Let us solve it in a few particular cases. We know that characteristics are given by

\[ \frac{dx(t)}{dt} = u(x(t), t) \]

But also on the characteristic

\[ \frac{\partial}{\partial t} u(x(t), t) = \left( \frac{\partial}{\partial x} u \right) \frac{dx}{dt} + \frac{1}{2} \frac{\partial^2}{\partial x^2} u = u u_x + u_t = 0 \]

Hence actually since source doesn't change on the characteristic

\[ \frac{dx}{dt} = u(x_0, 0) \quad \Rightarrow \quad x = u(x_0) t + x_0 \]

Let's look at a particular initial condition

\[ u_0(x) = \begin{cases} \frac{1}{2}, & 0 \leq x \leq 2 \\ 0, & x < 0 \text{ or } x > 2 \end{cases} \]

\[ x_0(x, t) = \begin{cases} x, & 0 \leq x < t \text{ or } x \geq t + 2 \\ \frac{t}{x - t}, & x - t < t + 2 \end{cases} \]

\[ u_0(x, t) = \begin{cases} \frac{1}{2}, & 0 \leq x < t \text{ or } x \geq t + 2 \\ 0, & x < 0 \text{ or } t < x < t + 2 \\ 1, & t + 2 \leq x \end{cases} \]
\[ \lim_{\mathbf{t} \to 0^+} \text{ (Not rigorous)} \quad \text{INITIAL DATA} \quad u_0(x) = \begin{cases} 0, & x < 0 \\ \frac{x}{t}, & 0 \leq x < t \\ 1, & t \leq x \end{cases} \]

\[ u(x, t) = \begin{cases} 0, & x < 0 \\ \frac{x}{t}, & 0 \leq x < t \\ 1, & t \leq x \end{cases} \]

Clearly, Regions I and III are solutions.

Let's check Region II

\[ \frac{\partial}{\partial t} \left( \frac{x}{t} \right) = -\frac{x}{t^2} \quad \frac{\partial}{\partial x} \left( \frac{x}{t} \right) = \frac{1}{t} \]

\[ u_t + u_x = -\frac{x}{t^2} + \left( \frac{1}{t} \right) \frac{x}{t} = 0 \quad \checkmark \]

So this is a solution, but there is a problem with characteristics looking like this.

No characteristics here.

This is called "rarefaction" even though the solution is not unique (due to non-linearity). Above solution is still physically correct.

Before discussing exactly why let us look at another example.
Consider initial condition:

\[ u_0(x) = \begin{cases} 1 & x < 0 \\ 0 & x > 0 \end{cases} \]

**Characteristics**

How do we interpret region where characteristics non-unique?

We will say a shock propagates where solution discontinuous.

To one side solution will have one value, to the other another.

Where \( \psi(t) \) is is yet undetermined.

How do we interpret discontinuous up stream as solution of PDE? Notice that for any smooth \( \psi \) there exists a solution of $u$ with $\psi(x,t) = 0$ for $x^2 + t^2 > r^2$ for some $r > 0$.

If $u$ solves PDE, then by integration by parts:

\[ \int_0^\infty \int_0^\infty \left[ u \psi_x + \frac{u^2}{2} \psi \right] dx dt = 0 \]

This is called a "weak" solution.

Let us try to figure out what requiring $u$ to be a weak solution implies for $\psi(t)$.
Consider \( F = \left( \frac{u}{2} \right) \) and
\[
\nabla \psi = \left( \frac{\partial \psi}{\partial x} \right)
\]

Applying Divergence Theorem
\[
\int_{L} \nabla \cdot (E \psi) \, dx \, dt + \int_{R} \nabla \cdot (E \psi) \, dx \, dt
\]
\[
= - \int_{x = \frac{\partial}{\partial t}} \left( u^{+} \psi n_{x} + \frac{(u^{+})^{2}}{2} \psi n_{x} \right) \, dl + \int_{x = \frac{\partial}{\partial t}} \left( u^{-} \psi n_{x} + \frac{(u^{-})^{2}}{2} \psi n_{x} \right) \, dl
\]

Where \( n = (u_{x}, u_{t}) \) is unit normal to \( \frac{\partial}{\partial t} \) pointing left

By fact \( u \) is weak soln. and product rule then
\[
0 = - \int_{L} \left( u^{+} + u_{x} u \right) \psi \, dx \, dt - \int_{R} \left( u^{-} + u_{x} u \right) \psi \, dx \, dt
\]
\[
- \int_{x = \frac{\partial}{\partial t}} \left( u^{+} n_{x} + \frac{(u^{+})^{2}}{2} n_{x} \right) \psi \, dl + \int_{x = \frac{\partial}{\partial t}} \left( u^{-} n_{x} + \frac{(u^{-})^{2}}{2} n_{x} \right) \psi \, dl
\]

But on L and R regions separately \( u^{+} - u_{x} u = 0 \)

So
\[
\int_{x = \frac{\partial}{\partial t}} \left( u^{+} n_{x} + \frac{(u^{+})^{2}}{2} n_{x} \right) \psi \, dl = \int_{x = \frac{\partial}{\partial t}} \left( u^{-} n_{x} + \frac{(u^{-})^{2}}{2} n_{x} \right) \psi \, dl
\]

Since \( \psi \) arbitrary
\[
u^{+} n_{x} + \frac{(u^{+})^{2}}{2} n_{x} = u^{-} n_{x} + \frac{(u^{-})^{2}}{2} n_{x}
\]
\[
u^{+} \left( - \frac{n_{x}}{n_{t}} \right) - u^{-} \left( \frac{m_{x}}{m_{t}} \right) = \frac{(u^{+})^{2}}{2} - \frac{(u^{-})^{2}}{2}
\]

\[
S = \frac{dS}{dt} = \left( - \frac{m_{x}}{m_{t}} \right) = \frac{1}{2} \left( \frac{(u^{+})^{2} - (u^{-})^{2}}{u^{+} - u^{-}} \right)
\]

This is Rankin-Hugoniot formula

Now that we have slope of shock we can solve our example problem

Notice characteristics go "into" shock.
This is physically correct and is called the Lax Entropy Condition

\[ u^{-} > s > u^{+} \]

If we try shock soln., in rarefaction case, mathematically correct but fails entropy condition.
Consider a string with displacement $u(x,t)$ from equilibrium. The string has density $\rho$ and tension $T(x,t)$ vector.

The magnitude of the vector along the string is assumed to be transverse to the string.

**Second Order Eqns**

$$F = ma$$

$$T(x,t) = \rho \frac{1}{\sqrt{1 + U_x^2}}$$

1. For long waves:

$$\left. \frac{T}{\sqrt{1 + U_x^2}} \right|_{x_0}^{x_1} = 0$$

2. For transverse waves:

$$\left. \frac{T U_x}{\sqrt{1 + U_x^2}} \right|_{x_0}^{x_1} = \int_{x_0}^{x_1} \rho u_{tt} \, dx$$

Assume $|U_x| \ll 1$.

By Taylor series expansion:

$$\sqrt{1 + U_x^2} = 1 + O(U_x^2) \approx 1$$

Eqn (1) says $T$ constant.

Eqn (2) (assuming $T$ independent of $t$):

$$(T U_x)_x = \rho u_{tt}$$

$$u_{tt} = c^2 U_{xx}$$

$c = \sqrt{T/\rho}$
VIBRATIONS OR A DRUM HEAD

Consider region D, horizontal components of Newton's law say tension constant to evaluate vertical components.

Do the following integral:

\[
F = \int_D T \delta u \, d\sigma = \iint_D \rho u_{tt} \, dx \, dy = ma
\]

\[
\frac{\delta u}{\delta n} = n \cdot \nabla u
\]

By Greens theorem:

\[
\iint_D \nabla \cdot (T \nabla u) \, dx \, dy = \iiint_D \rho u_{tt} \, dx \, dy
\]

\[
u_{tt} = c^2 \nabla \cdot \nabla u = \Delta u = c^2(u_{xx} + u_{yy})
\]

DIFFUSION THINK OF CONCENTRATION OF DYE IN PIPE END WITH NO FLOW BUT THERMAL DIFFUSION

\[
M(t) = \int_{x_0}^{x_1} u(x,t) \, dx
\]

\[
\frac{\partial M}{\partial t} = \int_{x_0}^{x_1} \frac{\partial u}{\partial t} \, dx
\]

\[
\int_{x_0}^{x_1} \frac{\partial u}{\partial t} = \frac{\partial M}{\partial t} = \text{flow in} - \text{flow out} = kU_x(x_1,t) - kU_x(x_0,t)
\]

\[
\text{Diff} u, u_{xx}, x_1
\]

\[
u_x = ku_{xx}
\]

\[
\int_D u_t \, dx 
\]

\[
\iint_D u_t \, dx \, dy = \iiint_D k(u \cdot \nabla u) \, d\sigma = \iiint_D \rho \, k(u \cdot \nabla u) \, d\sigma = k \iint_D uu_t \, d\sigma
\]

\[
u_t = k \Delta u
\]

Read rest of chapter 1.3