1. Linear Systems

Proof. If $\lambda = a + ib$, it follows by induction that

$$
\begin{bmatrix}
 a & -b \\
 b & a
\end{bmatrix}^k
= \begin{bmatrix}
 \text{Re}(\lambda^k) & -\text{Im}(\lambda^k) \\
 \text{Im}(\lambda^k) & \text{Re}(\lambda^k)
\end{bmatrix}
$$

where \(\text{Re}\) and \(\text{Im}\) denote the real and imaginary parts of the complex number \(\lambda\) respectively. Thus,

$$
e^A = \sum_{k=0}^{\infty} \begin{bmatrix}
 \text{Re} \left( \frac{\lambda^k}{k!} \right) & -\text{Im} \left( \frac{\lambda^k}{k!} \right) \\
 \text{Im} \left( \frac{\lambda^k}{k!} \right) & \text{Re} \left( \frac{\lambda^k}{k!} \right)
\end{bmatrix}
\begin{bmatrix}
 e^\lambda & e^{-\lambda} \\
 e^{-\lambda} & e^\lambda
\end{bmatrix}
\begin{bmatrix}
 \cos b & -\sin b \\
 \sin b & \cos b
\end{bmatrix}
\begin{bmatrix}
 e^\lambda & e^{-\lambda} \\
 e^{-\lambda} & e^\lambda
\end{bmatrix}
\begin{bmatrix}
 \cos b & -\sin b \\
 \sin b & \cos b
\end{bmatrix}
$$

Note that if \(a = 0\) in Corollary 3, then \(e^A\) is simply a rotation through \(b\) radians.

Corollary 4. If

$$A = \begin{bmatrix}
 a & b \\
 0 & a
\end{bmatrix}$$

then

$$e^A = e^a \begin{bmatrix}
 1 & b \\
 0 & 1
\end{bmatrix}.$$  

Proof. Write \(A = aI + B\) where

$$B = \begin{bmatrix}
 0 & b \\
 0 & 0
\end{bmatrix}.$$  

Then \(aI\) commutes with \(B\) and by Proposition 2,

$$e^A = e^{aI}e^B = e^a e^B.$$  

And from the definition

$$e^B = I + B + B^2/2! + \cdots = I + B$$

since by direct computation \(B^2 = B^3 = \cdots = 0\).

We can now compute the matrix \(e^{	ext{At}}\) for any \(2 \times 2\) matrix \(A\). In Section 1.8 of this chapter it is shown that there is an invertible \(2 \times 2\) matrix \(P\) (whose columns consist of generalized eigenvectors of \(A\)) such that the matrix

$$B = P^{-1}AP$$

has one of the following forms

$$B = \begin{bmatrix}
 \lambda & 0 \\
 0 & \mu
\end{bmatrix}, \quad B = \begin{bmatrix}
 \lambda & 1 \\
 0 & \lambda
\end{bmatrix} \quad \text{or} \quad B = \begin{bmatrix}
 a & -b \\
 b & a
\end{bmatrix}.$$  

1.3. Exponentials of Operators

It then follows from the above corollaries and Definition 2 that

$$e^{At} = \begin{bmatrix}
 e^{At} & 0 \\
 0 & e^{At}
\end{bmatrix}, \quad e^{Bt} = e^{At} \begin{bmatrix}
 1 & t \\
 0 & 1
\end{bmatrix} \quad \text{or} \quad e^{Bt} = e^{At} \begin{bmatrix}
 \cos bt & -\sin bt \\
 \sin bt & \cos bt
\end{bmatrix}$$

respectively. And by Proposition 1, the matrix \(e^{At}\) is then given by

$$e^{At} = P e^{Bt} P^{-1}.$$  

As we shall see in Section 1.4, finding the matrix \(e^{At}\) is equivalent to solving the linear system (1) in Section 1.1.

Problem Set 3

1. Compute the operator norm of the linear transformation defined by the following matrices:

(a) \(\begin{bmatrix}
 2 & 0 \\
 0 & -3
\end{bmatrix}\)

(b) \(\begin{bmatrix}
 1 & 2 \\
 0 & -1
\end{bmatrix}\)

(c) \(\begin{bmatrix}
 1 & 0 \\
 5 & 1
\end{bmatrix}\).

Hint: In (c) maximize \(|A \mathbf{x}|^2 = 26x_1^2 + 10x_1x_2 + x_2^2\) subject to the constraint \(x_1^2 + x_2^2 = 1\) and use the result of Problem 2; or use the fact that \(\|A\| = [\text{Max eigenvalue of } A^T A]^{1/2}\). Follow this same hint for (b).

2. Show that the operator norm of a linear transformation \(T\) on \(\mathbb{R}^n\) satisfies

$$\|T\| = \max_{\|x\| = 1} \|T(x)\| = \sup_{x \neq 0} \frac{|T(x)|}{\|x\|}.$$  

3. Use the lemma in this section to show that if \(T\) is an invertible linear transformation then \(\|T^{-1}\| > 0\) and

$$\|T^{-1}\| \geq \frac{1}{\|T\|}.$$  

4. If \(T\) is a linear transformation on \(\mathbb{R}^n\) with \(\|T - I\| < 1\), prove that \(T\) is invertible and that the series \(\sum_{k=0}^{\infty} (I - T)^k\) converges absolutely to \(T^{-1}\).

Hint: Use the geometric series.

5. Compute the exponentials of the following matrices:

\(\begin{bmatrix}
 2 & 0 \\
 0 & -3
\end{bmatrix}\) \(\begin{bmatrix}
 1 & 2 \\
 0 & -1
\end{bmatrix}\) \(\begin{bmatrix}
 1 & 0 \\
 5 & 1
\end{bmatrix}\)
1. Linear Systems

(d) \[
\begin{bmatrix}
5 & -6 \\
3 & -4
\end{bmatrix}
\]

(e) \[
\begin{bmatrix}
2 & -1 \\
1 & 2
\end{bmatrix}
\]

(f) \[
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\]

6. (a) For each matrix in Problem 5 find the eigenvalues of \(e^A\).

(b) Show that if \(x\) is an eigenvector of \(A\) corresponding to the eigenvalue \(\lambda\), then \(x\) is also an eigenvector of \(e^A\) corresponding to the eigenvalue \(e^\lambda\).

(c) If \(A = P\text{diag}(\lambda_j)P^{-1}\), use Corollary 1 to show that
\[
\det e^A = e^{\text{trace}A}.
\]

Also, using the results in the last paragraph of this section, show that this formula holds for any \(2 \times 2\) matrix \(A\).

7. Compute the exponentials of the following matrices:

(a) \[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 3
\end{bmatrix}
\]

(b) \[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 2 & 1 \\
0 & 0 & 2
\end{bmatrix}
\]

(c) \[
\begin{bmatrix}
2 & 0 & 0 \\
1 & 2 & 0 \\
0 & 1 & 2
\end{bmatrix}
\]

Hint: Write the matrices in (b) and (c) as a diagonal matrix \(S\) plus a matrix \(N\). Show that \(S\) and \(N\) commute and compute \(e^S\) as in part (a) and \(e^N\) by using the definition.

8. Find \(2 \times 2\) matrices \(A\) and \(B\) such that \(e^{A+B} \neq e^Ae^B\).

9. Let \(T\) be a linear operator on \(\mathbb{R}^n\) that leaves a subspace \(E \subset \mathbb{R}^n\) invariant; i.e., for all \(x \in E\), \(T(x) \in E\). Show that \(e^T\) also leaves \(E\) invariant.

1.4 The Fundamental Theorem for Linear Systems

Let \(A\) be an \(n \times n\) matrix. In this section we establish the fundamental fact that for \(x_0 \in \mathbb{R}^n\) the initial value problem
\[
\begin{align*}
x &= Ax \\
x(0) &= x_0
\end{align*}
\]
has a unique solution for all \(t \in \mathbb{R}\) which is given by
\[
x(t) = e^{At}x_0.
\]

Notice the similarity in the form of the solution (2) and the solution \(x(t) = e^{At}x_0\) of the elementary first-order differential equation \(\dot{x} = ax\) and initial condition \(x(0) = x_0\).

In order to prove this theorem, we first compute the derivative of the exponential function \(e^{At}\) using the basic fact from analysis that two convergent limit processes can be interchanged if one of them converges uniformly. This is referred to as Moore's Theorem; cf. Graves [G], p. 100 or Rudin [R], p. 149.

**Lemma.** Let \(A\) be a square matrix, then
\[
\frac{d}{dt}e^{At} = Ae^{At}.
\]

**Proof.** Since \(A\) commutes with itself, it follows from Proposition 2 and Definition 2 in Section 3 that
\[
\frac{d}{dt}e^{At} = \lim_{h \to 0} \frac{e^{A(t+h)} - e^{At}}{h} = \lim_{h \to 0} e^{At}(e^{Ah} - I) = e^{At} \lim_{h \to 0} \frac{e^{Ah} - 1}{h} = e^{At} \lim_{h \to 0} \frac{e^{Ah} - 1}{h} = Ae^{At}.
\]

The last equality follows since by the theorem in Section 1.3 the series defining \(e^{Ah}\) converges uniformly for \(|h| \leq 1\) and we can therefore interchange the two limits.

**Theorem (The Fundamental Theorem for Linear Systems).** Let \(A\) be an \(n \times n\) matrix. Then for a given \(x_0 \in \mathbb{R}^n\), the initial value problem
\[
\begin{align*}
x &= Ax \\
x(0) &= x_0
\end{align*}
\]
has a unique solution given by
\[
x(t) = e^{At}x_0.
\]

**Proof.** By the preceding lemma, if \(x(t) = e^{At}x_0\), then
\[
x'(t) = \frac{d}{dt}e^{At}x_0 = Ae^{At}x_0 = Ax(t)
\]
for all \(t \in \mathbb{R}\). Also, \(x(0) = Ix_0 = x_0\). Thus \(x(t) = e^{At}x_0\) is a solution. To see that this is the only solution, let \(x(t)\) be any solution of the initial value problem (1) and set
\[
y(t) = e^{-At}x(t).
\]
Then from the above lemma and the fact that $x(t)$ is a solution of (1)
\[ y'(t) = -A e^{-At} x(t) + e^{-At} x'(t) 
    = -A e^{-At} x(t) + e^{-At} A x(t) 
    = 0 \]
for all $t \in \mathbb{R}$ since $e^{-At}$ and $A$ commute. Thus, $y(t)$ is a constant. Setting $t = 0$ shows that $y(t) = x_0$ and therefore any solution of the initial value problem (1) is given by $x(t) = e^{At} y(t) = e^{At} x_0$. This completes the proof of the theorem.

**Example.** Solve the initial value problem
\[ \dot{x} = Ax \]
\[ x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \]
for
\[ A = \begin{bmatrix} -2 & -1 \\ 1 & -2 \end{bmatrix} \]
and sketch the solution curve in the phase plane $\mathbb{R}^2$. By the above theorem and Corollary 3 of the last section, the solution is given by
\[ x(t) = e^{At} x_0 = e^{-2t} \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = e^{-2t} \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}. \]
It follows that $|x(t)| = e^{-2t}$ and that the angle $\theta(t) = \tan^{-1} x_2(t)/x_1(t) = t$. The solution curve therefore spirals into the origin as shown in Figure 1 below.

**Figure 1**

1.4. The Fundamental Theorem for Linear Systems

**Problem Set 4**

1. Use the forms of the matrix $e^{Bt}$ computed in Section 1.3 and the theorem in this section to solve the linear system $\dot{x} = Bx$ for

(a) $B = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$

(b) $B = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$

(c) $B = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.

2. Solve the following linear system and sketch its phase portrait
\[ \dot{x} = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} x. \]
The origin is called a stable focus for this system.

3. Find $e^{At}$ and solve the linear system $\dot{x} = Ax$ for

(a) $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$.

Cf. Problem 1 in Problem Set 2.

4. Given
\[ A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & -1 \end{bmatrix}. \]
Compute the $3 \times 3$ matrix $e^{At}$ and solve $\dot{x} = Ax$. Cf. Problem 2 in Problem Set 2.

5. Find the solution of the linear system $\dot{x} = Ax$ where

(a) $A = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}$

(b) $A = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$

(c) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(d) $A = \begin{bmatrix} -2 & 0 & 0 \\ 1 & -2 & 0 \\ 0 & 1 & -2 \end{bmatrix}$.