Five kinds of maximal monotonicity

by Stephen Simons

Abstract

We discuss Gossez's "type (D)" maximal monotone multifunctions and the newer "type (ED)" subfamily (for which an analog of the Brøndsted–Rockafellar property holds). We then discuss the "locally maximal monotone" (= type (FP)) and "maximal monotone locally" (= type (FPV)) multifunctions of Fitzpatrick–Phelps and Fitzpatrick–Phelps–Verona–Verona. Finally, we discuss the strongly maximal monotone multifunctions. We prove that every maximal monotone multifunction of type (D) is locally maximal monotone, and every maximal monotone multifunction of type (ED) is both maximal monotone locally and strongly maximal monotone.

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0. Introduction

Let E be a real Banach space with dual E^*, and $S\colon\ E \mapsto 2^{E^*}$. Write

$$G(S) := \{(s, s^*)\colon\ s \in E,\ s^* \in Sx\}.$$

We suppose that S is *nontrivial*, that is to say $G(S) \neq \emptyset$. S is said to be *monotone* if

$$(s, s^*) \text{ and } (w, w^*) \in G(S) \quad\Longrightarrow\quad \langle s - w, s^* - w^*\rangle \geq 0.$$

S is said to be *maximal monotone* if S is monotone and, whenever $(w, w^*) \in E \times E^*$ and

$$(s, s^*) \in G(S) \quad\Longrightarrow\quad \langle s - w, s^* - w^*\rangle \geq 0,$$

then

$$(w, w^*) \in G(S).$$

The first two subclasses of the maximal monotone multifunctions that we consider in this paper are the maximal monotone multifunctions of type (D) and (ED). Maximal monotone multifunctions of "type (D)" were introduced by Gossez in order to generalize to nonreflexive spaces some of the results previously known for reflexive spaces (see Gossez, [6], Lemme 2.1, p. 375 and Phelps, [7], Section 3 for an exposition). The maximal monotone multifunctions of "type (ED)", a subclass of Gossez's class, were introduced in [13], (where they were called maximal monotone multifunctions of "type (DS)") and their study was continued in [14], where it was proved that they possess an analog of the Brøndsted–Rockafellar property of subdifferentials. This fact can be used to give an example of a continuous maximal monotone linear operator that is not of type (ED): let $E = \ell^1$, and define $T\colon\ \ell^1 \mapsto \ell^\infty = E^*$ by $(Tx)_n = \sum_{k\geq n} x_k$ (T is the "tail" operator). However, we do not know the solution to the following problem:

Problem 1. Is every maximal monotone multifunction of type (D) necessarily of type (ED)?

The third and fourth subclasses of the maximal monotone multifunctions that we consider in this paper are the maximal monotone multifunctions of type (FP) and type (FPV). Maximal monotone multifunctions of "type (FP)" were introduced by Fitzpatrick–Phelps in [4], Section 3 under the name of "locally maximal monotone" multifunctions. The motivation for their introduction was as follows. If E is reflexive then every maximal monotone operator on E can be approximated by "nicer" maximal monotone operators using the Moreau-Yosida approximation. If E is nonreflexive then every subdifferential can also be approximated by "nicer" subdifferentials by using the operation of inf–convolution. So the question arises whether a general maximal monotone operators on a nonreflexive space can also be approximated by "nicer" maximal monotone operators in some appropriate sense. Fitzpatrick–Phelps defined an appropriate sense of approximation in [4], and showed that the multifunctions of type (FP) can be approximated by "nicer" maximal monotone operators in their sense. Maximal monotone multifunctions of "type (FPV)" were introduced independently by Fitzpatrick–Phelps and Verona–Verona in [5], p. 65 and [15], p. 268 by dualizing the definition of "type (FP)". We shall see in Section 2 that most of the "common" maximal monotone multifunctions are in both classes. While the tail operator is (using results from [1]) not of type (FP), we do not know the solution to the following problem:

Problem 2. Is every maximal monotone multifunction of type (FPV)?

By way of introduction to the fifth subfamily of the maximal monotone operators that we wish to discuss, let $f\colon E \mapsto \mathbb{R}\cup\{\infty\}$ be a proper, convex lower semicontinuous function, and consider whether a given point $w \in E$ is a global minimizer of f. This is equivalent to the assertion that $0 \in \partial f(w)$, where $\partial f\colon E \mapsto 2^{E^*}$ is the associated subdifferential mapping. Since (as was proved by Rockafellar in [10]) ∂f is maximal monotone, it suffices that

$$(s, s^*) \in G(\partial f) \quad \Longrightarrow \quad \langle s - w, s^*\rangle \geq 0,$$

which is formally a much weaker condition. Now suppose that C is a nonempty $w(E, E^*)$–compact convex subset of E, and consider whether f attains a global minimum at some point of C. This is equivalent to the assertion that $(\partial f)^{-1}0 \cap C \neq \emptyset$. An analog of the above sufficient condition would be that

$$(s, s^*) \in G(\partial f) \quad \Longrightarrow \quad \text{there exists } w \in C \text{ such that } \langle s - w, s^*\rangle \geq 0.$$

In fact, this condition is sufficient for $(\partial f)^{-1}0\cap C \neq \emptyset$. This follows because ∂f is a *strongly maximal monotone multifunction* in the sense that we will make specific in Section 3.

We shall see in Section 3 that the "common" maximal monotone multifunctions are strongly maximal monotone. However, we do not know the solution to the following problem:

Problem 3. Is every maximal monotone multifunction strongly maximal monotone?

The main results of this paper are then (in Theorem 15) that *every maximal monotone multifunction of type (ED) is necessarily strongly maximal monotone*, (in Theorem 17) that *every maximal monotone multifunction of type (D) is necessarily of type (FP)*, and (in Theorem 20) that *every maximal monotone multifunction of type (ED) is necessarily of type (FPV)*.

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1. Maximal monotone multifunctions of types (D) and (ED)

We start off by defining a new topology on the bidual, E^{**}, of E.

Definition 4. We write $\mathcal{CLB}(E)$ for the set of all real convex functions on E that are Lipschitz on the bounded subsets of E, or equivalently bounded on the bounded subsets of E. (See [14], Definition 2.7, p. 262.) We define the topology $\mathcal{T}_{\mathcal{CLB}}(E^{**})$ on E^{**} to be the coarsest topology on E^{**} making all the functions $f^{**}\colon E^{**} \mapsto \mathbb{R}$ $(f \in \mathcal{CLB}(E))$ continuous. (See [14], Section 3.) We also write $\mathcal{T}_{\|\ \|}$ for "norm topology of".

We will give more properties of the topology $\mathcal{T}_{\mathcal{CLB}}(E^{**})$ in Lemma 13.

In order to define maximal monotone multifunctions of types (D) and (ED), we must introduce a concept due to Gossez: if $S\colon E \mapsto 2^{E^*}$, we define the multifunction $\overline{S}\colon E^{**} \mapsto 2^{E^*}$ by:

$$x^* \in \overline{S}x^{**} \quad \iff \quad \inf_{(s,s^*)\in G(S)} \langle s^* - x^*, \widehat{s} - x^{**}\rangle \geq 0,$$

where $\widehat{s}$ is the canonical image of s in E^{**}.

In what follows, $R(S) := \bigcup_{x\in E} Sx$.

Definition 5. Let $S\colon E \mapsto 2^{E^*}$ be maximal monotone.

S is *of type (D)* if, for all $(x^{**}, x^*) \in G(\overline{S})$, there exists a bounded net $\{(s_\gamma, s_\gamma^*)\}$ of elements of $G(S)$ such that $(\widehat{s_\gamma}, s_\gamma^*) \to (x^{**}, x^*)$ in $w(E^{**}, E^*) \times \mathcal{T}_{\|\ \|}(E^*)$.

S is *of type (ED)* if, for all $(x^{**}, x^*) \in G(\overline{S})$, there exists a net $\{(s_\gamma, s_\gamma^*)\}$ of elements of $G(S)$ such that $(\widehat{s_\gamma}, s_\gamma^*) \to (x^{**}, x^*)$ in $\mathcal{T}_{\mathcal{CLB}}(E^{**}) \times \mathcal{T}_{\|\ \|}(E^*)$.

It is easy to see that
• *every maximal monotone multifunction of type (ED) is of type (D)*
and
• *if E is reflexive then every maximal monotone multifunction $S\colon E \mapsto 2^{E^*}$ is of type (ED).*
It was essentially proved by Gossez in [6] (see Phelps, [7], Theorem 3.8, p. 221 for an exposition) that
• *if S is maximal monotone of type (D) then $\overline{R(S)}$ is convex,*
it was proved in [13], Theorem 35.3, p. 139 and [14], Theorem 12.6(b), p. 287 that
• *if $f\colon E \mapsto \mathbb{R} \cup \{\infty\}$ is proper, convex and lower semicontinuous then $\partial f\colon E \mapsto 2^{E^*}$ is maximal monotone of type (ED),*
and it was proved in [14], Theorem 11.1, p. 282 that
• *if $D(T)$ is a subspace of E and $T\colon D(T) \mapsto E^*$ is a maximal monotone linear operator of type (D) then T is of type (ED).*
Thus, even though the "tail operator" is not of type (ED), it is still true that many common maximal monotone multifunctions are.

2. Maximal monotone multifunctions of type (FP) and (FPV)

Definition 6. A monotone multifunction S is *of type (FP)* or *locally maximal monotone* provided the following holds: for any open convex subset U of E^* such that $U \cap R(S) \neq \emptyset$, if $(v, v^*) \in E \times U$ is such that

$$(s, s^*) \in G(S) \text{ and } s^* \in U \quad \Longrightarrow \quad \langle s - v, s^* - v^*\rangle \geq 0$$

then $(v, v^*) \in G(S)$. (If we take $U = E^*$, we see that every multifunction of type (FP) is maximal monotone.)

It was proved in [5], Theorem 3.7, p. 67 that
• *if S is maximal monotone and $R(S) = E^*$ then S is of type (FP),*
in [4], Theorem 3.5, p. 585 that
• *if S is maximal monotone of type (FP) then $\overline{R(S)}$ is convex,*
in [11], Main theorem, p. 470 and [13], Theorem 30.3, p. 120 that
• *if $f\colon\ E \mapsto \mathbb{R} \cup \{\infty\}$ is proper, convex and lower semicontinuous then $\partial f\colon E \mapsto 2^{E^*}$ is maximal monotone of type (FP),*
in [1], Theorem 4.1 (see also [8], Theorem 8.1, p. 327) that
• *if $T\colon\ E \mapsto E^*$ is a continuous maximal monotone linear operator then T is of type (D) $\iff$ T is of type (FP),*
in [8], Theorem 6.7, p. 320 that
• *if $D(T)$ is a subspace of E and $T\colon\ D(T) \mapsto E^*$ is a maximal monotone linear operator of type (D) then T is of type (FP),*
and, finally, in [4], Proposition 3.3, p. 585 that
• *if E is reflexive then every maximal monotone multifunction $S\colon\ E \mapsto 2^{E^*}$ is of type (FP).*

The parallels between the results known for multifunctions of type (D) and type (FP) will be clarified by Theorem 17.

In what follows, $D(S) := \{x \in E\colon\ Sx \neq \emptyset\}$.

Definition 7. A monotone multifunction S is *of type (FPV)* or *maximal monotone locally* provided the following holds: for any open convex subset U of E such that $U \cap D(S) \neq \emptyset$, if $(v, v^*) \in U \times E^*$ is such that

$$(s, s^*) \in G(S) \text{ and } s \in U \quad \Longrightarrow \quad \langle s - v, s^* - v^* \rangle \geq 0$$

then $(v, v^*) \in G(S)$. (If we take $U = E$, we see that every multifunction of type (FPV) is maximal monotone.)

It was proved in [5], Theorem 3.10, p. 68 that
• *if S is maximal monotone and $D(S) = E$ then S is of type (FPV),*
in [13], Theorem 26.3, p. 103 that
• *if S is maximal monotone of type (FPV) then $\overline{D(S)}$ is convex,*
in [5], Corollary 3.4, p. 66 and [15], Theorem 3, p. 269 (see also [13], Theorem 31.3, p. 122) that
• *if $f\colon\ E \mapsto \mathbb{R} \cup \{\infty\}$ is proper, convex and lower semicontinuous then $\partial f\colon E \mapsto 2^{E^*}$ is maximal monotone of type (FPV),*
in [13], Theorem 38.2, p. 146 that
• *if $D(T)$ is a subspace of E and $T\colon\ D(T) \mapsto E^*$ is* any *maximal monotone linear operator then T is of type (FPV),*
and, finally, in [4], Proposition 3.3, p. 585 that
• *if E is reflexive then every maximal monotone multifunction $S\colon\ E \mapsto 2^{E^*}$ is of type (FPV).*

3. Strongly maximal monotone multifunctions

Definition 8. Let $S\colon E \mapsto 2^{E^*}$ be monotone. We say that S is *strongly maximal monotone* if, whenever C is a nonempty $w(E^*,E)$–compact convex subset of E^* and $w \in E$ are such that

$$(s,s^*) \in G(S) \quad \Longrightarrow \quad \text{there exists } w^* \in C \text{ such that } \quad \langle s-w, s^*-w^* \rangle \geq 0$$

then

$$Sw \cap C \neq \emptyset \tag{8.1}$$

and, further, whenever C, a nonempty $w(E,E^*)$–compact convex subset of E, and $w^* \in E^*$ are such that

$$(s,s^*) \in G(S) \quad \Longrightarrow \quad \text{there exists } w \in C \text{ such that } \quad \langle s-w, s^*-w^* \rangle \geq 0$$

then

$$S^{-1}w^* \cap C \neq \emptyset. \tag{8.2}$$

It is clear (by taking C to be a singleton) that every strongly maximal monotone multifunction is maximal monotone.

It was proved in [12] Theorems 6.1 and 6.2, p. 1386 and, in a different way, in [13], Theorem 32.5, p. 128 that

• *if $f\colon E \mapsto \mathrm{I\!R} \cup \{\infty\}$ is proper, convex and lower semicontinuous then ∂f is strongly maximal monotone,*

in [2], Theorem 1.1, p. 166 that

• *if $D(T)$ is a subspace of E and $T\colon D(T) \mapsto E^*$ is linear and maximal monotone then T is strongly maximal monotone,*

and, finally, in [9], Theorem 3.2, p. 151 that

• *if E is reflexive then every maximal monotone multifunction from E into 2^{E^*} is strongly maximal monotone.*

Thus, as stated in the introduction, all the "common" maximal monotone multifunctions are strongly maximal monotone.

4. Preliminary results

We shall use the following classical minimax theorem. It follows from a result of Fan — see [3], or [13], Theorem 3.1, p. 17. It is important that the set A not be required to have any topological structure.

Lemma 9. *Let A be a nonempty convex subset of a vector space, B be a nonempty convex subset of a vector space and B also be a compact Hausdorff topological space. Let $h\colon A \times B \mapsto \mathrm{I\!R}$ be convex on A, and concave and upper semicontinuous on B. Then*

$$\inf_A \max_B h = \max_B \inf_A h.$$

Lemma 10 is an existence theorem for bounded linear functionals that does not assume the existence of any *a priori* bounds — its proof can be found in [13], Theorem 7.2, p. 27 or [14], Theorem 5.2, p. 269. Lemma 10 has other applications to the theory of multifunctions (see [13], Lemma 20.1, p. 77).

Lemma 10. *Let A be a nonempty convex subset of a vector space, F be a Banach space, $f\colon A \mapsto \mathbb{R}$ be convex and $g\colon A \mapsto F$ be affine. Then* (10.1)$\iff$(10.2).

$$a \in A \quad \Longrightarrow \quad f(a) + \|g(a)\|^2 \geq 0. \tag{10.1}$$

$$\left.\begin{array}{c}\textit{There exists } Z^* \in F^* \textit{ such that}\\ a \in A \quad \Longrightarrow \quad f(a) - 2\langle g(a), Z^*\rangle \geq \|Z^*\|^2.\end{array}\right\} \tag{10.2}$$

The construction described in Lemma 11 is explained in more detail in [13], Lemma 9.1, p. 33 or [14], Lemma 6.1, p. 272 (and the preceding text).

Lemma 11. *Let $T\colon E \mapsto 2^{E^*}$ be nontrivial. Then T is monotone if, and only if, there exist a vector space V, linear operators $p\colon V \mapsto E$, $q\colon V \mapsto E^*$ and $r\colon V \mapsto \mathbb{R}$, and a map $\delta\colon E \times E^* \mapsto V$ such that*

$$(y, y^*) \in E \times E^*, \quad \Longrightarrow \quad p(\delta(y, y^*)) = y, \quad q(\delta(y, y^*)) = y^* \quad \text{and} \quad r(\delta(y, y^*)) = \langle y, y^*\rangle$$

and, writing $\mathcal{CO}(T)$ for the convex hull of $\delta(G(T))$,

$$\mu \in \mathcal{CO}(T) \quad \Longrightarrow \quad r(\mu) \geq \langle p(\mu), q(\mu)\rangle. \tag{11.1}$$

5. Type (ED) implies strong maximal monotonicity

We prove two preliminary lemmas before establishing the main result of this section, Theorem 15. Lemma 12 will be used not only in Theorem 15, but is also a model for Lemmas 16 and 19.

Lemma 12. *Let $T\colon E \mapsto 2^{E^*}$ be nontrivial and monotone.*
(a) Let B be a nonempty $w(E^*, E)$–compact convex subset of E^*. Then there exist $w^* \in E^*$, $y^* \in B$ and $z^{**} \in E^{**}$ such that, for all $(t, t^*) \in G(T)$ and $u^* \in B$,

$$2\langle t, t^*\rangle - 2\langle t, w^*\rangle - 2\langle t^* + u^*, z^{**}\rangle \geq \|w^* + y^*\|^2 + \|z^{**}\|^2. \tag{12.1}$$

(b) Let B be a nonempty $w(E^*, E)$–compact convex subset of E^*. Then there exist $w^* \in E^*$, $y^* \in B$ and $z^{**} \in E^{**}$ such that, for all $(t, t^*) \in G(T)$ and $u^* \in B$,

$$2\langle t^* - w^*, \widehat{t} - z^{**}\rangle + 2\langle y^* - u^*, z^{**}\rangle \geq \|w^* + y^*\|^2 + 2\langle w^* + y^*, z^{**}\rangle + \|z^{**}\|^2 \geq 0. \tag{12.2}$$

(c) Let B be a nonempty $w(E, E^*)$–compact convex subset of E. Then there exist $w^{**} \in E^{**}$, $y \in B$ and $z^* \in E^*$ such that, for all $(t, t^*) \in G(T)$ and $u \in B$,

$$2\langle t^* - z^*, \widehat{t} - w^{**}\rangle + 2\langle y - u, z^*\rangle \geq \|z^*\|^2 + 2\langle z^*, w^{**} + \widehat{y}\rangle + \|w^{**} + \widehat{y}\|^2 \geq 0. \tag{12.3}$$

Proof. Let p, q, r and $\mathcal{CO}(T)$ be as in Lemma 11.

(a) For all $u \in E$, let $\varphi(u) := \max\langle u, B\rangle$. Let $F := E \times E^*$ with $\|(x, x^*)\| := \sqrt{\|x\|^2 + \|x^*\|^2}$, $A := \mathcal{CO}(T) \times B$ and, for all $(\mu, u^*) \in A$,

$$f(\mu, u^*) := 2r(\mu) + 2\varphi(p(\mu)) \in \mathbb{R} \quad \text{and} \quad g(\mu, u^*) := (p(\mu), q(\mu) + u^*) \in F.$$

Then, using (11.1), for all $(\mu, u^*) \in A$,

$$\begin{aligned} f(\mu, u^*) + \|g(\mu, u^*)\|^2 &= 2r(\mu) + 2\varphi(p(\mu)) + \|p(\mu)\|^2 + \|q(\mu) + u^*\|^2 \\ &\geq 2r(\mu) + 2\varphi(p(\mu)) - 2\langle p(\mu), q(\mu) + u^*\rangle \\ &\geq 2r(\mu) - 2\langle p(\mu), q(\mu)\rangle + 2\varphi(p(\mu)) - 2\langle p(\mu), u^*\rangle \geq 0. \end{aligned}$$

Thus, from Lemma 10, there exists $Z^* \in F^*$ such that:

$$(\mu, u^*) \in A \quad \Longrightarrow \quad f(\mu, u^*) - 2\langle g(\mu, u^*), Z^*\rangle \geq \|Z^*\|^2.$$

Now there exists $(z^*, z^{**}) \in E^* \times E^{**}$ such that $Z^* = (z^*, z^{**})$ and

$$\|Z^*\| = \sqrt{\|z^*\|^2 + \|z^{**}\|^2}.$$

It follows that

$$(\mu, u^*) \in A \quad \Longrightarrow \quad 2r(\mu) + 2\varphi(p(\mu)) - 2\langle p(\mu), z^*\rangle - 2\langle q(\mu) + u^*, z^{**}\rangle \geq \|z^*\|^2 + \|z^{**}\|^2.$$

We now define h: $A \times E^* \mapsto \mathbb{R}$ by

$$h((\mu, u^*), y^*) := 2r(\mu) + 2\langle p(\mu), y^*\rangle - 2\langle p(\mu), z^*\rangle - 2\langle q(\mu) + u^*, z^{**}\rangle.$$

Then the implication above is simply that

$$\inf_A \max_B h \geq \|z^*\|^2 + \|z^{**}\|^2.$$

The sets A and B are convex, and the function h is affine on A, and affine and $w(E^*, E)$-continuous on B. Thus, from Lemma 9,

$$\max_B \inf_A h \geq \|z^*\|^2 + \|z^{**}\|^2,$$

and so there exists $y^* \in B$ such that:

$$(\mu, u^*) \in A \Longrightarrow \quad 2r(\mu) + 2\langle p(\mu), y^*\rangle - 2\langle p(\mu), z^*\rangle - 2\langle q(\mu) + u^*, z^{**}\rangle \geq \|z^*\|^2 + \|z^{**}\|^2.$$

Making the substitution $w^* = z^* - y^* \in E^*$, we have:

$$(\mu, u^*) \in A \quad \Longrightarrow \quad 2r(\mu) - 2\langle p(\mu), w^*\rangle - 2\langle q(\mu) + u^*, z^{**}\rangle \geq \|w^* + y^*\|^2 + \|z^{**}\|^2.$$

We now obtain (a) by taking $\mu = \delta(t, t^*)$, where δ is as in Lemma 11.

(b) We obtain the first inequality in (12.2) by adding $2\langle w^* + y^*, z^{**}\rangle$ to both sides of (12.1) and rearranging the terms. The second inequality follows since

$$\begin{aligned} \|w^* + y^*\|^2 + 2\langle w^* + y^*, z^{**}\rangle + \|z^{**}\|^2 &\geq \|w^* + y^*\|^2 - 2\|w^* + y^*\|\|z^{**}\| + \|z^{**}\|^2 \\ &= (\|w^* + y^*\| - \|z^{**}\|)^2 \geq 0 \end{aligned}$$

This completes the proof of (b).

(c) We apply (a) with E replaced by E^*, T replaced by the monotone multifunction $\widehat{T^{-1}}$: $E^* \mapsto E^{**}$ and B replaced by $\widehat{B} \subset E^{**}$, and obtain $w^{**} \in E^{**}$, $y^{**} \in \widehat{B}$ and $z^{***} \in E^{***}$ such that, for all $(t^*, t^{**}) \in G(\widehat{T^{-1}})$ and $u^{**} \in \widehat{B}$,

$$2\langle t^*, t^{**}\rangle - 2\langle t^*, w^{**}\rangle - 2\langle t^{**} + u^{**}, z^{***}\rangle \geq \|w^{**} + y^{**}\|^2 + \|z^{***}\|^2.$$

Equivalently, $w^{**} \in E^{**}$, $y \in B$, $z^{***} \in E^{***}$ and, for all $(t, t^*) \in G(T)$ and $u \in B$,

$$2\langle t, t^*\rangle - 2\langle t^*, w^{**}\rangle - 2\langle \widehat{t} + \widehat{u}, z^{***}\rangle \geq \|w^{**} + \widehat{y}\|^2 + \|z^{***}\|^2.$$

Now let z^* be the image of z^{***} under the adjoint of the canonical map $\widehat{\ }$ (in other words, the "restriction" of z^{***} to E). Then $w^{**} \in E^{**}$, $y \in B$, $z^* \in E^*$ and, for all $(t, t^*) \in G(T)$ and $u \in B$,

$$2\langle t, t^*\rangle - 2\langle t^*, w^{**}\rangle - 2\langle t + u, z^*\rangle \geq \|w^{**} + \widehat{y}\|^2 + \|z^{***}\|^2 \geq \|w^{**} + \widehat{y}\|^2 + \|z^*\|^2.$$

The first inequality in (12.3) now follows by adding $2\langle z^*, w^{**} + \widehat{y}\rangle$ to both sides and rearranging the terms, and the second inequality follows as in (b). This completes the proof of (c). ▌

We collect together in the following lemma the main properties of $\mathcal{T}_{\mathcal{CLB}}(E^{**})$ that we shall need in Theorems 15 and 20.

Lemma 13. *Let $\{(t_\gamma^{**}, t_\gamma^*)\}$ be a net of elements of $E^{**} \times E^*$ such that $(t_\gamma^{**}, t_\gamma^*) \to (x^{**}, x^*)$ in $\mathcal{T}_{\mathcal{CLB}}(E^{**}) \times \mathcal{T}_{\|\ \|}(E^*)$. Then*
(a) $\langle t_\gamma^*, t_\gamma^{**}\rangle \to \langle x^*, x^{**}\rangle$.
(b) $\langle t_\gamma^* - x^*, t_\gamma^{**} - x^{**}\rangle \to 0$.
(c) If B is a nonempty $w(E^*, E)$–compact convex subset of E^* then

$$\sup\langle B, t_\gamma^{**}\rangle \to \sup\langle B, x^{**}\rangle. \tag{13.1}$$

(d) If B is a nonempty $w(E, E^*)$–compact convex subset of E then

$$\sup\langle B, t_\gamma^*\rangle \to \sup\langle B, x^*\rangle.$$

(e) If $x \in E$ then $\|t_\gamma^{**} - \widehat{x}\| \to \|x^{**} - \widehat{x}\|$.

Proof. (a) was proved in [14], Lemma 3.1(e), p. 263. In order to establish (b), we note from (a) and [14], Lemma 3.1(a), p. 263 that

$$\begin{aligned}\langle t_\gamma^* - x^*, t_\gamma^{**} - x^{**}\rangle &= \langle t_\gamma^*, t_\gamma^{**}\rangle - \langle x^*, t_\gamma^{**}\rangle - \langle t_\gamma^*, x^{**}\rangle + \langle x^*, x^{**}\rangle \\ &\to \langle x^*, x^{**}\rangle - \langle x^*, x^{**}\rangle - \langle x^*, x^{**}\rangle + \langle x^*, x^{**}\rangle = 0.\end{aligned}$$

Now define φ: $E \mapsto \mathbb{R}$ by $\varphi(x) := \max\langle x, B\rangle$. Then $\varphi \in \mathcal{CLB}(E)$ and, for all $x^{**} \in E^{**}$, $\varphi^{**}(x^{**}) = \sup\langle B, x^{**}\rangle$, so (c) follows from the definition of $\mathcal{T}_{\mathcal{CLB}}(E^{**})$. (d) is immediate since $t_\gamma^* \to x^*$ uniformly on B. Finally, (e) was proved in [14], Lemma 3.1(b), p. 263. ▌

Remark 14. The reason why we need S to be of type (ED) rather than of type (D) in our proof of Theorem 15 below is that (13.1) may fail if we only know that the net $\{t_\gamma^{**}\}$ is bounded and $t_\gamma^{**} \to x^{**}$ in $w(E^{**}, E^*)$. To see this take $E = c_0$, B to be the unit ball of E^*, $t_n^{**} = e_n \in \ell^\infty$ and $x^{**} = 0$.

Theorem 15. *Let $S\colon E \mapsto 2^{E^*}$ be maximal monotone of type (ED). Then S is strongly maximal monotone.*

Proof. We suppose first that C is a nonempty $w(E^*, E)$–compact convex subset of E^*, $w \in E$ and,

$$\text{for all } (s, s^*) \in G(S), \quad \text{there exists } w^* \in C \text{ such that} \quad \langle s - w, s^* - w^* \rangle \geq 0$$

and we will prove that (8.1) is satisfied, that is, $Sw \cap C \neq \emptyset$. Define $T\colon E \mapsto 2^{E^*}$ by $Tz := S(z + w)$. Then,

$$\text{for all } (t, t^*) \in G(T), \quad \text{there exists } w^* \in C \text{ such that} \quad \langle t, t^* - w^* \rangle \geq 0.$$

Let $B := -C$. Then, for all $(t, t^*) \in G(T)$, there exists $u^* \in B$ such that $\langle t, t^* + u^* \rangle \geq 0$, from which

$$(t, t^*) \in G(T) \quad \Longrightarrow \quad \langle t, t^* \rangle + \max\langle t, B \rangle \geq 0. \tag{15.1}$$

From Lemma 12(b), there exist $w^* \in E^*$, $y^* \in B$ and $z^{**} \in E^{**}$ such that, for all $(t, t^*) \in G(T)$ and $u^* \in B$,

$$2\langle t^* - w^*, \widehat{t} - z^{**} \rangle + 2\langle y^* - u^*, z^{**} \rangle \geq \|w^* + y^*\|^2 + 2\langle w^* + y^*, z^{**} \rangle + \|z^{**}\|^2 \geq 0. \tag{12.2}$$

In particular,

$$(t, t^*) \in G(T) \text{ and } u^* \in B \quad \Longrightarrow \quad \langle t^* - w^*, \widehat{t} - z^{**} \rangle + \langle y^* - u^*, z^{**} \rangle \geq 0. \tag{15.2}$$

Since $y^* \in B$, it follows by putting $u^* = y^*$ that

$$\inf_{(t, t^*) \in G(T)} \langle t^* - w^*, \widehat{t} - z^{**} \rangle \geq 0 \tag{15.3}$$

and so $(z^{**}, w^*) \in G(\overline{T})$. Since S is of type (ED), the same is true of T (see [14], Theorem 4.4, p. 268) and so there exists a net $\{(t_\gamma, t_\gamma^*)\}$ of elements of $G(T)$ such that $(\widehat{t_\gamma}, t_\gamma^*) \to (z^{**}, w^*)$ in $\mathcal{T}_{\mathcal{CLB}}(E^{**}) \times \mathcal{T}_{\|\ \|}(E^*)$. From Lemma 13(b), $\langle t_\gamma^* - w^*, \widehat{t_\gamma} - z^{**} \rangle \to 0$, and so, putting $(t, t^*) = (t_\gamma, t_\gamma^*)$ and passing to the limit in (15.2), for all $u^* \in B$, $\langle y^* - u^*, z^{**} \rangle \geq 0$, from which

$$\langle y^*, z^{**} \rangle = \sup\langle B, z^{**} \rangle. \tag{15.4}$$

Using (15.1),

$$\text{for all } \gamma, \quad \langle t_\gamma^*, \widehat{t_\gamma} \rangle + \sup\langle B, \widehat{t_\gamma} \rangle = \langle t_\gamma, t_\gamma^* \rangle + \max\langle t_\gamma, B \rangle \geq 0.$$

Passing to the limit in this and using (a) and (c) of Lemma 13, $\langle w^*, z^{**}\rangle + \sup\langle B, z^{**}\rangle \geq 0$. Combining this with (15.4), we derive that $\langle w^* + y^*, z^{**}\rangle \geq 0$. If we now substitute this into (12.2), we obtain that, for all $(t, t^*) \in G(T)$ and $u^* \in B$,

$$2\langle t^* - w^*, \widehat{t} - z^{**}\rangle + 2\langle y^* - u^*, z^{**}\rangle \geq \|w^* + y^*\|^2 + \|z^{**}\|^2.$$

Going through the same argument as above, that is, setting $u^* = y^*$, $(t, t^*) = (t_\gamma, t^*_\gamma)$ and passing to the limit, we have $\|w^* + y^*\|^2 + \|z^{**}\|^2 \leq 0$, hence $z^{**} = 0$ and $w^* + y^* = 0$. Substituting the first of these equalities into (15.3) yields $\inf_{(t,t^*)\in G(T)}\langle t^* - w^*, \widehat{t} - 0\rangle \geq 0$, that is to say,

$$\inf_{(t,t^*)\in G(T)} \langle t - 0, t^* - w^*\rangle \geq 0.$$

Since T is maximal monotone, this implies that $w^* \in T0$, and so $w^* \in Sw$. Finally, $w^* = -y^*$ and $y^* \in B$, thus $w^* \in C$. This gives (8.1).

Now we suppose that C is a nonempty $w(E, E^*)$–compact convex subset of E, $w^* \in E^*$ and,

$$\text{for all } (s, s^*) \in G(S), \quad \text{there exists } w \in C \text{ such that } \quad \langle s - w, s^* - w^*\rangle \geq 0$$

and we will prove that (8.2) is satisfied, that is, $S^{-1}w^* \cap C \neq \emptyset$. Let $T := S - w^*$. Then,

$$\text{for all } (t, t^*) \in G(T), \quad \text{there exists } w \in C \text{ such that } \quad \langle t - w, t^*\rangle \geq 0.$$

Let $B := -C$. Then, for all $(t, t^*) \in G(T)$, there exists $u \in B$ such that $\langle t + u, t^*\rangle \geq 0$, from which

$$(t, t^*) \in G(T) \quad \Longrightarrow \quad \langle t, t^*\rangle + \max\langle B, t^*\rangle \geq 0. \tag{15.5}$$

From Lemma 12(c), there exist $w^{**} \in E^{**}$, $y \in B$ and $z^* \in E^*$ such that, for all $(t, t^*) \in G(T)$ and $u \in B$,

$$2\langle t^* - z^*, \widehat{t} - w^{**}\rangle + 2\langle y - u, z^*\rangle \geq \|z^*\|^2 + 2\langle z^*, w^{**} + \widehat{y}\rangle + \|w^{**} + \widehat{y}\|^2. \tag{12.3}$$

As in (a),

$$(t, t^*) \in G(T) \text{ and } u \in B \quad \Longrightarrow \quad \langle t^* - z^*, \widehat{t} - w^{**}\rangle + \langle y - u, z^*\rangle \geq 0, \tag{15.6}$$

and then

$$\inf_{(t,t^*)\in G(T)} \langle t^* - z^*, \widehat{t} - w^{**}\rangle \geq 0, \tag{15.7}$$

and so $(w^{**}, z^*) \in G(\overline{T})$. Since S is of type (ED), the same is true of T (see [14], Theorem 4.4 again) and so there exists a net $\{(t_\gamma, t^*_\gamma)\}$ of elements of $G(T)$ such that $(\widehat{t_\gamma}, t^*_\gamma) \to (w^{**}, z^*)$ in $\mathcal{T}_{\mathcal{CLB}}(E^{**}) \times \mathcal{T}_{\|\ \|}(E^*)$. From Lemma 13(b), $\langle t^*_\gamma - z^*, \widehat{t_\gamma} - w^{**}\rangle \to 0$, and so, putting $(t, t^*) = (t_\gamma, t^*_\gamma)$ and passing to the limit in (15.6), for all $u \in B$, $\langle y - u, z^*\rangle \geq 0$, from which

$$\langle y, z^*\rangle = \max\langle B, z^*\rangle. \tag{15.8}$$

Using (15.5),

$$\text{for all } \gamma, \quad \langle t_\gamma^*, \widehat{t_\gamma}\rangle + \max\langle B, t_\gamma^*\rangle = \langle t_\gamma, t_\gamma^*\rangle + \max\langle B, t_\gamma^*\rangle \geq 0.$$

Passing to the limit in this and using (a) and (d) of Lemma 13, $\langle z^*, w^{**}\rangle + \max\langle B, z^*\rangle \geq 0$. Combining this with (15.8), we derive that

$$\langle z^*, w^{**} + \widehat{y}\rangle = \langle z^*, w^{**}\rangle + \langle y, z^*\rangle \geq 0.$$

Exactly as in (a), we now deduce that $z^* = 0$ and $w^{**} + \widehat{y} = 0$. Let $w = -y \in C$, so we now have $z^* = 0$ and $w^{**} = \widehat{w}$. Substituting both of these equalities into (15.7) yields $\inf_{(t,t^*)\in G(T)}\langle t^* - 0, \widehat{t} - \widehat{w}\rangle \geq 0$, that is to say,

$$\inf_{(t,t^*)\in G(T)} \langle t - w, t^* - 0\rangle \geq 0.$$

Since T is maximal monotone, this implies that $0 \in Tw$, and so $w \in S^{-1}w^*$. This gives (8.2), and completes the proof of the theorem. ▌

6. Type (D) implies type (FP)

We prove a preliminary lemma before establishing the main result of this section, Theorem 17. We have separated Lemma 16 into two parts since Lemma 16(a) will be used again in Lemma 19.

Lemma 16. *Let $T\colon E \mapsto 2^{E^*}$ be nontrivial and monotone, $\tau^* \in R(T)$ and $\varepsilon > 0$ and let $B := [0, \tau^*] + \{x^* \in E^*\colon \|x^*\| \leq \varepsilon\} \subset E^*$.*
(a) There exist $w^{**} \in E^{**}$, $y^{**} \in E^{**}$ and $z^* \in B$ such that, for all $(t, t^*) \in G(T)$ and $u^* \in B$,

$$2\langle t, t^*\rangle - 2\langle t, z^*\rangle - 2\langle t^*, w^{**}\rangle - 2\langle u^*, y^{**}\rangle \geq \|z^*\|^2 + \|w^{**} + y^{**}\|^2. \tag{16.1}$$

(b) Continuing on from (a), for all $(t, t^*) \in G(T)$ and $u^* \in B$,

$$2\langle t^* - z^*, \widehat{t} - w^{**}\rangle + 2\langle z^* - u^*, y^{**}\rangle \geq \|z^*\|^2 + 2\langle z^*, w^{**} + y^{**}\rangle + \|w^{**} + y^{**}\|^2 \geq 0. \tag{16.2}$$

Proof. Since $\tau^* \in R(T)$, we can fix $\tau \in E$ so that $(\tau, \tau^*) \in G(T)$. We then write

$$M := 1 \vee \|\tau\| \vee \|\tau^*\| \vee \langle \tau, \tau^*\rangle \quad \text{and} \quad N := 2M^2/\varepsilon.$$

For all $u \in E$, let $\varphi(u) := \max\langle u, B\rangle$. Then, by direct computation,

$$u \in E \quad \Longrightarrow \quad \varphi(u) \geq \langle u, \tau^*\rangle + \varepsilon\|u\|. \tag{16.3}$$

Now let p, q, r and $\mathcal{CO}(T)$ be as in Lemma 11 and $A := \mathcal{CO}(T) \times E \times B$. We first prove that, for all $(\mu, u, u^*) \in A$,

$$2r(\mu) + 2\varphi(u) + 2N\|q(\mu) - u^*\| + \|p(\mu) + u\|^2 + \|q(\mu)\|^2 \geq 0. \tag{16.4}$$

So let us suppose that $(\mu, u, u^*) \in A$. If $\|u\| \le N$ then, from (11.1) and the definition of φ,

$$\begin{aligned}
2r(\mu) + 2\varphi(u) + 2N\|q(\mu) - u^*\| + \|p(\mu) + u\|^2 + \|q(\mu)\|^2 \\
&\ge 2r(\mu) + 2\varphi(u) + 2N\|q(\mu) - u^*\| + 2\|p(\mu) + u\|\|q(\mu)\| \\
&\ge 2r(\mu) + 2\varphi(u) + 2\langle u, q(\mu) - u^*\rangle - 2\langle p(\mu) + u, q(\mu)\rangle \\
&\ge 2r(\mu) - 2\langle p(\mu), q(\mu)\rangle + 2\varphi(u) - 2\langle u, u^*\rangle \ge 0,
\end{aligned}$$

which gives (16.4). Suppose, on the other hand, that $\|u\| > N$. Then, from (16.3),

$$\varphi(u) - \langle u, \tau^*\rangle \ge \varepsilon\|u\| \ge \varepsilon N = 2M^2. \tag{16.5}$$

Since T is monotone, for all $(t, t^*) \in G(T)$,

$$\begin{aligned}
r(\delta(t,t^*)) - \langle p(\delta(t,t^*)), \tau^*\rangle - \langle \tau, q(\delta(t,t^*))\rangle + \langle \tau, \tau^*\rangle &= \langle t, t^*\rangle - \langle t, \tau^*\rangle - \langle \tau, t^*\rangle + \langle \tau, \tau^*\rangle \\
&= \langle t - \tau, t^* - \tau^*\rangle \ge 0
\end{aligned}$$

thus, from the linearity of p, q, and r,

$$r(\mu) - \langle p(\mu), \tau^*\rangle - \langle \tau, q(\mu)\rangle + \langle \tau, \tau^*\rangle \ge 0. \tag{16.6}$$

But then, since $\|p(\mu)+u\|^2 - 2M\|p(\mu)+u\| + M^2 \ge 0$ and $\|q(\mu)\|^2 - 2M\|q(\mu)\| + M^2 \ge 0$,

$$\begin{aligned}
2r(\mu) + 2\varphi(u) + \|p(\mu) + u\|^2 + \|q(\mu)\|^2 \\
&\ge 2r(\mu) + 2\varphi(u) + 2M\|p(\mu) + u\| + 2M\|q(\mu)\| - 2M^2 \\
&\ge 2r(\mu) + 2\varphi(u) - 2\langle p(\mu) + u, \tau^*\rangle - 2\langle \tau, q(\mu)\rangle - 2M^2 \\
&= 2[r(\mu) + \varphi(u) - \langle p(\mu) + u, \tau^*\rangle - \langle \tau, q(\mu)\rangle - M^2] \\
&\ge 2[r(\mu) + \varphi(u) - \langle p(\mu) + u, \tau^*\rangle - \langle \tau, q(\mu)\rangle + \langle \tau, \tau^*\rangle - 2M^2] \\
&= 2[r(\mu) - \langle p(\mu), \tau^*\rangle - \langle \tau, q(\mu)\rangle + \langle \tau, \tau^*\rangle + \varphi(u) - \langle u, \tau^*\rangle - 2M^2] \ge 0,
\end{aligned}$$

where the last inequality follows by adding (16.5) and (16.6). This completes the proof of (16.4).

Now let $F = E \times E^*$ be as in the proof of Lemma 12(a) and, for all $(\mu, u, u^*) \in A$,

$$f(\mu, u, u^*) := 2r(\mu) + 2\varphi(u) + 2N\|q(\mu) - u^*\| \in \mathbb{R} \quad \text{and} \quad g(\mu, u, u^*) := (p(\mu) + u, q(\mu)) \in F.$$

Arguing as in Lemma 12(a), we obtain $(z^*, z^{**}) \in E^* \times E^{**}$ such that, for all $(\mu, u, u^*) \in A$,

$$2r(\mu) + 2\varphi(u) + 2N\|q(\mu) - u^*\| - 2\langle p(\mu) + u, z^*\rangle - 2\langle q(\mu), z^{**}\rangle \ge \|z^*\|^2 + \|z^{**}\|^2. \tag{16.7}$$

For the moment fix μ and u^*. It follows that $\inf_E[\varphi - z^*] > -\infty$. Since $\varphi - z^*$ is positively homogeneous on E, we derive that, for all $u \in E$, $\langle u, z^*\rangle \le \varphi(u) = \max\langle u, B\rangle$ and so the

$w(E^*,E)$–compactness of B and the bipolar theorem imply that $z^* \in B$, as required. Now define $H := \mathcal{CO}(T) \times B$. Putting $u = 0$ in (16.7), we derive that:

$$(\mu, u^*) \in H \quad\Longrightarrow\quad 2r(\mu) + 2N\|q(\mu) - u^*\| - 2\langle p(\mu), z^*\rangle - 2\langle q(\mu), z^{**}\rangle \geq \|z^*\|^2 + \|z^{**}\|^2.$$

The minimax argument of Lemma 12(a) (with A replaced by H and B replaced by $\{x^{**} \in E^{**}\colon\ \|x^{**}\| \leq N\}$) now yields $y^{**} \in E^{**}$ such that:

$$(\mu, u^*) \in H \quad\Longrightarrow\quad 2r(\mu) + 2\langle q(\mu) - u^*, y^{**}\rangle - 2\langle p(\mu), z^*\rangle - 2\langle q(\mu), z^{**}\rangle \geq \|z^*\|^2 + \|z^{**}\|^2.$$

Making the substitution $w^{**} = z^{**} - y^{**}$:

$$(\mu, u^*) \in H \quad\Longrightarrow\quad 2r(\mu) - 2\langle p(\mu), z^*\rangle - 2\langle q(\mu), w^{**}\rangle - 2\langle u^*, y^{**}\rangle \geq \|z^*\|^2 + \|w^{**} + y^{**}\|^2.$$

We now obtain (a) by taking $\mu = \delta(t, t^*)$, where δ is as in Lemma 11.

(b) We obtain the first inequality in (16.2) by adding $2\langle z^*, w^{**} + y^{**}\rangle$ to both sides of (16.1) and rearranging the terms, and the second inequality follows as in Lemma 12(b). This completes the proof of Lemma 16. ▌

Theorem 17. *Let $S\colon\ E \mapsto 2^{E^*}$ be maximal monotone of type (D). Then S is of type (FP).*

Proof. Let U be an open convex subset of E^* such that $U \cap R(S) \neq \emptyset$ and $(v, v^*) \in E \times U$ be such that

$$(s, s^*) \in G(S) \text{ and } s^* \in U \quad\Longrightarrow\quad \langle s - v, s^* - v^*\rangle \geq 0.$$

We want to prove that $(v, v^*) \in G(S)$. Now define $T\colon\ E \mapsto 2^{E^*}$ by $G(T) := G(S) - (v, v^*)$. Further, writing $V := U - v^*$, we have that V is an open convex subset of E^* such that $V \ni 0$, $V \cap R(T) \neq \emptyset$ and

$$(t, t^*) \in G(T) \text{ and } t^* \in V \quad\Longrightarrow\quad \langle t, t^*\rangle \geq 0 \tag{17.1}$$

and now what we must prove is that

$$(0, 0) \in G(T). \tag{17.2}$$

We first find $\tau^* \in V \cap R(T)$ and choose $\varepsilon > 0$ so that

$$B := [0, \tau^*] + \{x^* \in E^*\colon\ \|x^*\| \leq \varepsilon\} \subset V.$$

Using Lemma 16, we then find $w^{**} \in E^{**}$, $y^{**} \in E^{**}$ and $z^* \in B$ such that, for all $(t, t^*) \in G(T)$ and $u^* \in B$,

$$2\langle t^* - z^*, \widehat{t} - w^{**}\rangle + 2\langle z^* - u^*, y^{**}\rangle \geq \|z^*\|^2 + 2\langle z^*, w^{**} + y^{**}\rangle + \|w^{**} + y^{**}\|^2 \geq 0. \tag{16.2}$$

Since $z^* \in B$, it follows by putting $u^* = z^*$ that

$$2 \inf_{(t,t^*) \in G(T)} \langle t^* - z^*, \widehat{t} - w^{**}\rangle \geq \|z^*\|^2 + 2\langle z^*, w^{**} + y^{**}\rangle + \|w^{**} + y^{**}\|^2 \geq 0, \tag{17.3}$$

and so $(w^{**}, z^*) \in G(\overline{T})$. Since S is of type (D), the same is true of T and so there exists a bounded net $\{(t_\gamma, t_\gamma^*)\}$ of elements of $G(T)$ such that $(\widehat{t_\gamma}, t_\gamma^*) \to (w^{**}, z^*)$ in $w(E^{**}, E^*) \times \mathcal{T}_{\|\ \|}(E^*)$. This implies that $\langle t_\gamma^* - z^*, \widehat{t_\gamma} - w^{**}\rangle \to 0$, and so putting $(t, t^*) = (t_\gamma, t_\gamma^*)$ and passing to the limit in (16.2), for all $u^* \in B$, $\langle z^* - u^*, y^{**}\rangle \geq 0$ thus

$$\langle z^*, y^{**}\rangle \geq \sup\langle B, y^{**}\rangle \geq \varepsilon\|y^{**}\|. \tag{17.4}$$

Now $z^* \in B \subset V$ and $t_\gamma^* \to z^*$ in $\mathcal{T}_{\|\ \|}(E^*)$ so, truncating the net $\{(t_\gamma, t_\gamma^*)\}$ if necessary, we may suppose that, for all γ, $t_\gamma^* \in V$. Using (17.1), we now derive that,

$$\text{for all } \gamma, \quad \langle t_\gamma^*, \widehat{t_\gamma}\rangle = \langle t_\gamma, t_\gamma^*\rangle \geq 0.$$

Passing to the limit in this, $\langle z^*, w^{**}\rangle \geq 0$. Combining with (17.4) and substituting into (17.3), we obtain that

$$2 \inf_{(t,t^*)\in G(T)} \langle t^* - z^*, \widehat{t} - w^{**}\rangle \geq \|z^*\|^2 + 2\varepsilon\|y^{**}\| + \|w^{**} + y^{**}\|^2 \geq 0. \tag{17.5}$$

Going through the same argument as above, that is, setting $(t, t^*) = (t_\gamma, t_\gamma^*)$ and passing to the limit, we have $\|z^*\|^2 + 2\varepsilon\|y^{**}\| + \|w^{**} + y^{**}\|^2 \leq 0$, hence $z^* = 0$ and $w^{**} = y^{**} = 0$. Substituting back into (17.5) yields $\inf_{(t,t^*)\in G(T)}\langle t^* - 0, \widehat{t} - 0\rangle \geq 0$, that is to say,

$$\inf_{(t,t^*)\in G(T)} \langle t - 0, t^* - 0\rangle \geq 0.$$

Since T is maximal monotone, this gives (17.2) and completes the proof of Theorem 17. ▌

We do not know the solution to the following problem:

Problem 18. Is every maximal monotone multifunction of type (FP) necessarily of type (D)?

7. Type (ED) implies type (FPV)

As in the previous section, we will prove a preliminary lemma before establishing the main result, Theorem 20. The technical details are somewhat harder in this case.

Lemma 19. *Let T: $E \mapsto 2^{E^*}$ be nontrivial and monotone, $\tau \in D(T)$ and $\varepsilon > 0$ and let $B = [0, \widehat{\tau}] + \{x^{**} \in E^{**}\colon \|x^{**}\| \leq \varepsilon\} \subset E^{**}$. Then there exist $w^* \in E^*$, $y^* \in E^*$ and $z^{**} \in B$ such that, for all $(t, t^*) \in G(T)$ and $u^{**} \in B$,*

$$2\langle t^* - w^*, \widehat{t} - z^{**}\rangle + 2\langle y^*, z^{**} - u^{**}\rangle \geq \|w^* + y^*\|^2 + 2\langle w^* + y^*, z^{**}\rangle + \|z^{**}\|^2 \geq 0. \tag{19.1}$$

Proof. We apply Lemma 16(a) with E replaced by E^*, T replaced by the monotone multifunction $\widehat{T^{-1}}$: $E^* \mapsto E^{**}$, τ^* replaced by $\widehat{\tau}$ and obtain $w^{***} \in E^{***}$, $y^{***} \in E^{***}$ and $z^{**} \in B$ such that, for all $(t^*, t^{**}) \in G(\widehat{T^{-1}})$ and $u^{**} \in B$,

$$2\langle t^*, t^{**}\rangle - 2\langle t^*, z^{**}\rangle - 2\langle t^{**}, w^{***}\rangle - 2\langle u^{**}, y^{***}\rangle \geq \|z^{**}\|^2 + \|w^{***} + y^{***}\|^2.$$

In particular, for all $(t^*, t^{**}) \in G(\widehat{T^{-1}})$ and $u \in B_0 := [0, \tau] + \{x \in E\colon\ \|x\| \le \varepsilon\} \subset E$,

$$2\langle t^*, t^{**}\rangle - 2\langle t^*, z^{**}\rangle - 2\langle t^{**}, w^{***}\rangle - 2\langle \widehat{u}, y^{***}\rangle \ge \|z^{**}\|^2 + \|w^{***} + y^{***}\|^2.$$

Equivalently, for all $(t, t^*) \in G(T)$ and $u \in B_0$,

$$2\langle t, t^*\rangle - 2\langle t^*, z^{**}\rangle - 2\langle \widehat{t}, w^{***}\rangle - 2\langle \widehat{u}, y^{***}\rangle \ge \|z^{**}\|^2 + \|w^{***} + y^{***}\|^2.$$

Now let w^* and y^* be the images of w^{***} and y^{***}, respectively, under the adjoint of the canonical map $\widehat{\ }$. Then $w^* \in E^*$, $y^* \in E^*$ and, for all $(t, t^*) \in G(T)$ and $u \in B_0$,

$$\begin{aligned} 2\langle t, t^*\rangle - 2\langle t^*, z^{**}\rangle - 2\langle t, w^*\rangle - 2\langle u, y^*\rangle &\ge \|z^{**}\|^2 + \|w^{***} + y^{***}\|^2 \\ &\ge \|z^{**}\|^2 + \|w^* + y^*\|^2. \end{aligned}$$

Consequently, for all $(t, t^*) \in G(T)$

$$2\langle t, t^*\rangle - 2\langle t^*, z^{**}\rangle - 2\langle t, w^*\rangle - 2\sup\langle B_0, y^*\rangle \ge \|z^{**}\|^2 + \|w^* + y^*\|^2.$$

Now, for all $u^{**} \in B$,

$$\langle y^*, u^{**}\rangle \le \langle y^*, \widehat{\tau}\rangle \vee 0 + \varepsilon\|y^*\| = \langle \tau, y^*\rangle \vee 0 + \varepsilon\|y^*\| = \sup\langle B_0, y^*\rangle$$

and so, for all $(t, t^*) \in G(T)$ and $u^{**} \in B$,

$$2\langle t, t^*\rangle - 2\langle t^*, z^{**}\rangle - 2\langle t, w^*\rangle - 2\langle y^*, u^{**}\rangle \ge \|z^{**}\|^2 + \|w^* + y^*\|^2.$$

The first inequality in (19.1) now follows by adding $2\langle w^* + y^*, z^{**}\rangle$ to both sides and rearranging the terms, and the second inequality follows as in Lemma 12(b). This completes the proof of Lemma 19. ▌

Theorem 20. *Let $S\colon\ E \mapsto 2^{E^*}$ be maximal monotone of type (ED). Then S is of type (FPV).*

Proof. Let U be an open convex subset of E such that $U \cap D(S) \ne \emptyset$ and $(v, v^*) \in U \times E^*$ be such that

$$(s, s^*) \in G(S) \text{ and } s \in U \quad \Longrightarrow \quad \langle s - v, s^* - v^*\rangle \ge 0.$$

We want to prove that $(v, v^*) \in G(S)$. Now define $T\colon\ E \mapsto 2^{E^*}$ by $G(T) := G(S) - (v, v^*)$. Further, writing $V := U - v$, we have that V is an open convex subset of E such that $V \ni 0$, $V \cap D(T) \ne \emptyset$ and

$$(t, t^*) \in G(T) \text{ and } t \in V \quad \Longrightarrow \quad \langle t, t^*\rangle \ge 0 \tag{20.1}$$

and now what we must prove is that

$$(0, 0) \in G(T). \tag{20.2}$$

We first find $\tau \in V \cap D(T)$ and choose $\varepsilon > 0$ so that $[0, \tau] + \{x \in E\colon \|x\| \le 2\varepsilon\} \subset V$, and let $B = [0, \widehat{\tau}] + \{x^{**} \in E^{**}\colon \|x^{**}\| \le \varepsilon\} \subset E^{**}$ as in Lemma 19. Using Lemma 19, we then find $w^* \in E^*$, $y^* \in E^*$ and $z^{**} \in B$ such that, for all $(t, t^*) \in G(T)$ and $u^{**} \in B$,

$$2\langle t^* - w^*, \widehat{t} - z^{**}\rangle + 2\langle y^*, z^{**} - u^{**}\rangle \ge \|w^* + y^*\|^2 + 2\langle w^* + y^*, z^{**}\rangle + \|z^{**}\|^2 \ge 0. \quad (19.1)$$

Since $z^{**} \in B$, it follows by putting $u^{**} = z^{**}$ that

$$2 \inf_{(t,t^*) \in G(T)} \langle t^* - w^*, \widehat{t} - z^{**}\rangle \ge \|w^* + y^*\|^2 + 2\langle w^* + y^*, z^{**}\rangle + \|z^{**}\|^2 \ge 0, \qquad (20.3)$$

and so $(z^{**}, w^*) \in G(\overline{T})$. Since S is of type (ED), the same is true of T (see [14], Theorem 4.4, p. 268) and so there exists a net $\{(t_\gamma, t^*_\gamma)\}$ of elements of $G(T)$ such that $(\widehat{t_\gamma}, t^*_\gamma) \to (z^{**}, w^*)$ in $\mathcal{T}_{\mathcal{CLB}}(E^{**}) \times \mathcal{T}_{\|\ \|}(E^*)$. From Lemma 13(b), $\langle t^*_\gamma - w^*, \widehat{t_\gamma} - z^{**}\rangle \to 0$, and so, putting $(t, t^*) = (t_\gamma, t^*_\gamma)$ and passing to the limit in (19.1) for all $u^{**} \in B$, $\langle y^*, z^{**} - u^{**}\rangle \ge 0$ thus

$$\langle y^*, z^{**}\rangle \ge \max\langle y^*, B\rangle \ge \varepsilon\|y^*\|. \qquad (20.4)$$

Since $z^{**} \in B$, there exists $\lambda \in [0, 1]$ such that $\|z^{**} - \lambda\widehat{\tau}\| \le \varepsilon$. On the other hand, from Lemma 13(e), $\|\widehat{t_\gamma} - \lambda\widehat{\tau}\| \to \|z^{**} - \lambda\widehat{\tau}\|$, that is to say, $\|t_\gamma - \lambda\tau\| \to \|z^{**} - \lambda\widehat{\tau}\|$. Thus, truncating the net $\{(t_\gamma, t^*_\gamma)\}$ if necessary, we may suppose that, for all γ, $\|t_\gamma - \lambda\tau\| \le 2\varepsilon$, from which $t_\gamma \in V$. Using (20.1), we now derive that,

$$\text{for all } \gamma, \quad \langle t^*_\gamma, \widehat{t_\gamma}\rangle = \langle t_\gamma, t^*_\gamma\rangle \ge 0.$$

Passing to the limit in this, $\langle w^*, z^{**}\rangle \ge 0$. Combining with (20.4) and substituting into (20.3), we obtain that

$$2 \inf_{(t,t^*) \in G(T)} \langle t^* - w^*, \widehat{t} - z^{**}\rangle \ge \|w^* + y^*\|^2 + 2\varepsilon\|y^*\| + \|z^{**}\|^2.$$

Going through the same argument as above, that is, setting $(t, t^*) = (t_\gamma, t^*_\gamma)$ and passing to the limit, we have $\|w^* + y^*\|^2 + 2\varepsilon\|y^*\| + \|z^{**}\|^2 \le 0$, hence $z^{**} = 0$ and $w^* = y^* = 0$. Substituting back into (20.3) yields $\inf_{(t,t^*) \in G(T)} \langle t^* - 0, \widehat{t} - 0\rangle \ge 0$, that is to say,

$$\inf_{(t,t^*) \in G(T)} \langle t - 0, t^* - 0\rangle \ge 0.$$

Since T is maximal monotone, this gives (20.2) and completes the proof of Theorem 20. ▌

References

[1] H. H. Bauschke and J. M. Borwein, *Maximal monotonicity of dense type, local maximal monotonicity, and monotonicity of the conjugate are all the same for continuous linear operators*, Pacific J. Math. **189** (1999), 1–20.

[2] H. H. Bauschke and S. Simons, *Stronger maximal monotonicity properties of linear operators*, Bull. Austral. Math. Soc. **60** (1999), 163–174.

[3] K. Fan, *Minimax theorems*, Proc. Nat. Acad. Sci. U.S.A. **39** (1953), 42–47.

[4] S. P. Fitzpatrick and R. R. Phelps, *Bounded approximants to monotone operators on Banach spaces*, Ann. Inst. Henri Poincaré, Analyse non linéaire **9** (1992), 573–595.

[5] S. P. Fitzpatrick and R. R. Phelps, *Some properties of maximal monotone operators on nonreflexive Banach spaces*, Set–Valued Analysis **3**(1995), 51–69.

[6] J.- P. Gossez, *Opérateurs monotones non linéaires dans les espaces de Banach non réflexifs*, J. Math. Anal. Appl. **34** (1971), 371–395.

[7] R. R. Phelps, *Lectures on Maximal Monotone Operators*, Extracta Mathematicae **12** (1997), 193–230.

[8] R. R. Phelps and S. Simons, *Unbounded linear monotone operators on nonreflexive Banach spaces*, J. Convex Analysis, **5** (1998), 303–328.

[9] M. Przeworski and D. Zagrodny, *Constrained equilibrium point of maximal monotone operator via variational inequality*, J. Appl. Anal. **5** (1999), 147–152.

[10] R. T. Rockafellar, *On the maximal monotonicity of subdifferential mappings*, Pac. J. Math. **33** (1970), 209-216.

[11] S. Simons, *Subdifferentials are locally maximal monotone*, Bull. Australian Math. Soc. **47** (1993), 465–471.

[12] S. Simons, *Subtangents with controlled slope*, Nonlinear Analysis **22** (1994), 1373–1389.

[13] S. Simons, *Minimax and monotonicity*, Lecture Notes in Mathematics **1693** (1998), Springer–Verlag.

[14] S. Simons, *Maximal monotone multifunctions of Brøndsted–Rockafellar type*, Set–Valued Anal. **7** (1999), 255–294.

[15] A. and M. E. Verona, *Remarks on subgradients and ε-subgradients*, Set–Valued Analysis **1**(1993), 261–272.

Department of Mathematics
University of California
Santa Barbara
CA 93106-3080