1.3.1 Consider forces on the interval \((x_0, x_1)\). The situation is the same as what is derived in the book however we want to add an additional drag force that will act in the opposite direction of the motion of the string. Since we are assuming the motion is only transverse, this implies the forces for the longitudinal component are unchanged, and still imply a constant tension magnitude \(T\). In the transverse direction we have

\[
\frac{T u_x}{\sqrt{1 + u_x^2}} \bigg|_{x_0}^{x_1} + \int_{x_0}^{x_1} (-k) u_t dx = \int_{x_0}^{x_1} \rho u_{tt} dx,
\]

where \(k\) is a positive constant characterizing the strength of the damping. The negative sign indicates that the direction of the drag force opposes the direction of motion. Again we assume that \(u_x \ll 1\) and use a Taylor expansion. We get,

\[
T u_x \bigg|_{x_0}^{x_1} - \int_{x_0}^{x_1} ku_t dx = \int_{x_0}^{x_1} \rho u_{tt} dx.
\]

We now take a derivative with respect to \(x_1\) to get,

\[
Tu_{xx} - ku_t = \rho u_{tt}.
\]

1.3.2 First we calculate the magnitude of the tension at each point in the chain. Recalling that the length of a curve \(f(x)\) on some interval \((a,b)\) is given by \(L_{a,b} = \int_a^b \sqrt{1 + (f'(s))^2} ds\) we get,

\[
T(x) = lpg - \rho \int_0^x \sqrt{1 + u_x^2(s,t)} ds.
\]

Above, we express the tension as the total weight of the chain minus the length that is above a given position \(x\). Here \(g\) is the acceleration due to gravity. Again we assume small displacements from equilibrium \((u_x \ll 1)\) and expand in \(u_x\),
\[ T(x) = l\rho g - \rho g \int_0^x \left( 1 + O\left(u_x^2\right) \right) \, dx \]
\[ \sim \rho g \left(l - x\right). \]

We set up the equation for the transverse forces in the same way we as was done in the book and lecture, except we will now be conscious of the fact that the tension is no longer constant.

\[ \frac{T(x)u_x}{\sqrt{1 + u_x^2}} \bigg|_{x_0}^{x_1} = \int_{x_0}^{x_1} \rho u_t \, dx \]
\[ g\rho(l - x)u_x \bigg|_{x_0}^{x_1} = \int_{x_0}^{x_1} \rho u_t \, dx \]

Again we differentiate with respect to \(x_1\) getting,

\[ g \frac{\partial}{\partial x} \left((l - x) \, u_x\right) = u_{tt}. \]

1.3.6 As derived in your text the heat equation in three dimensions is given by,

\[ u_t = k\Delta u, \]

where in Cartesian coordinates the Laplacian is given by,

\[ \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \]

We wish to re-express this equation as an equation in cylindrical coordinates. However our job is simplified by the fact that by assumption the heat distribution does not depend on \(\theta\) and \(z\), but rather only on \(r\). Hence we can write,

\[ u(t, x, y, z) = U(t, r(x, y)). \]
Let us compute the derivatives in the new variable using chain rule,

\[ \frac{\partial U}{\partial x} = U_r \frac{\partial r}{\partial x} = U_r \frac{x}{\sqrt{x^2 + y^2}} \]

\[ \frac{\partial^2 U}{\partial x^2} = \frac{\partial}{\partial x} \left( U_r \frac{x}{\sqrt{x^2 + y^2}} \right) = U_{rr} \frac{x^2}{x^2 + y^2} + \left( -\frac{x^2}{(x^2 + y^2)^{3/2}} + \frac{1}{\sqrt{x^2 + y^2}} \right) U_r \]

\[ \frac{\partial^2 U}{\partial y^2} = U_{rr} \frac{y^2}{x^2 + y^2} + \left( -\frac{y^2}{(x^2 + y^2)^{3/2}} + \frac{1}{\sqrt{x^2 + y^2}} \right) U_r \]

\[ \frac{\partial^2 U}{\partial z^2} = 0. \]

Hence adding the two expressions above and replacing instances using \( r^2 = x^2 + y^2 \) we get,

\[ U_t = k \left( U_{rr} + \frac{1}{r} U_r \right). \]

1.3.7 We proceed in the same way as the previous problem hence steps will be listed without detailed explanation.

\[ u_t = k \Delta u \]
\[ u(x, y, z) = U(r(x, y, z)) \]

\[ \frac{\partial U}{\partial x} = U_r \frac{\partial r}{\partial x} = U_r \frac{x}{\sqrt{x^2 + y^2 + z^2}} \]

\[ \frac{\partial^2 U}{\partial x^2} = \frac{\partial}{\partial x} \left( U_r \frac{x}{\sqrt{x^2 + y^2 + z^2}} \right) = U_{rr} \frac{x^2}{r^2} U_r + \left( -\frac{x^2}{r^3} + \frac{1}{r} \right) U_r \]

\[ \frac{\partial^2 U}{\partial y^2} = \frac{y^2}{r^2} U_{rr} + \left( -\frac{y^2}{r^3} + \frac{1}{r} \right) U_r \]

\[ \frac{\partial^2 U}{\partial z^2} = \frac{z^2}{r^2} U_{rr} + \left( -\frac{z^2}{r^3} + \frac{1}{r} \right) U_r \]
\[ U_t = k \left( \frac{x^2 + y^2 + z^2}{r^2} U_{rr} + \left( \frac{-x^2 - y^2 - z^2}{r^3} + \frac{3}{r} \right) U_r \right) \]
\[ = k \left( U_{rr} + \frac{2}{r} U_r \right) \]

1.3.9 First we manually evaluate the left hand side:

\[ \nabla \cdot \mathbf{F} = \text{grad}(x^2 + y^2 + z^2) \cdot (xi + yj + zk) + (x^2 + y^2 + z^2) \text{div} (xi + yj + zk) \]
\[ = 2(xi + yj + zk) \cdot (xi + yj + zk) + 3(x^2 + y^2 + z^2) \]
\[ = 5(x^2 + y^2 + z^2) \]
\[ = 5r^2. \]

Hence we get,

\[ \int \int \int_D \nabla \cdot \mathbf{F} \, dx = \int_0^a 5r^2(4\pi r^2) \, dr \]
\[ = 4\pi a^5, \]

where above we have done the angle integrals in the spherical coordinate system first, obtaining the surface area of the sphere $4\pi r^2$. We now evaluate the right hand side. On $\partial D$, the unit normal vector is $\mathbf{n} = \mathbf{x}/|x| = \mathbf{x}/r$,

\[ \mathbf{F} \cdot \mathbf{n} = r^2\mathbf{x} \cdot \frac{\mathbf{x}}{r} \]
\[ = r^3 \]
\[ \int \int_{\partial D} \mathbf{F} \cdot \mathbf{n} \, dS = \int \int_{\partial D} a^3 dS_a \]
\[ = a^3 \int \int_{\partial D} dS_a \]
\[ = a^3(4\pi a^2) \]
\[ = 4\pi a^5 \]

1.3.10 Take the $D$ in the divergence theorem to be a ball of radius $a$,

\[ \left| \int \int \int_{B_a} \nabla \cdot \mathbf{f} \, dV \right| = \left| \int \int_{\partial B_a} \mathbf{f} \cdot \mathbf{n} \, dS_a \right| \leq \int \int_{\partial B_a} |\mathbf{f} \cdot \mathbf{n}| \, dS_a \]
\[ \leq \int \int_{\partial B_a} |\mathbf{f}| |\mathbf{n}| dS_a \]
\[ \leq \int \int_{\partial B_a} |\mathbf{f}| dS_a \]
\[ \leq \int \int_{\partial B_a} \frac{1}{a^3 + 1} dS_a \]
\[
\frac{4\pi a^2}{a^3 + 1}
\]

Now we take the limit as \( a \to \infty \),
\[
\lim_{a \to \infty} \frac{4\pi a^2}{a^3 + 1} = \lim_{a \to \infty} \frac{8\pi a}{3a^2} = \lim_{a \to \infty} \frac{8\pi}{3a} = 0.
\]

1.4.6  a. The mathematical problem for heat flow in these rods is given by

\[
\frac{\partial u_1}{\partial t} = k_1 \frac{\partial^2 u_1}{\partial x^2}, \quad x \in (-L_1, 0)
\]
\[
\frac{\partial u_2}{\partial t} = k_2 \frac{\partial^2 u_2}{\partial x^2}, \quad x \in (0, L_2)
\]
\[
u_1(-L_1, t) = 0 \\
u_2(L_2, t) = T \\
u_1(0, t) = u_2(0, t)
\]
\[
k_1 \left. \frac{\partial u_1}{\partial x} \right|_{x=0} = k_2 \left. \frac{\partial u_2}{\partial x} \right|_{x=0}.
\]

We want to analyse the steady state problem hence we set the time derivatives equal to zero. The general solution for the two functions \( u_1 \) and \( u_2 \) then becomes,
\[
u_1(x) = c_1 x + c_2 \\
u_2(x) = d_1 x + d_2.
\]

We have four undetermined constants hence we want to find four equations. We use the boundary conditions above and plug in our expressions for \( u_1(x) \) and \( u_2(x) \),
\[
0 = u_1(-L_1) = -L_1 c_1 + c_2 \\
T = u_2(L_2) = L_2 d_1 + d_2 \\
0 = u_1(0) - u_2(0) = c_2 - d_2 \\
0 = k_1 u_1'(0) - k_2 u_2'(0) = k_1 c_1 - k_2 d_1.
\]
We write them down in matrix form:

\[
\begin{pmatrix}
0 \\
T \\
0
\end{pmatrix}
= 
\begin{pmatrix}
-L_1 & 1 & 0 & 0 \\
0 & 0 & L_2 & 1 \\
k_1 & 0 & -k_2 & 0 \\
k_2 & 1 & 0 & -1
\end{pmatrix}
\begin{pmatrix}
c_1 \\
c_2 \\
d_1 \\
d_2
\end{pmatrix}.
\]

We solve this system by for example Cramer’s rule to obtain,

\[
\begin{pmatrix}
c_1 \\
c_2 \\
d_1 \\
d_2
\end{pmatrix}
= 
\frac{1}{k_2 L_1 + k_1 L_2}
\begin{pmatrix}
k_2 T \\
k_2 TL_1 \\
k_1 T \\
k_2 TL_1
\end{pmatrix}.
\]

And hence the final solution is given by

\[
u(x) = \begin{cases} 
k_2 T (L_1 + x) + k_1 L_2, & x \leq 0, \\
\frac{k_2 TL_1}{k_2 L_1 + k_1 L_2}, & x > 0,
\end{cases}
\]

b. Plugging in the values in the question we get,

\[
u(x) = \begin{cases} 
\frac{10(3+x)}{7}, & x \leq 0, \\
\frac{10}{7}(3 + 2x), & x > 0
\end{cases}
\]

Note that this is different than the back of the book simply because here the origin of the coordinate system was defined as the weld point, whereas the book used the left end of the left rod. We could recover the books answer by shifting the coordinate system to a new variable \(\xi = x + 3\).
1.6.1 a.

\[ 0 = u_{xx} - u_{xy} + 2u_y + u_{yy} - 3u_{yx} + 4u \]

“Lower Order Terms" = \[ u_{xx} + 2(-2)u_{xy} + u_{yy} \]

\[ a_{11} = 1 \]
\[ a_{12} = -2 \]
\[ a_{22} = 1 \]

\[ D = a_{12}^2 - a_{11}a_{22} = (-2)^2 - (1)(1) = 3 \]

So we have \( D > 0 \) hence the equation is Hyperbolic.

b. We proceed in the same manner.

\[ D = 3^2 - (9)(1) = 0 \]

Hence equation is Parabolic.

1.6.2 First lets figure out what the coefficients \( a_{11} \), etc. are,

\[ a_{11}(x, y) = 1 + x \]
\[ a_{12}(x, y) = xy \]
\[ a_{22}(x, y) = -y^2 \]

\[ D(x, y) = x^2y^2 + (1 + x)y^2. \]

First consider where the equation is Parabolic.

\[ 0 = D(x, y) \]
\[ 0 = y^2(x^2 + (1 + x)) \]

You can check that the terms in parentheses have no solutions for \( x \in \mathbb{R} \). Therefore the only solution is \( y = 0 \). On this region the equation is Parabolic. Both above and below this line \( D > 0 \) and hence everywhere except \( y = 0 \) the equation is Hyperbolic.
1.6.5 We make the substitution the text suggests.

\[ u = v e^{\alpha x + \beta y} \]
\[ 0 = u_{xx} + 3u_{yy} - 2u_x + 24u_y + 5u \]
\[ 0 = \frac{\partial^2}{\partial x^2} (v e^{\alpha x + \beta y}) + 3 \frac{\partial^2}{\partial y^2} (v e^{\alpha x + \beta y}) - 2 \frac{\partial}{\partial x} (v e^{\alpha x + \beta y}) + 24 \frac{\partial}{\partial y} (v e^{\alpha x + \beta y}) + 5 (v e^{\alpha x + \beta y}) \]
\[ = v_{xx} e^{\alpha x + \beta y} + \alpha^2 v e^{\alpha x + \beta y} + 2\alpha v_x e^{\alpha x + \beta y} + 5v e^{\alpha x + \beta y} \]
\[ - 2 \left( \alpha v e^{\alpha x + \beta y} + v_x e^{\alpha x + \beta y} \right) + 24 \left( \beta v e^{\alpha x + \beta y} + v_y e^{\alpha x + \beta y} \right) \]
\[ + 3 \left( \beta^2 v e^{\alpha x + \beta y} + 2\beta v y e^{\alpha x + \beta y} + v_{yy} e^{\alpha x + \beta y} \right) \]
\[ 0 = v (\alpha^2 - 2\alpha^2 + 3\beta^2 + 24\beta + 5) + (2\alpha - 2)v_x + v_{xx} + (6\beta + 24)v_y + 3v_{yy} \]
\[ 0 = 2\alpha - 2 \]
\[ \alpha = 1 \]
\[ 0 = 6\beta + 24 \]
\[ \beta = -4 \]
\[ 0 = v_{xx} + 3v_{yy} - 44v \]
\[ \hat{y} = \frac{1}{\sqrt{3}} y \]
\[ \hat{v}(x, y) = V(x, \hat{y}(y)) \]
\[ \frac{\partial}{\partial y} v = \frac{\partial \hat{v}}{\partial y} \frac{\partial}{\partial \hat{y}} V \]
\[ = \frac{1}{\sqrt{3}} V_y \]
\[
\frac{\partial^2}{\partial y^2} \nu = \frac{1}{3} V_{\dot{y} \dot{y}} \\
0 = V_{xx} + V_{yy} - 44V
\]