\Omega^\alpha\phi(r\omega)|^4\,d\omega\right)^{1/4}.$$

Multiply this by $r\sigma^{(\lambda+1)/2}(r)$ and apply Lemma 4.1.

Actually, this result will only be used with $\lambda = 1$.

5. Main Estimate

We now proceed to derive the $L^\infty - L^2$ estimates.

THEOREM 5.1. *Let σ and τ be as in* (2.8). *Then for all $u \in X^4(T)$ and $v \in \tilde{X}^4(T)$,*

$$(1 + r)\sigma(t,r)|\nabla\partial u(t,x)| \leqq C\left[E_4(u(t))^{1/2} + t\|\Box u(t,\cdot)\|_2\right], \tag{5.1}$$

$$(1 + r)\tau(t,r)|\nabla\partial v(t,x)| \leqq C\left[E_4(v(t))^{1/2} + t\|Lv(t,\cdot)\|_2\right]. \tag{5.2}$$

Proof: In order to simplify equations, we will adopt the convention of suppressing the space variable when taking the L^2 norm. Thus, for example, we will write $\|\sigma(t)u(t)\|$ for $\|\sigma(t,\cdot)u(t,\cdot)\|$. By the classical Sobolev inequality, the boundedness of the derivatives of σ, and (3.1), we have the estimates

$$\begin{aligned}
\sigma(t,r)|\nabla\partial u(t,x)| &\leqq C\sum_{|\beta|\leqq 2}\|\nabla^\beta[\sigma(t)\nabla\partial u(t)]\| \\
&\leqq C\sum_{|\beta|\leqq 2}\|\sigma(t)\nabla\partial\nabla^\beta u(t)\| \\
&\leqq C\sum_{|\beta|\leqq 2}\left[E_2(\nabla^\beta u(t))^{1/2} + t\|\Box\nabla^\beta u(t)\|\right] \\
&\leqq C\left[E_4(u(t))^{1/2} + t\|\Box u(t)\|_2\right].
\end{aligned} \tag{5.3}$$

On the other hand, because the derivatives of $\sigma(t,r)$ are bounded independently of t, we may apply Lemma 4.2 with $\phi = \nabla\partial u$ and $\lambda = 1$. Thus,

$$\begin{aligned}
r\sigma(t,r)|\nabla\partial u(t,x)| &\leqq C\left[\sum_{|\beta|\leqq 2}\|\sigma(t)\Omega^\beta\nabla\partial u(t)\| + \sum_{|\beta|\leqq 1}\|\sigma(t)\partial_r\Omega^\beta\nabla\partial u(t)\|\right] \\
&\leqq C\sum_{|\beta|\leqq 2}\|\sigma(t)\Gamma^\beta\nabla\partial u(t)\| \\
&\leqq C\sum_{|\beta|\leqq 2}\|\sigma(t)\nabla\partial\Gamma^\beta u(t)\|.
\end{aligned}$$

We apply (3.1) to each derivative $\Gamma^\beta u$, and continue the estimate to obtain

$$\begin{aligned}
r\sigma(t,r)|\nabla\partial u(t,x)| &\leqq C\sum_{|\beta|\leqq 2}\left[E_2(\Gamma^\beta u(t))^{1/2} + t\|\Box\Gamma^\beta u(t)\|\right] \\
&\leqq C\sum_{|\beta|\leqq 2}\left[E_2(\Gamma^\beta u(t))^{1/2} + t\|\Gamma^\beta\Box u(t)\|\right] \\
&\leqq C\left[E_4(u(t))^{1/2} + t\|\Box u(t)\|_2\right].
\end{aligned} \tag{5.4}$$

Combining (5.3) and (5.4) completes the proof of (5.1).

The second case is analogous. In fact, using (3.2), estimate (5.3) carries over directly with u, σ, and $\Box$ replaced by v, τ, and L, respectively. For the other estimate, we must temporarily introduce the simultaneous rotations $\tilde{\Omega}$ (see (1.4) and (1.5)) in order to have the proper commutation with the operator L. Using Lemma 4.2 and the equivalence of the Γ and $\tilde{\Gamma}$ norms, the estimate proceeds as follows:

$$\begin{aligned} r\,\tau(t,r)|\nabla\partial v(t,x)| &\leqq C\Bigg[\sum_{|\beta|\leqq 2}\|\tau(t)\Omega^\beta\nabla\partial v(t)\| \\ &\qquad + \sum_{|\beta|\leqq 1}\|\tau(t)\partial_r\Omega^\beta\nabla\partial v(t)\|\Bigg] \\ &\leqq C\sum_{|\beta|\leqq 2}\|\tau(t)\tilde{\Gamma}^\beta\nabla\partial v(t)\| \\ &\leqq C\sum_{|\beta|\leqq 2}\|\tau(t)\nabla\partial\tilde{\Gamma}^\beta v(t)\|\,. \end{aligned}$$

Now we use (3.2) to finish with

$$\begin{aligned} r\,\tau(t,r)|\nabla\partial v(t,x)| &\leqq C\sum_{|\beta|\leqq 2}\left[E_2(\tilde{\Gamma}^\beta v(t))^{1/2} + t\|L\tilde{\Gamma}^\beta v(t)\|\right] \\ &\leqq C\sum_{|\beta|\leqq 2}\left[E_2(\tilde{\Gamma}^\beta v(t))^{1/2} + t\|\tilde{\Gamma}^\beta L v(t)\|\right] \\ &\leqq C\left[E_4(v(t))^{1/2} + t\|Lv(t)\|_2\right]. \end{aligned} \tag{5.5}$$

Thus, (5.2) follows from the v analogue of (5.3) and (5.5).

COROLLARY 5.2. *Let σ and τ be as in* (2.8), *and let $|\alpha| \leqq k$. Then for all $u \in X^{k+4}(T)$ and $v \in \tilde{X}^{k+4}(T)$,*

$$(1+r)\sigma(t,r)|\nabla\partial\Gamma^\alpha u(t,x)| \leqq C\left[E_{k+4}(u(t))^{1/2} + t\|\Box u(t,\cdot)\|_{k+2}\right], \tag{5.6}$$

$$(1+r)\tau(t,r)|\nabla\partial\Gamma^\alpha v(t,x)| \leqq C\left[E_{k+4}(v(t))^{1/2} + t\|Lv(t,\cdot)\|_{k+2}\right]. \tag{5.7}$$

Proof: Apply Theorem 5.1 to $\Gamma^\alpha u$ and $\Gamma^\alpha v$.

COROLLARY 5.3. *Let σ and τ be as in* (2.8), *and let $|\alpha| \leqq k$. Then for all $u \in X^{k+4}(T)$ and $v \in \tilde{X}^{k+4}(T)$,*

$$(1+r)\sigma(t,r)|\partial\Gamma^\alpha u(t,x)| \leqq C\left[\mathscr{E}_{k+3}(u(t))^{1/2} + t\||D|^{-1}\Box u(t,\cdot)\|_{k+2}\right], \tag{5.8}$$

$$(1+r)\tau(t,r)|\partial\Gamma^\alpha v(t,x)| \leqq C\left[\mathscr{E}_{k+3}(v(t))^{1/2} + t\||D|^{-1}Lv(t,\cdot)\|_{k+2}\right]. \tag{5.9}$$

Proof: A brief appearance will be made by the Riesz transformations R_i, the operators of order zero with symbols $\xi^i/|\xi|$. Note that $I = R_iR_i = \partial_iR_i|D|^{-1}$, and that the R_i commute with ∂ and S whereas the commutator of an R with an Ω is either zero or another R. These statements can be verified with the use of the Fourier transform.

So we apply Theorem 5.1 to $R_i|D|^{-1}u$ and obtain

$$\begin{aligned}(1+r)\sigma(t,r)|\partial u(t,x)| &= (1+r)\sigma(t,r)|\partial\partial_iR_i|D|^{-1}u(t,x)| \\ &\leqq C\left[E_4(R_i|D|^{-1}u(t))^{1/2} + t\|\Box R_i|D|^{-1}u(t)\|_2\right] \\ &\leqq C\left[E_4(|D|^{-1}u(t))^{1/2} + t\||D|^{-1}\Box u(t)\|_2\right] \\ &\leqq C\left[\mathscr{E}_3(u(t))^{1/2} + t\||D|^{-1}\Box u(t)\|_2\right].\end{aligned}$$

The estimate (5.8) follows if this result is applied to $\Gamma^\alpha u$. The proof of (5.9) is obviously identical.

6. Almost Global Existence

We now give the almost global existence results.

The nonlinearities considered here are slightly more restrictive than in [4]–[6]. Here, the initial data are not required to have compact support, and are assumed to have fewer derivatives than in the previous work. However, we must take $\partial_tu(0) \in |D|H^k_\Lambda$ instead of the more natural H^{k-1}_Λ because of the use of the nonlocal energy $\mathscr{E}_{k-1}$.

THEOREM 6.1. *Let $C^i_{\alpha\beta}$ be constants for $\alpha,\beta = 0,\dots,3$ and $i = 1,2,3$. Let $u_0 \in H^k_\Lambda$ and $u_1 \in |D|H^k_\Lambda$ for $k \geqq 7$. Then there exist positive constants $A > 0$ and $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ the Cauchy problem,*

$$\Box u = \sum_{\alpha,\beta=0}^{3}\sum_{i=1}^{3} C^i_{\alpha\beta}\partial_i\left(\partial_\alpha u\partial_\beta u\right),$$

$$u(0) = \varepsilon u_0, \quad \partial_tu(0) = \varepsilon u_1,$$

has a unique solution $u \in X^k(T_\varepsilon)$ with a life span $T_\varepsilon > \exp(A/\varepsilon)$ satisfying the estimate

$$E_k(u(t))^{1/2} + \mathscr{E}_{k-1}(u(t))^{1/2} \leqq A\varepsilon.$$

Proof: We give a sketch of the proof, outlining the key a priori estimates leading to a bound for the energy $E_k(u(t)) + \mathscr{E}_{k-1}(u(t))$, for some fixed $k \geqq 7$.

The first step is to derive an energy estimate for $E_k(u(t))$, which can be done in the usual manner. For any $|\alpha| \leqq k-1$, apply the operator Γ^α to the equation and multiply by $\partial_t\Gamma^\alpha u$. If we choose ε_0 small enough, $|\partial u|$ will be sufficiently small, independently of ε. The top-order nonlinear terms can then be absorbed on the left-hand side after pulling out one time derivative. We write the nonlinearity as a sum of terms of the form $\partial u \nabla \partial u$ to obtain

$$\begin{aligned} E_k(u(t)) &\leqq E_k(u(0)) \exp\left[C \int_0^t \|\partial \Gamma u(s) \nabla \partial u(s)\|_{k-2} E_k(u(s))^{1/2}\, ds \right] \\ &\leqq E_k(u(0)) \exp\left[C \int_0^t |\partial u(s)|_{k_0} E_k(u(s))\, ds \right], \end{aligned} \tag{6.1}$$

with $k_0 = [(k-2)/2] + 1$.

Second, we apply the decay estimate, Corollary 5.3,

$$(1+t)|\partial u(t)|_{k_0} \leqq C\left[\mathscr{E}_{k_0+3}(u(t))^{1/2} + t\| \, |D|^{-1} \Box u(t)\|_{k_0+2} \right].$$

Because $\Box u$ is a spatial divergence, we obtain

$$\begin{aligned} \| \, |D|^{-1} \Box u(t)\|_{k_0+2} &\leqq C\|\partial u(t) \partial u(t)\|_{k_0+2} \\ &\leqq C|\partial u(t)|_{k_1} E_{k_0+3}(u(t))^{1/2}, \end{aligned}$$

where $k_1 = [(k_0+2)/2]$. Now $k_1 \leqq k_0$, and because $k \geqq 7$, we have $k_0 + 3 \leqq k - 1$. Thus for solutions with small enough $E_k(u(t))$, that is, for ε_0 small enough, we have derived

$$(1+t)|\partial u(t)|_{k_0} \leqq C\mathscr{E}_{k-1}(u(t))^{1/2}. \tag{6.2}$$

Finally, to close the chain of inequalities, we must estimate $\mathscr{E}_{k-1}(u(t))$. So for $|\alpha| \leqq k-1$, apply $|D|^{-1}\Gamma^\alpha$ to the equation and multiply by $\partial_t |D|^{-1} \Gamma^\alpha u(t)$ to get

$$\begin{aligned} \frac{d}{dt}\mathscr{E}_{k-1}(u(t)) &\leqq C\| \, |D|^{-1} \Box u(t)\|_{k-1} \mathscr{E}_{k-1}(u(t))^{1/2} \\ &\leqq C\|\partial u(t) \partial u(t)\|_{k-1} \mathscr{E}_{k-1}(u(t))^{1/2} \\ &\leqq C|\partial u(t)|_{[(k-1)/2]} E_k(u(t))^{1/2} \mathscr{E}_{k-1}(u(t))^{1/2} \\ &\leqq C|\partial u(t)|_{k_0} E_k(u(t))^{1/2} \mathscr{E}_{k-1}(u(t))^{1/2}. \end{aligned} \tag{6.3}$$

The inequalities (6.1)–(6.3) combine to show that $E_k(u(t)) + \mathscr{E}_{k-1}(u(t))$ remains of order ε^2 for a time of order $\exp(A/\varepsilon)$. This completes the outline of the proof.

The case of general quadratic nonlinearities with the same linearized behavior at the origin can be handled. The addition of higher-order terms creates no problem because in the energy inequality (6.1), the cubic term leads to the square of the *sup norm*, which is integrable in time in three dimensions. We remark that the only quadratic nonlinearities that are not permitted in Theorem 6.1 are those containing the combinations $\partial_i u \partial_j \partial_t u$ and $\partial u \partial_t^2 u$, because they cannot be written as the spatial divergence of some expression. (Of course, the term $\partial u \partial_t^2 u$ can always be absorbed on the left-hand side, at the expense of adding higher-order terms.)

We now turn to the equations of homogeneous isotropic hyperelasticity. See Gurtin [3] for a concise discussion of this system. Actually, the equations considered here are correct only to second order, but again, the higher-order corrections present no analytical difficulties. We mention that Ebin [2] has constructed a global solution for incompressible, neo-Hookean materials.

THEOREM 6.2. *Let $C^{ijk}_{\ell mn}$; $i, j, k, \ell, m, n = 1, 2, 3$, be constants with the symmetry properties*

$$C^{ijk}_{\ell mn} + C^{ikj}_{\ell nm} + C^{ijk}_{m\ell n} + C^{ikj}_{mn\ell} = C^{jik}_{\ell mn} + C^{jki}_{\ell nm} + C^{jik}_{m\ell n} + C^{jki}_{mn\ell}.$$

Let $v_0 \in H^k_\Lambda$ and $v_1 \in |D|H^k_\Lambda$ for $k \geqq 7$. Then there exist positive constants $A > 0$ and $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ the Cauchy problem,

$$(Lv)^i = \partial_t^2 v^i - c_2^2 \Delta v^i - (c_1^2 - c_2^2)\partial_i(\nabla \cdot v) = \sum_{j,k,\ell,m,n=1}^{3} C^{ijk}_{\ell mn}\partial_\ell(\partial_m v^j \partial_n v^k)$$

$$v(0) = \varepsilon v_0, \quad \partial_t v(0) = \varepsilon v_1,$$

has a unique solution $u \in \widetilde{X}^k(T_\varepsilon)$ with life span $T_\varepsilon > \exp(A/\varepsilon)$, satisfying the estimate

$$E_k(v(t))^{1/2} + \mathscr{E}_{k-1}(v(t))^{1/2} \leqq A\varepsilon.$$

Proof: The proof is essentially the same as above. The symmetry condition guarantees that the energy estimate (6.1) is true.

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