The Null Condition and Global Existence of Nonlinear Elastic Waves

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1. Introduction

The equations of motion for the displacement of an isotropic, homogeneous, hyperelastic material form a quasilinear hyperbolic system,

$$Lu = \partial_t^2 u - c_2^2 \Delta u - (c_1^2 - c_2^2)\nabla(\nabla \cdot u) = F(\nabla u)\nabla^2 u,$$

in three space dimensions, with wave speeds $0 < c_2 < c_1$ and a nonlinearity, the precise form of which will be spelled out in later sections. We shall prove that for certain classes of materials, small initial disturbances give rise to global smooth solutions. These special materials are distinguished by a null condition imposed on the quadratic portion of the nonlinearity.

It is known from the work of John [6], that the equations possess almost global solutions for small initial values. Moreover, John [4] has identified a genuine nonlinearity condition which, at least in the spherically symmetry case, leads to formation of singularities even for small data, see also [3]. The null condition, presented below, is the complementary case to genuine nonlinearity.

Thus, the situation is entirely analogous to the case of scalar nonlinear wave equations. Small solutions exist almost globally [7], [8]. Examples suggest that small solutions break down in finite time [4], [11], unless the quadratic terms in the nonlinearity satisfy a null condition, in which case they exist globally [1], [9]. In the present case, however, such results are not immediate generalizations of the scalar case. The original proofs of the existence results for the wave equation depended upon the Lorentz invariance of the equations, a property not shared by the equations of elasticity. Nevertheless, using various extensions of the ideas in [10], we obtain global existence with the null condition without the benefit of Lorentz invariance.

There are two new observations which lead to the global existence result. First, within the class of physically meaningful nonlinearities arising from the hyperelasticity assumption, there exists a null condition. The null condition leads to enhanced decay (i.e. $(1+t)^{-2}$ instead of $(1+t)^{-1}$) of the nonlinear terms along the light cones. Secondly, it is shown how to combine the weighted $L^\infty - L^2$ and $L^2 - L^2$ estimates obtained in [10] in order to obtain enhanced decay inside the light cones.

[1] Research partially supported by the National Science Foundation, DMS-9500284

A precise statement of the existence theorem is given in section 5, after introducing a bit of (mostly) standard notation in section 2, formulating the equations of motion in section 3, and exploring the nonlinearity in section 4. In order to avoid too much technicality, we consider a truncated version of the equations, since only the quadratically nonlinear terms matter in the construction of small solutions. The rest of the paper is devoted to the proof of the result, based on energy and decay estimates.

2. Notation

We begin with a brief description of the notation to be used, most of which is standard, leading to a definition of the function space in which solutions are to be constructed.

Partial derivatives will be written as

$$\partial_0 = \partial_t = \frac{\partial}{\partial t} \quad \text{and} \quad \partial_i = \frac{\partial}{\partial x^i}, \quad i = 1, 2, 3.$$

We will also abbreviate

$$\partial = (\partial_0, \partial_1, \partial_2, \partial_3), \quad \text{and} \quad \nabla = (\partial_1, \partial_2, \partial_3).$$

The angular momentum operators are the vector fields

$$\Omega = (\Omega_1, \Omega_2, \Omega_3) = x \wedge \nabla,$$

$\wedge$ being the usual vector cross product. Then the spatial partial derivatives can be conveniently decomposed into radial and angular components

$$\nabla = \frac{x}{r}\partial_r - \frac{x}{r^2} \wedge \Omega, \quad \text{where} \quad r = |x|, \quad \text{and} \quad \partial_r = \frac{x}{r} \cdot \nabla. \tag{2.1}$$

In the present context of isotropic elasticity, solutions are invariant under the simultaneous rotation of dependent and independent variables, whose generators are given by the vector fields

$$\widetilde{\Omega} = \Omega I + U$$

with

$$U_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad U_2 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad U_3 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The scaling operator is

$$S = t\,\partial_t + r\,\partial_r,$$

however, due to the scaling law of the forthcoming equations, it is more precise and more convenient to use

$$\widetilde{S} = S - 1.$$

The eight vector fields will be written as $\Gamma = (\Gamma_0, \ldots, \Gamma_7) = (\partial, \widetilde{\Omega}, \widetilde{S})$. The commutator of any two Γ's is either 0 or another Γ. By Γ^a, $|a| = \kappa$, will be meant an ordered product of κ vector fields $\Gamma_{a_1}\cdots\Gamma_{a_\kappa}$. (Note that we are not using the standard multi-index notation because of the noncommutativity of the Γ's.)

We will also have occasion to use the operators $|D|=\sqrt{-\Delta}$ and $|D|^{-1}$ which are most easily defined by their symbols: $|\xi|$ and $|\xi|^{-1}$, respectively.

In order to describe the solution space we also introduce the time independent analog of Γ. Set

$$\Lambda=(\Lambda_1,\ldots,\Lambda_7)=(\nabla,\widetilde{\Omega},r\partial_r-1).$$

Then the Λ's have the same commutation properties as the Γ's. Moreover, $|D|$ and $|D|^{-1}$ commute with $\Lambda_1,\ldots,\Lambda_6$, while

$$[\Lambda_7,|D|]=|D| \quad \text{and} \quad [\Lambda_7,|D|^{-1}]=-|D|^{-1}, \tag{2.2}$$

as can be checked using the Fourier transform. Define

$$H^\kappa_\Lambda=\{f\in L^2(\mathbf{R}^3)^3:\Lambda^a f\in L^2(\mathbf{R}^3)^3,\ |a|\le\kappa\},$$

with the norm

$$\|f\|^2_{H^\kappa_\Lambda}=\sum_{|a|\le\kappa}\|\Lambda^a f\|^2_{L^2(\mathbf{R}^3)}.$$

By the commutation relation (2.2), it follows that

$$|D|:H^\kappa_\Lambda\to H^{\kappa-1}_\Lambda.$$

The natural energy norm associated to the linear operator is

$$E_1(u(t))=\tfrac12\int_{\mathbf{R}^3}\left[|\partial_t u(t,x)|^2+c_2^2|\nabla u(t,x)|^2+(c_1^2-c_2^2)(\nabla\cdot u)^2\right]dx,$$

and higher order norms are defined through

$$E_\kappa(u(t))=\sum_{|a|\le\kappa-1}E_1(\Gamma^a u(t)).$$

In order to control the remaining derivatives up to order κ, we also introduce a nonlocal version of the energy

$$\mathcal{E}_0(u(t))=E_1(|D|^{-1}u(t)),\quad \text{and} \quad \mathcal{E}_\kappa(u(t))=\sum_{|a|\le\kappa}\mathcal{E}_0(\Gamma^a u(t)).$$

Thus, we have

$$\mathcal{E}_\kappa(u(t)),\quad \text{and} \quad \mathcal{E}_{\kappa-1}(\nabla u(t))\le CE_\kappa(u(t)),$$

as well as

$$\sum_{|a|\le\kappa}\|\Gamma^a u(t)\|^2_{L^2}\le C\mathcal{E}_\kappa(u(t)) \quad \text{and} \quad \sum_{|a|\le\kappa-1}\|\partial\Gamma^a u(t)\|^2_{L^2}\le CE_\kappa(u(t)).$$

The solution will be constructed in the space

$$X^\kappa(T)=\left\{u(t,x):u\in\bigcap_{j=0}^{\kappa}C^j([0,T);H^{\kappa-j}_\Lambda),\ \partial_t u\in\bigcap_{j=0}^{\kappa}C^j([0,T);|D|H^{\kappa-j}_\Lambda)\right\}.$$

For $u \in X^\kappa(T)$, the norms $E_\kappa(u(t))$ and $\mathcal{E}_\kappa(u(t))$ are finite, $0 \le t < T$.

3. The equations of motion

We now present an explicit description of the equations of motion for the displacement of an isotropic, homogeneous, hyperelastic material filling space, see [2]. The unknown of the problem is $\varphi(t,x)$, a smooth deformation of the material evolving with time. The deformation gradient is then $F = \nabla\varphi$, the matrix with components $F_{i\ell} = \partial_\ell \varphi^i$. For the types of materials under consideration, the potential energy density is characterized by a stored energy function, σ, which depends on F through the list of principal invariants $\imath_1, \imath_2, \imath_3$ of the (left) Cauchy-Green strain matrix $B = FF^T$. A basic requirement for the deformation is that $B > 0$.

The equations of motion are derived by applying Hamilton's principle to

$$\int\int \left[\tfrac{1}{2}\,|\partial_t\varphi|^2 - \sigma(\imath_1, \imath_2, \imath_3)\right] dxdt.$$

So the PDE's can be formulated as the nonlinear system

$$\frac{\partial^2 \varphi^i}{\partial t^2} - \frac{\partial}{\partial x^\ell}\frac{\partial \sigma}{\partial F_{i\ell}} = 0. \tag{3.1}$$

Here, and in the following, we adopt the summation convention. Appropriate conditions will be imposed on σ in order that (3.1) be hyperbolic.

We will consider only small displacements, $u(t,x) = \varphi(t,x) - x$, from the reference configuration. In three space dimensions, the global existence of small amplitude solutions to nonlinear hyperbolic systems hinges on the specific form of the quadratic portion of the nonlinearity in relation to the linear part. Such compatibility conditions are often referred to as null conditions. From the analytical point of view, therefore, it is enough to truncate (3.1) at third order in u, the higher order corrections having no influence on the existence of small solutions.

To this end, it is convenient to compute in terms of u, $G = F - I$, and $C = B - I = G + G^T + GG^T$ instead of φ, F, and B. We will expand the stress matrix $\dfrac{\partial\sigma}{\partial F_{i\ell}}$ to second order in G. Let $\jmath_1, \jmath_2, \jmath_3$ be the invariants of C. Since $C = B - I$, the $\jmath_k$ are linear expressions in the $\imath_k$, as can be checked by comparing the eigenvalues of B and C. This means that we may just as well regard σ as a function of the $\jmath_k$. Variations in F and G are the same, so we can write

$$\frac{\partial\sigma}{\partial F} = \frac{\partial\sigma}{\partial G} = \frac{\partial\sigma}{\partial \jmath_k}\frac{\partial \jmath_k}{\partial G}. \tag{3.2}$$

We make use of the following general formulas for the invariants of a 3×3 matrix:

$$\begin{aligned}
\jmath_1 &= \operatorname{tr} C\\
\jmath_2 &= \tfrac{1}{2}[(\operatorname{tr} C)^2 - \operatorname{tr} C^2]\\
\jmath_3 &= \tfrac{1}{6}[(\operatorname{tr} C)^3 - 3(\operatorname{tr} C)(\operatorname{tr} C^2) + 2\operatorname{tr} C^3].
\end{aligned}$$

Putting $C = G + G^T + GG^T$, we obtain, after discarding terms of fourth order and higher,

$$
\begin{aligned}
\jmath_1 &= 2\mathrm{tr}\, G + \mathrm{tr}\, GG^T \\
\jmath_2 &= 2(\mathrm{tr}\, G)^2 - \mathrm{tr}\, G^2 - \mathrm{tr}\, GG^T \\
&\quad +2(\mathrm{tr}\, G)(\mathrm{tr}\, GG^T) - \mathrm{tr}\, G^2G^T - \tfrac{1}{2}\mathrm{tr}\, GG^TG - \tfrac{1}{2}\mathrm{tr}\, GG^TG^T + \dots \\
\jmath_3 &= \tfrac{4}{3}(\mathrm{tr}\, G^3) - 2(\mathrm{tr}\, G)\mathrm{tr}\, (G^2 + GG^T) \\
&\quad +\tfrac{2}{3}\mathrm{tr}\, (G^3 + G^2G^T + GG^TG + GG^TG^T) + \dots,
\end{aligned}
\tag{3.3}
$$

with ... denoting higher order terms. From (3.3) it is apparent that $\jmath_k = O(G^k)$. Therefore, the relevant terms in the Taylor expansion of σ about $\jmath_k = 0$ are

$$
\sigma = \sigma_0 + \sigma_1\jmath_1 + \tfrac{1}{2}\sigma_{11}\jmath_1^2 + \sigma_2\jmath_2 + \tfrac{1}{6}\sigma_{111}\jmath_1^3 + \sigma_{12}\jmath_1\jmath_2 + \sigma_3\jmath_3 + \dots,
\tag{3.4}
$$

the constants σ_0, σ_1, etc., standing for the partial derivatives of σ at $\jmath_k = 0$. From (3.4), the significant terms for the derivatives $\dfrac{\partial\sigma}{\partial\jmath_k}$ are

$$
\begin{aligned}
\frac{\partial\sigma}{\partial\jmath_1} &= \sigma_1 + \sigma_{11}\jmath_1 + \tfrac{1}{2}\sigma_{111}\jmath_1^2 + \sigma_{12}\jmath_2 + \dots \\
\frac{\partial\sigma}{\partial\jmath_2} &= \sigma_2 + \sigma_{12}\jmath_1 + \dots \\
\frac{\partial\sigma}{\partial\jmath_3} &= \sigma_3 + \dots.
\end{aligned}
\tag{3.5}
$$

The derivatives $\dfrac{\partial\jmath_k}{\partial G}$ can be found from (3.3), after a bit of calculation,

$$
\begin{aligned}
\frac{\partial\jmath_1}{\partial G} &= 2(I + G) \\
\frac{\partial\jmath_2}{\partial G} &= 2\left[2(\mathrm{tr}\, G)I - G - G^T + (\mathrm{tr}\, GG^T)I + 2(\mathrm{tr}\, G)G \right. \\
&\qquad \left. - (GG^T + G^TG + G^2)\right] + \dots \\
\frac{\partial\jmath_3}{\partial G} &= 2\left[2(\mathrm{tr}\, G)^2 - 2(\mathrm{tr}\, G)(G + G^T) \right. \\
&\qquad \left. - (\mathrm{tr}\, G^2 + \mathrm{tr}\, GG^T)I + (G + G^T)^2\right] + \dots.
\end{aligned}
\tag{3.6}
$$

Into (3.2), we insert the expressions (3.5) and (3.6), keeping only terms of order one and two

$$
\begin{aligned}
\frac{\partial\sigma}{\partial G} &= 2\left\{\sigma_1(I+G)+\sigma_{11}(2\mathrm{tr}\ G)I+\sigma_2[2(\mathrm{tr}\ G)I-G-G^T]\right\} \\
&\quad +2\left\{\sigma_{11}[2(\mathrm{tr}\ G)G+(\mathrm{tr}\ GG^T)I]+\sigma_{111}[2(\mathrm{tr}\ G)^2 I]\right. \\
&\quad +\sigma_{12}[6(\mathrm{tr}\ G)^2 I-(2(\mathrm{tr}\ G)G+(\mathrm{tr}\ GG^T)I)-(2(\mathrm{tr}\ G)G^T+(\mathrm{tr}\ G^2)I)] \\
&\quad +\sigma_2[(2(\mathrm{tr}\ G)G+(\mathrm{tr}\ GG^T)I)-(G^2+GG^T+G^TG)] \\
&\quad +\sigma_3[2(\mathrm{tr}\ G)^2 I-(2(\mathrm{tr}\ G)G+(\mathrm{tr}\ GG^T)I)-(2(\mathrm{tr}\ G)G^T+(\mathrm{tr}\ G^2)I) \\
&\qquad \left.+(G^2+GG^T+G^TG)+G^TG^T]\right\}+\dots
\end{aligned}
\tag{3.7}
$$

Examine first the linear terms, which are grouped in the first line of (3.7):

$$2\sigma_1(I+G)+4(\sigma_{11}+\sigma_2)(\mathrm{tr}\ G)I-2\sigma_2(G+G^T).$$

We impose the condition $\sigma_1 = 0$, which implies that the reference configuration is a stress-free state. The Lamé constants $\lambda = 4(\sigma_{11}+\sigma_2)$ and $\mu = -2\sigma_2$ are assumed to be positive. (The density in the reference configuration has been taken to be 1.) This renders the linear part of the equation hyperbolic. Set $\lambda + 2\mu = c_1^2$ and $\mu = c_2^2$. Extraction of the linear terms in (3.1) yields the linear operator

$$Lu = \partial_t^2 u - c_2^2\Delta u - (c_1^2 - c_2^2)\nabla(\nabla\cdot u). \tag{3.8}$$

The constants c_1 and c_2 correspond to the speeds of spherical and rotational waves, respectively. We have also

$$\sigma_2 = -c_2^2/2 \quad \text{and} \quad \sigma_{11} = c_1^2/4. \tag{3.9}$$

The remaining quadratic terms in (3.7) contribute to the nonlinearity. The i^{th} component of the nonlinear term of (3.1) takes the general form $N(u,u)^i$ with

$$N(u,v)^i = D^{ijk}_{\ell mn}\partial_\ell(\partial_m u^j \partial_n v^k), \tag{3.10}$$

for certain constants $D^{ijk}_{\ell mn}$, to be specified in the next section. (Superscripts will be reserved for vector coordinates and subscripts for derivatives.) The result of our calculation is that the truncated equations of motion (3.1) are, succinctly,

$$Lu = N(u,u), \tag{3.11}$$

with L defined in (3.8) and $N(u,u)$ described in (3.10).

4. The nonlinearity

It is necessary to further analyze the coefficients $D^{ijk}_{\ell mn}$ of the nonlinearity. To begin, we will consider the nine different types of quadratic terms that occur in (3.7), each of which having the form ${}_pC^{ijk}_{\ell mn}G_{jm}G_{kn}$. The following table identifies the coefficients ${}_pC^{ijk}_{\ell mn}$, $p = 1, 2, \dots, 9$

p	${}_pC^{ijk}_{\ell mn}G_{jm}G_{kn}$	${}_pC^{ijk}_{\ell mn}$
1	$(\mathrm{tr}\ G)^2I$	$\delta^i_\ell\delta^j_m\delta^k_n$
2	$(\mathrm{tr}\ G)G$	$\frac{1}{2}(\delta^{ij}\delta^k_n\delta_{\ell m}+\delta^{ik}\delta^j_m\delta_{\ell n})$
3	$(\mathrm{tr}\ GG^T)I$	$\delta^i_\ell\delta^{jk}\delta_{mn}$
4	$(\mathrm{tr}\ G)G^T$	$\frac{1}{2}(\delta^i_m\delta^j_\ell\delta^k_n+\delta^i_n\delta^j_m\delta^k_\ell)$
5	$(\mathrm{tr}\ G^2)I$	$\delta^i_\ell\delta^j_n\delta^k_m$
6	G^2	$\frac{1}{2}(\delta^{ij}\delta^k_m\delta_{\ell n}+\delta^{ik}\delta^j_n\delta_{\ell m})$
7	GG^T	$\frac{1}{2}(\delta^{ij}\delta^k_\ell\delta_{mn}+\delta^{ik}\delta^j_\ell\delta_{mn})$
8	G^TG	$\frac{1}{2}(\delta^i_m\delta^{jk}\delta_{\ell n}+\delta^i_n\delta^{jk}\delta_{\ell m})$
9	G^TG^T	$\frac{1}{2}(\delta^i_m\delta^j_n\delta^k_\ell+\delta^i_n\delta^j_\ell\delta^k_m)$

As is natural given the form of (3.10), the ${}_pC^{ijk}_{\ell mn}$ have been chosen so that ${}_pC^{ijk}_{\ell mn} = {}_pC^{ikj}_{\ell nm}$. From (3.7), we have

$$
\begin{aligned}
-D^{ijk}_{\ell mn} &= 4(\sigma_{111}+3\sigma_{12}+\sigma_3)[{}_1C^{ijk}_{\ell mn}] \\
&\quad +2(\sigma_{11}-\sigma_{12}+\sigma_2-\sigma_3)[2({}_2C^{ijk}_{\ell mn})+{}_3C^{ijk}_{\ell mn}] \\
&\quad -2(\sigma_{12}+\sigma_3)[2({}_4C^{ijk}_{\ell mn})+{}_5C^{ijk}_{\ell mn}] \\
&\quad -2(\sigma_2-\sigma_3)[{}_6C^{ijk}_{\ell mn}+{}_7C^{ijk}_{\ell mn}+{}_8C^{ijk}_{\ell mn}] \\
&\quad -\sigma_3[{}_9C^{ijk}_{\ell mn}]
\end{aligned} \tag{4.1}
$$

So $D^{ijk}_{\ell mn}$ inherits the symmetry

$$D^{ijk}_{\ell mn} = D^{ikj}_{\ell nm}, \tag{4.2}$$

from the ${}_pC^{ijk}_{\ell mn}$, and from (3.10) it follows that

$$N(u,v) = N(v,u). \tag{4.3}$$

The coefficients satisfy two additional useful symmetries. The first is related to the variational origin of the problem and is crucial in the derivation of the energy estimates. Set

$$E^{ijk}_{\ell mn} = \tfrac{1}{2}(D^{ijk}_{\ell mn}+D^{ijk}_{m\ell n}) \quad \text{and} \quad F^{ijk}_{\ell mn} = \tfrac{1}{2}(D^{ijk}_{\ell mn}+D^{ijk}_{nm\ell}). \tag{4.4}$$

Then by (4.4) and (4.2),

$$E^{ijk}_{\ell mn} = E^{ijk}_{m\ell n} \quad \text{and} \quad F^{ijk}_{\ell mn} = F^{ijk}_{nm\ell}, \tag{4.5}$$

and the nonlinearity (3.10) can be expanded

$$\begin{aligned} N(u,v) &= D^{ijk}_{\ell mn}(\partial_\ell \partial_m u^j \partial_n v^k + \partial_m u^j \partial_\ell \partial_n v^k) \\ &= E^{ijk}_{\ell mn}\partial_\ell \partial_m u^j \partial_n v^k + F^{ijk}_{\ell mn}\partial_m u^j \partial_\ell \partial_n v^k. \end{aligned} \tag{4.6}$$

The energy symmetry is expressed through

$$E^{ijk}_{\ell mn} = E^{jik}_{\ell mn} = E^{jik}_{m\ell n} \quad \text{and} \quad F^{ijk}_{\ell mn} = F^{kji}_{\ell mn} = F^{kji}_{nm\ell}. \tag{4.7}$$

These properties follow from (4.1) and the explicit formulas given in the table. However, the symmetry does not hold for the ${}_pC^{ijk}_{\ell mn}$ separately. Rather, it is necessary to group the terms into the five units in which they appear in (4.1).

The second symmetry property is a consequence of isotropy, the invariance of the displacement under the transformation $u(t,x) \to \rho^{-1}u(t,\rho x)$ for any orthogonal matrix ρ. Let $\varepsilon_{\alpha\beta\gamma}$ be the usual tensor with value $+1$ when $\alpha\beta\gamma$ is an even permutation of 123, with value -1 when $\alpha\beta\gamma$ is an odd permutation of 123, and with value 0 otherwise. Then we have

$$D^{\beta jk}_{\ell mn}\varepsilon_{\alpha\beta i} + D^{i\beta k}_{\ell mn}\varepsilon_{\alpha\beta j} + D^{ij\beta}_{\ell mn}\varepsilon_{\alpha\beta k} + D^{ijk}_{\beta mn}\varepsilon_{\alpha\beta\ell} + D^{ijk}_{\ell\beta n}\varepsilon_{\alpha\beta m} + D^{ijk}_{\ell m\beta}\varepsilon_{\alpha\beta n} = 0. \tag{4.8}$$

This may also be verified from the formulas above, although it is the infinitesimal version of the isotropy assumption. This property will be used in a later section to derive simple expressions for angular derivatives of $N(u,v)$.

In order to obtain global solutions, the null condition must be imposed. Using the decomposition (2.1), it follows for any functions $u, v \in H^2_\Lambda$, that the leading term of $\partial_\ell(\partial_m u^j \partial_n v^k)$ is $\dfrac{x^\ell x^m x^n}{r^3}\partial_r(\partial_r u^j \partial_r v^k)$. The null condition is designed to cancel these leading terms. Therefore, we would like to have

$$D^{ijk}_{\ell mn}x^\ell x^m x^n = 0, \tag{4.9}$$

for all $x \in \mathbf{R^3}$, and all $i,j,k = 1,2,3$. This can be arranged, within the class of physically meaningful nonlinearities, by balancing the leading terms of the individual expressions ${}_pC^{ijk}_{\ell mn}\partial_\ell(\partial_m u^j \partial_n v^k)$. These leading terms fall into three different groups:

$${}_pC^{ijk}_{\ell mn}x^\ell x^m x^n = \begin{cases} x^i x^j x^k & p = 1,4,5,9 \\ \frac{1}{2}(\delta^{ij}+\delta^{ik})x^i x^j x^k & p = 2,6,7 \\ \delta^{jk}x^i x^j x^k & p = 3,8. \end{cases} \tag{4.10}$$

Rearranging the coefficients in (4.1), we have

$$
\begin{aligned}
-D^{ijk}_{\ell mn} \;=\;& 4(\sigma_{111}+3\sigma_{12})[{}_1C^{ijk}_{\ell mn}] \\
&-2\sigma_{12}[2({}_4C^{ijk}_{\ell mn})+{}_5C^{ijk}_{\ell mn}] \\
&+2(\sigma_{11}-\sigma_{12})[2({}_2C^{ijk}_{\ell mn})+{}_3C^{ijk}_{\ell mn}] \\
&+\sigma_2[2({}_2C^{ijk}_{\ell mn})-{}_6C^{ijk}_{\ell mn}-{}_7C^{ijk}_{\ell mn}+{}_3C^{ijk}_{\ell mn}-{}_8C^{ijk}_{\ell mn}] \\
&+\sigma_3[2({}_1C^{ijk}_{\ell mn})-2({}_4C^{ijk}_{\ell mn})-{}_5C^{ijk}_{\ell mn}+{}_9C^{ijk}_{\ell mn} \\
&\qquad -2({}_2C^{ijk}_{\ell mn})+{}_6C^{ijk}_{\ell mn}+{}_7C^{ijk}_{\ell mn}-{}_3C^{ijk}_{\ell mn}+{}_8C^{ijk}_{\ell mn}].
\end{aligned}
\tag{4.11}
$$

Note that in view of (4.10), the expressions with coefficients σ_2 and σ_3 are already null. In order to kill the remaining leading terms, we must take

$$\sigma_{11}=\sigma_{12} \quad \text{and} \quad 3\sigma_{12}+2\sigma_{111}=0. \tag{4.12}$$

Thus, we see that (4.12) implies (4.9).[2]

The following pointwise inequalities follow from (4.9). They are typical of the way in which the null condition will be used later on.

$$|D^{ijk}_{\ell mn}\partial_\ell\partial_m u^j\partial_n v^k| \;\le\; \frac{C}{r}\left[|\nabla\Omega u||\nabla v|+|\nabla^2 u||\Omega v|\right], \tag{4.13}$$

$$
\begin{aligned}
|D^{ijk}_{\ell mn}\partial_\ell u^i\partial_m v^j\partial_n w^k| \;\le\;& \frac{C}{r}\left[|\Omega u||\nabla v||\nabla w|+|\nabla u||\Omega v||\nabla w|\right. \\
&\left.+|\nabla u||\nabla v||\Omega w|\right].
\end{aligned}
\tag{4.14}
$$

The condition (4.9) is flexible in that analogous inequalities hold when the upper or lower indices of $D^{ijk}_{\ell mn}$ are permuted, provided the roles of u, v, and w are likewise interchanged.

The system of PDE's (3.11) support spherically symmetric solutions, namely, those for which $\widetilde{\Omega}u=0$. Such solutions have the form $u(t,x)=\frac{x}{r}\psi(t,r)$, where ψ is a scalar. In this case, when the derivatives are split into radial and angular components, the leading terms all reduce to $\dfrac{x^i}{r}\partial_r[(\partial_r\psi)^2]$. Thus in (4.11), the combination of the terms with coefficient σ_{12} is now also null. In the spherically symmetric case, the null condition reduces to

$$3\sigma_{11}+2\sigma_{111}=0. \tag{4.15}$$

Condition (4.15) is complementary to the genuine nonlinearity condition given by John in [3], although this is not immediately apparent because John works with the principal invariants of $\sqrt{FF^T}-I$ rather than FF^T-I. However, a change of variables shows that his genuine nonlinearity condition for spherically symmetric solutions is precisely the condition that $3\sigma_{11}+2\sigma_{111}\neq 0$.

[2]Recall that $\sigma_1=0$, $\sigma_{11}=c_1^2/4=(\lambda+2\mu)/4$, and $\sigma_2=-c_2^2/2=-\mu/2$ have already been fixed. So we have used all but one of the available degrees of freedom in the coefficients of σ.

5. The existence theorem

Having set down the necessary notation and formulated the initial value problem, we are now ready to record the main result to be proved.

Theorem 5.1. *Let $f \in H^\kappa_\Lambda$ and $g \in |D|H^\kappa_\Lambda$, $\kappa \geq 6$, be given functions with*

$$\|f\|_{H^\kappa_\Lambda} + \|g\|_{H^{\kappa-1}_\Lambda} + \||D|^{-1}g\|_{H^\kappa_\Lambda} < \varepsilon_0,$$

ε_0 sufficiently small. Assume that the coefficients $D^{ijk}_{\ell mn}$ defined by (4.1) satisfy (3.9) and the null condition (4.12).

Then the Cauchy problem

$$\begin{aligned} &Lu = N(u,u)\\ &u(0) = f, \quad \partial_t u(0) = g, \end{aligned}$$

with

$$\begin{aligned} &Lu = \partial_t^2 u - c_2^2 \Delta u - (c_1^2 - c_2^2)\nabla(\nabla \cdot u)\\ &N(u,v)^i = D^{ijk}_{\ell mn}\partial_\ell(\partial_m u^j \partial_n v^k) \end{aligned}$$

has a unique global solution $u \in X^\kappa(\infty)$ satisfying the bound

$$\sup_{0\leq t<\infty} [E_\kappa(u(t)) + (1+t)^{-\delta_0}\mathcal{E}_\kappa(u(t))] \leq 2\varepsilon_0,$$

for some $\delta_0 > 0$.

We remark that if the initial data are spherically symmetric, i.e. $\widetilde{\Omega} f = \widetilde{\Omega} g = 0$, then the solution remains spherically symmetric, and the theorem holds under the weaker assumption (4.15) instead of (4.12).

The use of the nonlocal energy $\mathcal{E}_\kappa$ requires $g \in |D|H^\kappa_\Lambda$ instead of the more natural space $H^{\kappa-1}_\Lambda$.

6. Commutation relations

Let $\Gamma^a = \Gamma_{a_1}\cdots\Gamma_{a_\kappa}$ be a product of any κ vector fields. If u is a solution of (3.11) in $X^\kappa(T)$, then we will show that

$$L\Gamma^a u = \sum_{b+c=a} N(\Gamma^b u, \Gamma^c u), \tag{6.1}$$

in which the sum extends over all ordered products Γ^b and Γ^c corresponding to partitions of Γ^a. This formula reflects the symmetries present in the equations of motion, and its usefulness in the forthcoming estimates motivates the choice of $\widetilde{\Omega}$, $\widetilde{S}$ over Ω, S. It is a consequence of the following commutation relations.

For the linear operator L in (3.8), we have

$$[\partial, L] = 0, \quad [\widetilde{\Omega}, L] = 0, \quad \text{and} \quad [\widetilde{S}, L] = -2L.$$

For the nonlinear term given by (3.10) and (4.1), we can also easily get

$$\partial N(u,v) = N(\partial u, v) + N(u, \partial v),$$

and

$$\widetilde{S} N(u,v) = N(\widetilde{S} u, v) + N(u, \widetilde{S} v) - 2N(u,v).$$

A slightly longer computation using (4.8) and the fact that

$$(\widetilde{\Omega}_j u)^i = \varepsilon_{jpq} x^p \partial_q u^i + \varepsilon_{jip} u^p$$

yields

$$\widetilde{\Omega}_j N(u,v) = N(\widetilde{\Omega}_j u, v) + N(u, \widetilde{\Omega}_j v).$$

Equation (6.1) is proven inductively by successively applying each Γ_{a_j}, $j = \kappa, \ldots, 1$ to the equation (3.11), and using the above observations.

As a simple consequence of (6.1), we point out that solutions of $Lu = N(u,u)$ satisfy $L\widetilde{\Omega}_j u = N(\widetilde{\Omega}_j u, u) + N(u, \widetilde{\Omega}_j u)$, $j = 1,2,3$, which shows by uniqueness that spherical symmetry is preserved, as mentioned above.

7. Decay estimates

In this section, we assemble the decay estimates that we will use. Lemmas 7.1 and 7.2 are of Sobolev type: a weighted L^∞ norm is controlled by a weighted L^2 norm with two more derivatives. Lemma 7.3 gives a bound for weighted L^2 norms of vectors u in terms of energy and Lu in L^2. Finally, after these preliminaries, we obtain the main decay estimates in Lemmas 7.4 and 7.5, valid for solutions of $Lu = N(u,u)$. These results are extensions of estimates given in [10].

Lemma 7.1. *Let $\tau(r) \geq 1$ be a C^2 radial function with*

$$\sup_{r \geq 0} [|\tau'(r)| + |\tau''(r)|] \leq Const.$$

For all sufficiently regular vector-valued functions ϕ on $\mathbf{R}^3$, we have

$$(1+r^2)^{1/2} \tau(r) |\phi(x)| \leq C \sum_{|a|+|b| \leq 2} \|\tau \nabla^a \widetilde{\Omega}^b \phi\|_{L^2}.$$

with $r = |x|$ and C depending only on $\sup_{r \geq 0} [|\tau'(r)| + |\tau''(r)|]$.

Proof: In Lemma 4.2 in [10], it was shown that

$$r\tau(r)|\phi(x)| \leq C \left[\sum_{|b| \leq 2} \|\tau \Omega^b \phi\|_{L^2} + \sum_{|b| \leq 1} \|\tau \partial_r \Omega^b \phi\|_{L^2} \right].$$

Of course, for vector-valued functions, we can use $\Omega I = \widetilde{\Omega} - U$ to get the same inequality with Ω replaced by $\widetilde{\Omega}$. Finally, in view of the classical Sobolev inequality in three dimensions, we have $\tau(r)|\phi(x)| \leq C\sum_{|a|\leq 2}\|\nabla^a(\tau\phi)\|_{L^2}$, and so by the boundedness of the derivatives of τ, we can replace r by $(1+r^2)^{1/2}$ above. This proves the Lemma.

We now define a weight function that captures the behavior of the solution near the two light cones. Set $\alpha(r) = (1+r^2)^{1/2}$, and define

$$\tau(t,r) = \frac{\alpha(c_1t-r)\alpha(c_2t-r)}{1+\alpha(c_1t-r)+\alpha(c_2t-r)}. \tag{7.1}$$

Then

$$\tfrac{1}{3}\min[\alpha(c_1t-r),\alpha(c_2t-r)] \leq \tau(t,r) \leq \min[\alpha(c_1t-r),\alpha(c_2t-r)], \text{ for } r \geq 0,$$

$$C(1+t) \leq \alpha(r)\tau(t,r), \text{ for } r \geq 0, \tag{7.2}$$

and

$$C(1+t) \leq \tau(t,r), \text{ for } 0 \leq r \leq c_2(1+t)/2. \tag{7.3}$$

Lemma 7.2. *Let $\tau(t,r)$ be defined by (7.1). For all $u \in X^\kappa(T)$,*

$$\alpha(r)\tau(t,r)|\partial^a u(t,x)| \leq C\sum_{|b|+|c|\leq 2}\|\tau(t)\partial^a\nabla^b\widetilde{\Omega}^c u(t)\|_{L^2}, \quad |a|+2\leq\kappa, \tag{7.4}$$

$$\alpha(r)|\partial^a u(t,x)| \leq C\sum_{|b|+|c|\leq 2}\|\partial^a\nabla^b\widetilde{\Omega}^c u(t)\|_{L^2}, \quad |a|+2\leq\kappa. \tag{7.5}$$

Proof: To prove (7.4), apply the previous Lemma to $\partial^a u(t,\cdot)$ for fixed t. Note that ∂^a and $\nabla^b\widetilde{\Omega}^c$ commute. The bound holds uniformly in t because $|\partial_r\tau(t,r)|+|\partial_r^2\tau(t,r)|$ is uniformly bounded.

If Lemma 7.1 is applied to $\partial^a u(t,\cdot)$ again for fixed t, and $\tau(r)$ is taken to be identically equal to 1, then (7.5) results.

Lemma 7.3. *For all $u \in X^\kappa(T)$,*

$$\|\tau(t)\partial\Gamma^a u(t)\|_{L^2}^2 \leq C\left[\mathcal{E}_\kappa(u(t)) + t^2\||D|^{-1}L\Gamma^a u(t)\|_{L^2}^2\right], \quad |a|+1\leq\kappa, \tag{7.6}$$

$$\|\tau(t)\partial\nabla\Gamma^a u(t)\|_{L^2}^2 \leq C\left[E_\kappa(u(t)) + t^2\|L\Gamma^a u(t)\|_{L^2}^2\right], \quad |a|+2\leq\kappa. \tag{7.7}$$

Proof: By Lemma 3.1 in [10], we have[3]

$$\|\tau(t)\partial\nabla u(t)\|_{L^2}^2 \leq C\left[E_2(u(t)) + t^2\|Lu(t)\|_{L^2}^2\right]. \tag{7.8}$$

Apply this to $\Gamma^a u$, $|a|\leq\kappa-2$, and note that $E_2(\Gamma^a u(t)) \leq E_\kappa(u(t))$ for $|a|+2\leq\kappa$ to get (7.7).

[3] In [10] the energy norm was based on S rather than $\widetilde{S}$, but the two norms are obviously equivalent.

Next, write

$$\begin{aligned}\|\tau(t)\partial\Gamma^a u(t)\|_{L^2}^2 &= \|\tau(t)\partial(-\Delta)(-\Delta)^{-1}\Gamma^a u(t)\|_{L^2}^2\\ &= \|\tau(t)\partial\partial_p(\partial_p|D|^{-1})|D|^{-1}\Gamma^a u(t)\|_{L^2}^2,\end{aligned}$$

and apply (7.8) to $(\partial_p|D|^{-1})|D|^{-1}\Gamma^a u$, $|a|\le\kappa-1$. Then since $\partial_p|D|^{-1}$ is bounded in L^2 and $\partial_p(-\Delta)^{-1}$ commutes with L, we get

$$\|\tau(t)\partial\Gamma^a u(t)\|_{L^2}^2 \le C\left[E_2(|D|^{-1}\Gamma^a u(t)) + t^2\||D|^{-1}L\Gamma^a u(t)\|_{L^2}^2\right].$$

However, $E_2(|D|^{-1}\Gamma^a u(t)) \le C\mathcal{E}_1(\Gamma^a u(t)) \le C\mathcal{E}_\kappa(u(t))$, which proves (7.6).

Lemma 7.4. *Let $u\in X^\kappa(T)$, $\kappa\ge 5$, solve $Lu = N(u,u)$, and suppose that*

$$\sup_{0\le t<T} E_\kappa(u(t)) \equiv \varepsilon_0$$

is sufficiently small. Then

$$\|\tau(t)\partial\Gamma^a u(t)\|_{L^2} \le C\mathcal{E}_\kappa(u(t)), \quad |a|+1\le\kappa, \tag{7.9}$$

$$\|\tau(t)\partial\nabla\Gamma^a u(t)\|_{L^2} \le CE_\kappa(u(t)), \quad |a|+2\le\kappa. \tag{7.10}$$

Proof: We first prove (7.10) using (7.7). Thus, we need to estimate $\|L\Gamma^a u(t)\|_{L^2}$. Let $|a|+2\le\kappa$. Using the differentiation formula (6.1) and (4.6), we have

$$\begin{aligned}L\Gamma^a u^i &= \sum_{b+c=a} N(\Gamma^b u,\Gamma^c u)^i\\ &= \sum_{b+c=a}\left[E^{ijk}_{\ell mn}\partial_\ell\partial_m\Gamma^b u^j\partial_n\Gamma^c u^k + F^{ijk}_{\ell mn}\partial_m\Gamma^b u^j\partial_\ell\partial_n\Gamma^c u^k\right],\end{aligned}$$

where, for instance, $\Gamma^b u^j$ is the j^{th} component of $\Gamma^b u$. By symmetry, we need only consider the first group of terms, which can be further separated into two sums depending on whether $|b|+1\le|c|$ or $|b|+1>|c|$.

If $|b|+1\le|c|$, then $|b|+2\le\kappa-2$ (since $\kappa\ge5$) and $|c|\le|a|<\kappa-1$. We have by (7.2) and (7.4),

$$\begin{aligned}t^2\|\partial_\ell\partial_m\Gamma^b u(t)\partial_n\Gamma^c u(t)\|_{L^2}^2 &\le C\|\alpha\tau(t)\partial_\ell\partial_m\Gamma^b u(t)\|_{L^\infty}^2\|\partial_n\Gamma^c u(t)\|_{L^2}^2\\ &\le C\sum_{|a|\le 2}\|\tau(t)\nabla^2\Gamma^a\Gamma^b u(t)\|_{L^2}^2\|\nabla\Gamma^c u(t)\|_{L^2}^2\\ &\le C\sum_{|a|\le\kappa-2}\|\tau(t)\nabla^2\Gamma^a u(t)\|_{L^2}^2 E_\kappa(u(t)).\end{aligned}$$

If, on the other hand, $|b|+1>|c|$, then $|b|\leq|a|\leq\kappa-2$ and $|c|+2\leq\kappa-1$ ($\kappa\geq 5$). Hence, by (7.2) and (7.5), we obtain

$$\begin{aligned} t^2\|\partial_\ell\partial_m\Gamma^b u(t)\partial_n\Gamma^c u(t)\|_{L^2}^2 &\leq C\|\tau(t)\partial_\ell\partial_m\Gamma^b u(t)\|_{L^2}^2\|\alpha\partial_n\Gamma^c u(t)\|_{L^\infty}^2 \\ &\leq C\|\tau(t)\nabla^2\Gamma^b u(t)\|_{L^2}^2\sum_{|a|\leq 2}\|\nabla\Gamma^a\Gamma^c u(t)\|_{L^2}^2 \\ &\leq C\sum_{|a|\leq\kappa-2}\|\tau(t)\nabla^2\Gamma^a u(t)\|_{L^2}^2 E_\kappa(u(t)). \end{aligned}$$

The preceding therefore yields,

$$t^2\|L\Gamma^a u(t)\|_{L^2}^2\leq C\sum_{|a|\leq\kappa-2}\|\tau(t)\nabla^2\Gamma^a u(t)\|_{L^2}^2 E_\kappa(u(t)),$$

which, if we combine with (7.7) and sum on $|a|\leq\kappa-2$, results in

$$\sum_{|a|\leq\kappa-2}\|\tau(t)\partial\nabla\Gamma^a u(t)\|_{L^2}^2\leq C\left[E_\kappa(u(t))+\varepsilon_0\sum_{|a|\leq\kappa-2}\|\tau(t)\nabla^2\Gamma^a u(t)\|_{L^2}^2\right].$$

So if $C\varepsilon_0\leq 1/2$, we get the bound (7.10).

The estimate (7.9) is simpler, since there is one less derivative to estimate in the nonlinearity:[4]

$$\begin{aligned} \||D|^{-1}L\Gamma^a u\|_{L^2} &= \|\sum_{b+c=a}|D|^{-1}N(\Gamma^b u,\Gamma^c u)\|_{L^2} \\ &\leq C\sum_{b+c=a}\||D|^{-1}\partial_\ell(\partial_m\Gamma^b u^j\partial_n\Gamma^c u^k)\|_{L^2} \qquad (7.11)\\ &\leq C\sum_{b+c=a}\|\nabla\Gamma^b u\nabla\Gamma^c u(t)\|_{L^2}. \end{aligned}$$

Then by (7.2),

$$t^2\||D|^{-1}L\Gamma^a u(t)\|_{L^2}^2\leq C\sum_{b+c=a}\|\alpha\tau(t)\nabla\Gamma^b u(t)\nabla\Gamma^c u(t)\|_{L^2}^2.$$

By symmetry, we can take $|b|\geq|c|$, so that $|b|\leq|a|\leq\kappa-1$ and $|c|+2\leq\kappa-1$ ($\kappa\geq 5$). Now by (7.4),

$$\begin{aligned} \|\alpha\tau(t)\nabla\Gamma^b u(t)\nabla\Gamma^c u(t)\|_{L^2}^2 &\leq C\|\nabla\Gamma^b u(t)\|_{L^2}^2\|\alpha\tau(t)\nabla\Gamma^c u(t)\|_{L^\infty}^2 \\ &\leq C\|\nabla\Gamma^b u(t)\|_{L^2}^2\sum_{|a|\leq 2}\|\tau(t)\nabla\Gamma^a\Gamma^c u(t)\|_{L^2}^2 \\ &\leq CE_\kappa(u(t))\sum_{|a|\leq\kappa-1}\|\tau(t)\nabla\Gamma^a u(t)\|_{L^2}^2. \end{aligned}$$

[4]The divergence form of the nonlinear terms is important here.

The proof of (7.9) is now completed through the use of (7.6).

To conclude this section, we note the following immediate corollary of Lemmas 7.2 and 7.4.

Lemma 7.5. *Let $u \in X^\kappa(T)$, $\kappa \geq 5$, solve $Lu = N(u,u)$, and suppose that*

$$\sup_{0\leq t<T} E_\kappa(u(t))$$

is sufficiently small. Then

$$|\alpha(r)\tau(t,r)\partial\Gamma^a u(t,x)| \leq C\mathcal{E}_\kappa(u(t))^{1/2}, \quad |a|+3\leq\kappa, \tag{7.12}$$

$$|\alpha(r)\tau(t,r)\partial\nabla\Gamma^a u(t,x)| \leq CE_\kappa(u(t))^{1/2}, \quad |a|+4\leq\kappa. \tag{7.13}$$

8. Energy estimates

We now come to the main part of the proof, the derivation of a priori energy estimates for small solutions. These estimates reflect the essential information required for a formal global existence proof based on iteration.

Let $u \in X^\kappa(T)$, $\kappa \geq 6$, have small initial data. The strategy will be to show that

$$\frac{d}{dt}E_\kappa(u(t)) \leq \frac{C}{(1+t)^2}E_\kappa(u(t))\mathcal{E}_\kappa(u(t))^{1/2}$$

and

$$\frac{d}{dt}\mathcal{E}_\kappa(u(t)) \leq \frac{C}{1+t}\mathcal{E}_\kappa(u(t))E_\kappa(u(t))^{1/2}.$$

Thus, for small enough initial values $E_\kappa(u(0))$ and $\mathcal{E}_\kappa(u(0))$, there is a $\delta > 0$ such that

$$E_\kappa(u(t)) + (1+t)^{-\delta}\mathcal{E}_\kappa(u(t)) \leq 2[E_\kappa(u(0)) + \mathcal{E}_\kappa(u(0))], \tag{8.1}$$

for all $0 \leq t < \infty$. Actually, as is standard, the differential inequalities will be proven for perturbed energies with a small cubic correction term which takes into account the quasilinear nature of the equations. Bounds for the quadratic energies E_κ and $\mathcal{E}_\kappa$ follow immediately. This type of coupled system of inequalities allowing for slow growth in a larger norm was used also in [9].

To obtain the first of these estimates, the nonlinearity will be estimated differently in the regions $r < c_2(t+1)/2$ and $r > c_2(t+1)/2$. In the first zone, the $(1+t)^{-2}$ decay factor comes from the weighted L^2 and L^∞ estimates. For large r, the weighted L^∞ estimate and the null condition are used. The inequality for the nonlocal energy depends on the divergence form of the nonlinearity and the weighted L^∞ estimate.

Recall the formula (6.1):

$$L\Gamma^a u = \sum_{b+c=a} N(\Gamma^b u, \Gamma^c u).$$

For any $|a| \le \kappa - 1$, take the dot product with $\partial_t \Gamma^a u$, integrate over $\mathbf{R}^3$, and sum on a to get

$$\frac{d}{dt} E_\kappa(u(t)) = \sum_{|a| \le \kappa-1} \sum_{b+c=a} \int \partial_t \Gamma^a u(t) \cdot N(\Gamma^b u(t), \Gamma^c u(t)) dx.$$

First, we estimate the terms with $|b|+2 \le \kappa$ and $|c|+2 \le \kappa$. Thus, either $|a|+2 \le \kappa$ or if $|a|+1 = \kappa$, then $b \ne a$ and $c \ne a$. For such terms, we have

$$\left| \int \partial_t \Gamma^a u(t) \cdot N(\Gamma^b u(t), \Gamma^c u(t)) dx \right| \le C \|\partial \Gamma^a u(t)\|_{L^2} \|N(\Gamma^b u(t), \Gamma^c u(t))\|_{L^2}.$$

Write

$$N(\Gamma^b u, \Gamma^c u) = E^{ijk}_{\ell mn} \partial_\ell \partial_m \Gamma^b u^j \partial_n \Gamma^c u^k + F^{ijk}_{\ell mn} \partial_m \Gamma^b u^j \partial_\ell \partial_n \Gamma^c u^k,$$

using (4.6). It is enough to estimate the first group of terms. Since $\kappa \ge 6$, we have either $|b| + 4 \le \kappa$ or $|c| + 3 \le \kappa$.

Suppose that $|b| + 4 \le \kappa$. Then in view of (7.3), (7.13), and (7.9), we have

$$\begin{aligned} &\int\limits_{r < c_2(t+1)/2} |\partial_\ell \partial_m \Gamma^b u^j(t) \partial_n \Gamma^c u^k(t)|^2 dx \\ &\qquad \le \frac{C}{(1+t)^4} \|\alpha \tau(t) \nabla^2 \Gamma^b u(t)\|^2_{L^\infty} \|\tau(t) \nabla \Gamma^c u(t)\|^2_{L^2} \\ &\qquad \le \frac{C}{(1+t)^4} E_\kappa(u(t)) \mathcal{E}_\kappa(u(t)). \end{aligned}$$

Or if $|c| + 3 \le \kappa$, then by (7.3), (7.12), and (7.10),

$$\begin{aligned} &\int\limits_{r < c_2(t+1)/2} |\partial_\ell \partial_m \Gamma^b u^j(t) \partial_n \Gamma^c u^k(t)|^2 dx \\ &\qquad \le \frac{C}{(1+t)^4} \|\tau(t) \nabla^2 \Gamma^b u(t)\|^2_{L^2} \|\alpha \tau(t) \nabla \Gamma^c u(t)\|^2_{L^\infty} \\ &\qquad \le \frac{C}{(1+t)^4} E_\kappa(u(t)) \mathcal{E}_\kappa(u(t)). \end{aligned}$$

On the other hand, when $r > c_2(t+1)/2$, we make use of (4.13), (7.12), and (7.13). Assume that $|b| + 4 \leq \kappa$. Then

$$
\begin{aligned}
&\int\limits_{r>c_2(t+1)/2} |E^{ijk}_{\ell mn}\partial_\ell\partial_m\Gamma^b u^j(t)\partial_n\Gamma^c u^k(t)|^2 dx \\
&\quad \leq C \int\limits_{r>c_2(t+1)/2} \frac{1}{r^2}\Big[|\nabla\widetilde{\Omega}\Gamma^b u(t)||\nabla\Gamma^c u(t)| + |\nabla^2\Gamma^b u(t)||\widetilde{\Omega}\Gamma^c u(t)| \\
&\qquad\qquad + |\nabla\Gamma^b u(t)||\nabla\Gamma^c u(t)| + |\nabla^2\Gamma^b u(t)||\Gamma^c u(t)|\Big]^2 dx \\
&\quad \leq \frac{C}{(1+t)^4}\Big[\|\alpha\tau(t)\nabla\widetilde{\Omega}\Gamma^b u(t)\|_{L^\infty}\|\nabla\Gamma^c u(t)\|_{L^2} \\
&\qquad\qquad + \|\alpha\tau(t)\nabla^2\Gamma^b u(t)\|_{L^\infty}\|\widetilde{\Omega}\Gamma^c u(t)\|_{L^2} \\
&\qquad\qquad + \|\alpha\tau(t)\nabla\Gamma^b u(t)\|_{L^\infty}\|\nabla\Gamma^c u(t)\|_{L^2} \\
&\qquad\qquad + \|\alpha\tau(t)\nabla^2\Gamma^b u(t)\|_{L^\infty}\|\Gamma^c u(t)\|_{L^2}\Big]^2 \\
&\quad \leq \frac{C}{(1+t)^4}E_\kappa(t)\mathcal{E}_\kappa(t),
\end{aligned}
$$

If $|c| + 3 \leq \kappa$, then the role of the L^2 and L^∞ norms is simply interchanged in the preceding lines, and (7.9) and (7.10) are used.

Now we proceed to the most singular case when $a = b$, $|a| = |b| = \kappa - 1$, and $c = 0$, which produces a derivative of order $\kappa + 1$. It can be absorbed on the right after an integration by parts, and it is here that the energy symmetry will enter. The following identity captures the basic idea. We state it first in a general form, since it will be used again when we consider the nonlocal energy. Let v, $w \in X^\kappa(T)$. Using

(4.6) and (4.7) with integration by parts

$$
\begin{aligned}
\int \partial_t v(t) \cdot N(v(t), w(t)) dx& \\
&= \int D^{ijk}_{\ell mn} \partial_t v^i(t) \partial_\ell (\partial_m v^j(t) \partial_n w^k(t)) dx \\
&= \int E^{ijk}_{\ell mn} \partial_t v^i(t) \partial_\ell \partial_m v^j(t) \partial_n w^k(t) dx \\
&\qquad + \int F^{ijk}_{\ell mn} \partial_t v^i(t) \partial_m v^j(t) \partial_\ell \partial_n w^k(t) dx \\
&= -\int E^{ijk}_{\ell mn} \partial_t \partial_\ell v^i(t) \partial_m v^j(t) \partial_n w^k(t) dx \\
&\qquad + \int (F^{ijk}_{\ell mn} - E^{ijk}_{\ell mn}) \partial_t v^i(t) \partial_m v^j(t) \partial_\ell \partial_n w^k(t) dx \\
&= -\frac{1}{2} \int E^{ijk}_{\ell mn} \partial_t (\partial_\ell v^i(t) \partial_m v^j(t)) \partial_n w^k(t) dx \\
&\quad + \int (F^{ijk}_{\ell mn} - E^{ijk}_{\ell mn}) \partial_t v^i(t) \partial_m v^j(t) \partial_\ell \partial_n w^k(t) dx \\
&= -\frac{1}{2} \frac{d}{dt} \int E^{ijk}_{\ell mn} \partial_\ell v^i(t) \partial_m v^j(t) \partial_n w^k(t) dx \\
&\quad + \frac{1}{2} \int E^{ijk}_{\ell mn} \partial_\ell v^i(t) \partial_m v^j(t) \partial_t \partial_n w^k(t) dx \\
&\quad + \int (F^{ijk}_{\ell mn} - E^{ijk}_{\ell mn}) \partial_t v^i(t) \partial_m v^j(t) \partial_\ell \partial_n w^k(t) dx.
\end{aligned}
\tag{8.2}
$$

Apply this identity with $v = \Gamma^a u$ and $w = u$, to get

$$
\begin{aligned}
\int \partial_t \Gamma^a u(t) \cdot N(\Gamma^a u(t), u(t)) dx& \\
&= -\frac{1}{2} \frac{d}{dt} \int E^{ijk}_{\ell mn} \partial_\ell \Gamma^a u^i(t) \partial_m \Gamma^a u^j(t) \partial_n u^k(t) dx \\
&+ \frac{1}{2} \int E^{ijk}_{\ell mn} \partial_\ell \Gamma^a u^i(t) \partial_m \Gamma^a u^j(t) \partial_t \partial_n u^k(t) dx \\
&+ \int (F^{ijk}_{\ell mn} - E^{ijk}_{\ell mn}) \partial_t \Gamma^a u^i(t) \partial_m \Gamma^a u^j(t) \partial_\ell \partial_n u^k(t) dx.
\end{aligned}
$$

An identical expression arises when $b = 0$ and $c = a$, thanks to (4.3).

The last two integrals are bounded by $C(1+t)^{-2} E_\kappa(u(t)) \mathcal{E}_\kappa(u(t))^{1/2}$ in the same way as above, using (4.13), (4.14), (7.9), and (7.13). Note that the presence of one time-derivative is permitted in (7.13).

The first term and its twin must be included in the energy. Set

$$\widetilde{E}_\kappa(u(t)) = E_\kappa(u(t)) + \sum_{|a|=\kappa-1} \int E^{ijk}_{\ell mn} \partial_\ell \Gamma^a u^i(t) \partial_m \Gamma^a u^j(t) \partial_n u^k(t) dx.$$

Provided that $\|\partial_n u^k(t)\|_{L^\infty}$ is small, we have

$$cE_\kappa(u(t)) \le \widetilde{E}_\kappa(u(t)) \le CE_\kappa(u(t)).$$

Thus, we have shown that

$$\frac{d}{dt}\widetilde{E}_\kappa(u(t)) \le \frac{C}{(1+t)^2} E_\kappa(u(t))\mathcal{E}_\kappa(u(t))^{1/2}. \tag{8.3}$$

The final step is to estimate the nonlocal energy. Apply $|D|^{-1}\Gamma^a$ to the equation, multiply by $\partial_t|D|^{-1}\Gamma^a u$, integrate in x, and sum over $|a| \le \kappa$ to get

$$\frac{d}{dt}\mathcal{E}_\kappa(u(t)) = \sum_{|a|=\kappa} \sum_{b+c=a} \int \partial_t |D|^{-1}\Gamma^a u(t) \cdot |D|^{-1} N(\Gamma^b u(t), \Gamma^c u(t)) dx.$$

We first consider the case in which $|b| + 1 \le \kappa$ and $|c| + 1 \le \kappa$. We can then estimate

$$\begin{aligned} &\left| \int \partial_t |D|^{-1}\Gamma^a u(t) \cdot |D|^{-1} N(\Gamma^b u(t), \Gamma^c u(t)) dx \right| \\ &\qquad \le C\|\partial_t |D|^{-1}\Gamma^a u(t)\|_{L^2} \||D|^{-1} N(\Gamma^b u(t), \Gamma^c u(t))\|_{L^2}. \end{aligned}$$

Recalling (7.11), we have

$$\||D|^{-1} N(\Gamma^b u(t), \Gamma^c u(t))\|_{L^2} \le C\|\nabla\Gamma^b u(t) \nabla\Gamma^c u(t)\|_{L^2}.$$

Since $\kappa \ge 6$, we have either $|b| + 3 \le \kappa$ or $|c| + 3 \le \kappa$. Assume the former, without loss of generality. Then by (7.2) and (7.12), we see that

$$\begin{aligned} \|\nabla\Gamma^b u(t) \nabla\Gamma^c u(t)\|_{L^2} &\le \|\nabla\Gamma^b u(t)\|_{L^\infty} \|\nabla\Gamma^c u(t)\|_{L^2} \\ &\le \frac{C}{1+t} \|\alpha\tau(t)\nabla\Gamma^b u(t)\|_{L^\infty} \|\nabla\Gamma^c u(t)\|_{L^2} \\ &\le \frac{C}{1+t} \mathcal{E}_\kappa(u(t))^{1/2} E_\kappa(u(t))^{1/2}. \end{aligned}$$

This gives the desired bound for all such terms.

Now take $a = b$, $c = 0$, and $|a| = \kappa$. Then integration by parts gives

$$
\begin{aligned}
&\int \partial_t |D|^{-1}\Gamma^a u(t) \cdot |D|^{-1} N(\Gamma^a u(t), u(t))dx \\
&\qquad = \int \partial_t |D|^{-2}\Gamma^a u(t) \cdot N(\Gamma^a u(t), u(t))dx \\
&\qquad = \int D^{ijk}_{\ell mn} \partial_t (-\Delta)^{-1}\Gamma^a u^i(t)\ \partial_\ell[\partial_m \Gamma^a u^j(t)\ \partial_n u^k(t)]dx \\
&\qquad = \int D^{ijk}_{\ell mn} \partial_t (-\Delta)^{-1}\Gamma^a u^i(t)\ \partial_\ell[\partial_m (-\Delta)(-\Delta)^{-1}\Gamma^a u^j(t)\ \partial_n u^k(t)]dx \\
&\qquad = -\int D^{ijk}_{\ell mn} \partial_t \partial_p (-\Delta)^{-1}\Gamma^a u^i(t)\ \partial_\ell[\partial_m \partial_p (-\Delta)^{-1}\Gamma^a u^j(t)\ \partial_n u^k(t)]dx \\
&\qquad\qquad - \int D^{ijk}_{\ell mn} \partial_t (-\Delta)^{-1}\Gamma^a u^i(t)\ \partial_\ell[\partial_m \partial_p (-\Delta)^{-1}\Gamma^a u^j(t)\ \partial_n \partial_p u^k(t)]dx \\
&\qquad = -\int \partial_t \partial_p (-\Delta)^{-1}\Gamma^a u(t) \cdot N(\partial_p (-\Delta)^{-1}\Gamma^a u(t), u(t))dx \\
&\qquad\qquad - \int D^{ijk}_{\ell mn} \partial_t |D|^{-1}\Gamma^a u^i(t)\ (|D|^{-1}\partial_\ell)[\partial_m \partial_p (-\Delta)^{-1}\Gamma^a u^j(t)\ \partial_n \partial_p u^k(t)]dx.
\end{aligned}
$$

Apply (8.2) to the first integral, with $v = \partial_p(-\Delta)^{-1}\Gamma^a u$ and $w = u$. This produces the identity

$$
\begin{aligned}
&\int \partial_t |D|^{-1}\Gamma^a u(t) \cdot |D|^{-1} N(\Gamma^a u(t)\ , u(t))dx \\
&\qquad = -\frac{1}{2}\frac{d}{dt}\int E^{ijk}_{\ell mn} \partial_\ell \partial_p (-\Delta)^{-1}\Gamma^a u^i(t) \partial_m \partial_p (-\Delta)^{-1}\Gamma^a u^j(t)\ \partial_n u^k(t)dx \\
&\qquad\qquad + \frac{1}{2}\int E^{ijk}_{\ell mn} \partial_\ell \partial_p (-\Delta)^{-1}\Gamma^a u^i(t)\ \partial_m \partial_p (-\Delta)^{-1}\Gamma^a u^j(t)\ \partial_t \partial_n u^k(t)dx \\
&\qquad\qquad + \int (F^{ijk}_{\ell mn} - E^{ijk}_{\ell mn}) \partial_t \partial_p (-\Delta)^{-1}\Gamma^a u^i(t)\ \partial_m \partial_p (-\Delta)^{-1}\Gamma^a u^j(t)\ \partial_\ell \partial_n u^k(t)dx \\
&\qquad\qquad - \int D^{ijk}_{\ell mn} \partial_t |D|^{-1}\Gamma^a u^i(t)\ (|D|^{-1}\partial_\ell)[\partial_m \partial_p (-\Delta)^{-1}\Gamma^a u^j(t)\ \partial_n \partial_p u^k(t)]dx.
\end{aligned}
$$

The last three integrals are bounded by $C(1+t)^{-1}\mathcal{E}_\kappa(u(t))E_\kappa(u(t))^{1/2}$, using (7.13). We absorb the first integral into the energy by setting

$$
\widetilde{\mathcal{E}}_\kappa(u(t)) = \mathcal{E}_\kappa(u(t)) + \int E^{ijk}_{\ell mn} \partial_\ell \partial_p (-\Delta)^{-1}\Gamma^a u^i(t) \partial_m \partial_p (-\Delta)^{-1}\Gamma^a u^j(t) \partial_n u^k(t)dx.
$$

Then for $\|\nabla u(t)\|_{L^\infty}$ small, we have

$$
c\mathcal{E}_\kappa(u(t)) \le \widetilde{\mathcal{E}}_\kappa(u(t)) \le C\mathcal{E}_\kappa(u(t)).
$$

And we obtain

$$\frac{d}{dt}\widetilde{\mathcal{E}}_\kappa(u(t)) \le \frac{C}{1+t}\mathcal{E}_\kappa(u(t))E_\kappa(u(t))^{1/2}. \tag{8.4}$$

Of course, we may replace the energies on the right hand side of (8.3) and (8.4) by the modified energies. The resulting differential inequalities yield the bound (8.1) for $\widetilde{E}_\kappa(u(t))$ and $\widetilde{\mathcal{E}}_\kappa(u(t))$, but then also for $E_\kappa(u(t))$ and $\mathcal{E}_\kappa(u(t))$, provided the initial values are small.

Acknowledgement

The author acknowledges with pleasure several helpful conversations with Sergiu Klainerman. In particular, this work depends crucially on our joint paper [10].

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