Notes on the Deuring-Heilbronn Phenomenon

Jeffrey Stopple

Introduction

Analytic number theory studies L-functions, generalizations of the Riemann zeta function $\zeta(s)$. It can be difficult to see why this is number theory. In fact, the Class Number Formula (6) of Dirichlet gives the number $h(-d)$ of classes of binary quadratic forms of discriminant $-d$ as the value of such an L-function at $s = 1$. The location of the zeros is important: since the functions are continuous, the value is influenced by any zero of the function near $s = 1$. Such a zero would of course contradict the Generalized Riemann Hypothesis (GRH).

The Deuring-Heilbronn phenomenon says that such a counterexample to the GRH for one L-function would influence the horizontal and vertical distribution of the zeros of *other* L-functions. They would be forced to lie on the critical line $s = 1/2 + it$, at least up to some height. This is the "Local GRH". More surprisingly, the imaginary parts t would be restricted to a set which is very nearly periodic. This is a very beautiful result indeed. Standard analogies interpret the imaginary parts t as frequencies; the Deuring-Heilbronn phenomenon means these frequencies are in harmony.

We give an overview of the proof, first in the case $h(-d) = 1$ before treating $h(-d) > 1$. Even though the class number 1 problem is now solved, the essential features of the general problem are visible there. We also look at some examples which indicate that even in the absence of a contradiction to GRH, "near contradictions" still cause a tendency towards such a phenomenon.

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Notations

The complex variable s is written $\sigma + it$. We write

$$f(x) \ll h(x) \quad \text{resp.} \quad f(x) = g(x) + O(h(x))$$

if there is some constant C so that

$$|f(x)| \le Ch(x) \quad \text{resp.} \quad f(x) - g(x) \ll h(x)$$

usually for x approaching some limiting value, which may not be explicitly stated. We write

$$f(x) \sim h(x) \quad \text{if} \quad f(x)/h(x) \to 1.$$

Binary Quadratic Forms

Algebraic number theory has its roots in the beautiful theorem of Fermat that an odd prime p is the sum of two squares,

$$p = x^2 + y^2 \quad \Leftrightarrow \quad p \equiv 1 \bmod 4.$$

Euler, Lagrange, and Gauss developed many generalizations of this; for example, $p \neq 7$ can be written as

$$p = x^2 + xy + 2y^2 \quad \Leftrightarrow \quad p \equiv 1, 2, 4 \bmod 7.$$

The necessity of the congruence is the easy half, as it is in Fermat's theorem ($4p = (2x + y)^2 + 7y^2$, now reduce modulo 7). In general one studies positive definite binary quadratic forms, functions

$$Q(x,y) = ax^2 + bxy + cy^2,$$
$$\text{with} \quad -d = b^2 - 4ac < 0$$

the discriminant. (We build the minus sign into the notation to simplify later when we want square root or logarithm of the absolute value.)

Two forms Q and Q' are said to be *equivalent* if there is an integer matrix M with determinant 1 such that

$$Q'(x,y) = Q((x,y)M).$$

Such forms obviously have the same range as functions, i.e., represent the same integers. A calculation shows that equivalent forms have the same discriminant, and it is not difficult to show the number of classes $h(-d)$ is finite. In fact, a very deep result of Gauss is that they form a finite abelian group $C(-d)$.

Below we will need the Kronecker symbol χ_{-d} attached to a discriminant $-d$. In the simplest case that d is an odd prime, χ_{-d} reduces to the Legendre symbol

$$\chi_{-d}(n) = \begin{cases} 0, & \text{if } d|n \\ 1, & \text{if } n \equiv \text{a square} \\ -1, & \text{otherwise.} \end{cases}$$

Of course, this definition works just fine for positive discriminants as well. For odd composite discriminants we can define a Jacobi symbol via multiplicativity. This no longer detects squares, for example,

$$\chi_{-15}(2) = \chi_{-3}(2)\chi_5(2) = -1 \cdot -1 = 1,$$

but 2 is not a square modulo 15. It *does* however, detect whether a prime is represented by some form of discriminant $-d$, just as in the example with discriminant -7 above. For primes p with $\chi_{-15}(p) = +1$ then

$$p = x^2 + xy + 4y^2 \quad \Leftrightarrow \quad p \equiv 1, 4 \bmod 15$$
$$p = 2x^2 + xy + 2y^2 \quad \Leftrightarrow \quad p \equiv 2, 8 \bmod 15.$$

Primes with $\chi_{-15}(p) = -1$ are not represented by any form of discriminant -15. The natural (slightly complicated) extension of this function to even discriminants as well is called the Kronecker symbol.

A weaker relation than equivalence is also useful: two forms are in the same *genus* if they represent the same congruence classes in the multiplicative group modulo d. For example,

$$x^2 + 14y^2 \qquad \text{and} \qquad 2x^2 + 7y^2$$

both have discriminant -56; they must be in different classes since the first represents 1 while the second does not. However, they are in the same genus since

$$2 \cdot 5^2 + 7 \cdot 1^2 = 57 \equiv 1 \bmod 56.$$

In this case $h(-56) = 4$; the other two classes of forms are

$$3x^2 \pm 2xy + 5y^2.$$

Both these forms represent 3, while neither of the first two forms can represent an integer congruent to 3 mod 56 (reduce modulo 7). So there are two genera each consisting of two classes. For more details on the algebraic theory of binary quadratic forms, see [2].

To bring analysis to the study of binary quadratic forms, we introduce the classical Riemann zeta function and the Dirichlet L-function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p (1 - p^{-s})^{-1}$$
$$L(s, \chi_{-d}) = \sum_{n=1}^{\infty} \frac{\chi_{-d}(n)}{n^s} = \prod_p (1 - \chi_{-d}(p)p^{-s})^{-1}.$$

We will also want the Epstein zeta function

$$\zeta_Q(s) = \frac{1}{2} \sum_{(x,y)}{}' Q(x,y)^{-s}$$

where the $'$ in the sum means omit the term $(0,0)$. (The factor of $1/2$ in the definition accounts for the automorphism $(x,y) \to (-x,-y)$. This is in fact the only automorphism, if we assume $d > 4$, which we do from now on. Forms with *positive* discriminant have infinitely many automorphisms.) For another way of writing this, we group together all the terms in which the form takes on the same value, and count them with the representation numbers $r_Q(n)$:

$$r_Q(n) = \frac{1}{2} \cdot \#\{(x,y) \mid Q(x,y) = n\}.$$

This gives

$$\zeta_Q(s) = \sum_{n=1}^{\infty} r_Q(n) n^{-s}.$$

Asymptotic Behavior

We begin by generalizing the proof that

$$\zeta(s) = \frac{1}{s-1} + O(1).$$

The idea is that the sum $\sum_{n<B} r_Q(n)$ of the representation numbers is one half the number of lattice points inside the ellipse $Q(x,y) = B$, which, as $B \to \infty$, is approximately the area. The change of variables that converts the ellipse to a circle has Jacobian $2/\sqrt{d}$, *independent* of Q, which gives

$$\sum_{n<B} r_Q(n) \sim \frac{\pi}{\sqrt{d}} B.$$

In fact one can show the error is $O(B^{1/2})$. From this and a calculus identity for n^{-s}, we can compute a residue:

$$\begin{aligned}\zeta_Q(s) &= \sum_{n=1}^{\infty} r_Q(n)\, s \int_n^{\infty} x^{-s-1}\,dx \\ &= s\int_1^{\infty}\left(\sum_{n<x} r_Q(n)\right) x^{-s-1}\,dx \\ &= s\int_1^{\infty}\left(\frac{\pi}{\sqrt{d}}x + O(\sqrt{x})\right) x^{-s-1}\,dx.\end{aligned}$$

The integral of the error term is finite for $\sigma > 1/2$, which gives

$$\zeta_Q(s) = \frac{\pi}{\sqrt{d}}\frac{1}{s-1} + O(1). \tag{1}$$

The key fact is that the residue is the same for all forms Q of discriminant $-d$.

This same circle of ideas will actually give us a little more information: there is a constant C so that, for $\sigma > 1$, we have

$$\left|\zeta_Q(s) - a^{-s}\zeta(2s)\right| \le \left(\frac{d}{4}\right)^{1/2-\sigma}\frac{C}{\sigma - 1}. \tag{2}$$

Here's a sketch of a proof. The expression $a^{-s}\zeta(2s)$ is exactly the contribution to $\zeta_Q(s)$ of the terms $Q(m,n)^{-s}$ with $n = 0$. So we want to bound the remaining terms

$$\sum_{m\in\mathbb{Z}}\sum_{n=1}^{\infty} Q(m,n)^{-s}$$

We use the fact that the sum is absolutely convergent for $\sigma > 1$, and the usual trick of comparing a sum to an integral. (There are some things to check carefully here which is why this is only a sketch.) We may as well assume the middle coefficient $b = 0$; if not a real rotation by $\phi = \arctan(b/(a-c))/2$ makes it so, without changing the value of the integral or the discriminant. So we want to estimate

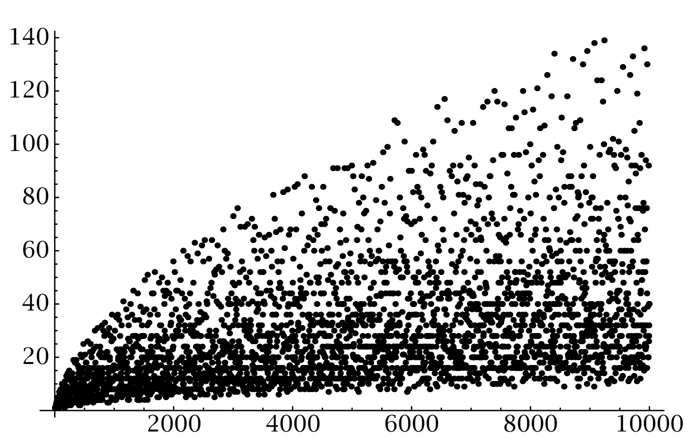


Figure 1. Discriminants vs. class numbers.

$$\iint (ax^2 + cy^2)^{-\sigma}\,dxdy,$$

where the annular region of integration $r > 1, \phi \le \theta \le \phi + \pi$ contains our (rotated) lattice points. A change of variables

$$x = \sqrt{c}\,r\cos(\theta) \qquad y = \sqrt{a}\,r\sin(\theta)$$

with Jacobian $\sqrt{ac}\,r = \sqrt{d/4}\,r$ converts this to

$$\left(\frac{d}{4}\right)^{1/2-\sigma}\pi\int_1^{\infty} r^{1-2\sigma}dr = \left(\frac{d}{4}\right)^{1/2-\sigma}\frac{\pi/2}{\sigma-1}.$$

The point of (2) is that for s in the region $\sigma > 1$, we get

$$\zeta_Q(s) \to a^{-s}\zeta(2s) \qquad \text{as} \qquad \sqrt{d}/a \to \infty. \tag{3}$$

We will see below that we can extend (3) to the larger region $\sigma > 0, s \ne 1$. This is in some sense the reason why the Deuring-Heilbronn phenomenon occurs.

Class Number Formula

The representation numbers $r_Q(n)$ of the individual forms are mysterious, but there is a nice expression for $r_{-d}(n)$, the total number of ways n is represented by *any* form of discriminant $-d$:

$$\sum_{[Q]} r_Q(n) = \sum_{m|n}\chi_{-d}(m). \tag{4}$$

The proof uses the Chinese Remainder Theorem. The right side above is the Dirichlet convolution of the multiplicative functions χ_{-d} and 1, the constant function. Together with (4), this implies

$$\begin{aligned}\sum_{[Q]}\zeta_Q(s) &= \sum_{n=1}^{\infty}\left\{\sum_{m|n}\chi_{-d}(m)\right\} n^{-s} \\ &= \zeta(s)L(s,\chi_{-d}).\end{aligned} \tag{5}$$

The previous calculation (1) of the residues at $s = 1$ for the Epstein zeta function gives us, when we sum over classes $[Q]$, Dirichlet's Analytic Class Number Formula

$$L(1,\chi_{-d}) = \frac{\pi h(-d)}{\sqrt{d}}. \tag{6}$$

From this it is not too hard to prove upper bounds on the class number:

$$h(-d) \ll \log(d)\sqrt{d}.$$

(See [12] for an exposition). Figure 1 shows a scatter plot of discriminants and class numbers for $d < 10000$. You can see the square root upper bound in the upper envelope of the points, roughly a parabola.

Lower bounds are much much harder. Gauss conjectured in Art. 303 of *Disquisitiones Arithmeticae*

> ...the series of [discriminants] corresponding to the same given classification (i.e., the given number of both genera and classes) always seems to terminate with a finite number...There seems to be no doubt that the preceding series does in fact terminate, and by analogy it is permissible to extend the same conclusion to any other classifications...However *rigorous* proofs of these observations seem to be very difficult.

Getting good answers to these questions are still the main open problems in the theory.

Fourier Expansion

So far we have viewed the forms $Q(x, y) = ax^2 + bxy + cy^2$ as discrete objects. A slightly different point of view lets us view them as sitting in a continuous family parametrized by a variable z in the complex upper half plane $\mathcal{H}$. We dehomogenize the form to find roots (Heegner points)

$$az^2 + bz + c = 0, \qquad z = \frac{-b + i\sqrt{d}}{2a}.$$

Then

$$Q(m, n) = a|m - nz|^2,$$

$$\zeta_Q(s) = a^{-s} \sum_{(m,n)}{}' |m - nz|^{-2s}.$$

As a function of z in the complex upper half plane, this is invariant under $z \to z + 1$, since

$$Q(m, n) \sim Q\left((m, n)\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\right),$$

and thus the Epstein zeta function has a Fourier expansion in $x = \mathrm{Re}(z)$.

Aphorism. *[Hecke] A periodic function should always be expanded into its Fourier series.*

In fact, we will end up with a Fourier cosine series since our Epstein zeta function is an even function of the $x = b/2a$ parameter: the forms

$$Q(m, n) = am^2 + bmn + cn^2$$
$$\text{and} \quad Q^{-1}(m, n) = am^2 - bmn + cn^2$$

represent the same integers, so their Epstein zeta functions are identical. The Fourier coefficients will each be a function of the Dirichlet series variable s, as well as $\mathrm{Im}(z) = \sqrt{d}/2a$. To see what the Fourier expansion looks like, we will need the K-Bessel function $K_\nu(y)$, which is the solution to the second order ODE (derivatives with respect to y)

$$u'' + \frac{1}{y}u' - (1 + \frac{\nu^2}{y^2})u = 0$$

which tends to 0 as $y \to \infty$. In fact

$$(7) \quad K_\nu(y) = \left(\frac{\pi}{2y}\right)^{1/2} \exp(-y)\left(1 + O(y^{-1})\right).$$

Also significant for us is that it has the integral representation (Mellin transform)

$$(8) \quad K_\nu(y) = \frac{1}{2}\int_0^\infty \exp(-y/2(\tau + 1/\tau))\tau^\nu \frac{d\tau}{\tau},$$

A good reference for this is [5]. The change of variables $\tau \to \tau^{-1}$ in the integral (8) gives

$$(9) \quad K_\nu(y) = K_{-\nu}(y).$$

Theorem. *[Chowla-Selberg]*

$$\Lambda_Q(s) \stackrel{\text{def.}}{=} \left(\frac{\sqrt{d}}{2\pi}\right)^{s-1/2} \Gamma(s)\zeta_Q(s)$$
$$= T(s) + T(1 - s) + U(s),$$

where

$$T(s) = \left(\frac{\sqrt{d}}{2\pi}\right)^{s-1/2} a^{-s}\Gamma(s)\zeta(2s)$$

and

$$U(s) = \frac{4\sqrt{\pi}}{\sqrt{a}} \sum_{n=1}^{\infty} n^{s-1/2}\sigma_{1-2s}(n) \times K_{s-1/2}\left(\frac{\pi n\sqrt{d}}{a}\right)\cos(n\pi b/a).$$

The divisor function $\sigma_\nu(n)$ is defined by $\sum_{m|n} m^\nu$. Notice that each term $n^{s-1/2}\sigma_{1-2s}(n)$ is invariant under $s \to 1 - s$, as is the K-Bessel function by (9), and this gives as a Corollary the analytic continuation and functional equation

$$\Lambda_Q(s) = \Lambda_Q(1 - s).$$

Proof (extremely sketchy): The appearance of the term $T(s)$ is not surprising; just as before it is exactly the contribution of pairs (m, n) that have $n = 0$. The remaining terms contribute a sum over m in $\mathbb{Z}$ and n in $\mathbb{N}$. Each summand can be written as a Mellin transform, and the sum pulled through the integral. Poisson Summation gives the integral (8) for the K-Bessel function. □

We wrote the Fourier expansion this way, isolating the constant term $T(s) + T(1 - s)$, because this term will be dominant. The details are messy because the implicit constant in (7) depends on s, but it is shown in [1] that for $0 \le \sigma \le 1$

$$|U(s)| \le \frac{2}{d^{1/4}} \frac{\Gamma(\sigma)}{|\Gamma(s)|} \exp(-\pi\sqrt{d}/a).$$

On the critical line $\sigma = 1/2$ one can get from (8) an estimate independent of t:

$$|U(1/2+it)| \leq \frac{6}{d^{1/4}} \exp(-\pi\sqrt{d}/a).$$

An important consequence of this is that we extend the asymptotic behavior (3)

$$\zeta_Q(s) \to a^{-s}\zeta(2s) \quad \text{as} \quad \sqrt{d}/a \to \infty \tag{10}$$

to s in the region $\sigma > 0$, and $s \neq 1$. This has to fail at $s = 1$, of course; $\zeta_Q(s)$ has a pole there and $\zeta(2s)$ does not. It is slightly surprising where this pole appears in our Fourier expansion: it is in the $T(1-s)$ term; the function $\Gamma(s)$ has a simple pole at $s = 0$.

The Siegel Zero and Consequences

Hecke's aphorism pays off now by showing us a deep connection between the arithmetic and the analysis in the following three theorems.

Theorem. *[Chowla-Selberg] For $d/a^2 > 200$,*

(i) The Epstein zeta function has a real zero in $(1/2, 1)$.
(ii) If $h(-d) = 1$, then the Generalized Riemann Hypothesis (GRH) is false.

Proof: We start by computing $\Lambda_Q(s)$ at $s = 1/2$. Although the term $\zeta(2s)$ has a pole at $s = 1/2$, a calculation will show that $T(s) + T(1-s)$ has a removable singularity at $s = 1/2$:

$$\zeta(2s) = \frac{1/2}{s-1/2} + \gamma + O(s-1/2)$$
$$\Gamma(s) = \sqrt{\pi} + \sqrt{\pi}(-\gamma - \log(4))(s-1/2) + O(s-1/2)^2$$
$$a^{-s}\left(\frac{\sqrt{d}}{2\pi}\right)^{s-1/2} = \frac{1}{\sqrt{a}} + \frac{\log(\sqrt{d}/(2\pi a))}{\sqrt{a}}(s-1/2) + O(s-1/2)^2.$$

Here γ is Euler's constant, and these calculations are classical enough that *Mathematica* can do them for us. So $T(s)$ is

$$\frac{\sqrt{\pi/a}/2}{s-1/2} + \sqrt{\pi/a}/2\left(\gamma + \log(\sqrt{d}/(8\pi a))\right) + O(s-1/2).$$

Adding the expansion of $T(1-s)$ kills off all powers of $s - 1/2$ with odd exponent, including the pole, and we find

$$T(s) + T(1-s)|_{s=1/2} = \sqrt{\pi/a}\left(\gamma + \log(\sqrt{d}/(8\pi a))\right).$$

Since $U(1/2)$ is exponentially small, $\Lambda_Q(1/2) > 0$ for $d/a^2 \gg 1$ (in fact bigger than 200). But recall

$$\zeta_Q(s) = \frac{\pi}{\sqrt{d}} \cdot \frac{1}{s-1} + O(1)$$

is negative for $s \to 1^-$. By the Intermediate Value Theorem, $\zeta_Q(s)$ has a real zero in $(1/2, 1)$. This proves (i).

Now suppose $h(-d) = 1$, so by genus theory d is a prime congruent to 3 mod 4, and

$$Q(x, y) = x^2 + xy + \frac{1+d}{4}y^2, \qquad a = 1.$$

We make use of (5), which now says $\zeta_Q(s) = \zeta(s)L(s, \chi_{-d})$. The fact that the Riemann zeta function has no real zeros in $(0, 1)$ follows from Euler's identity

$$(1 - 2^{1-s})\zeta(s) = \sum_{n=1}^{\infty} (-1)^{n-1} n^{-s}.$$

The series converges for real $s > 0$ by the Alternating Series Test; now group the terms in pairs to see that the sum is positive. So $L(s, \chi_{-d})$ has a real zero in $(1/2, 1)$, a *Siegel zero*.

The Argument Principle tells us the number of zeros minus the number of poles, inside the circle of say radius $1/4$ around, $s = 1$, is given by

$$\frac{1}{2\pi i}\int_{|s-1|=1/4} \frac{\zeta_Q'(s)}{\zeta_Q(s)} ds.$$

By (10), for sufficiently large d this is

$$= \frac{1}{2\pi i}\int_{|s-1|=1/4} \frac{\zeta'(2s)}{\zeta(2s)} ds = 0$$

because $\zeta(2s)$ has neither zeros nor poles near $s = 1$. Of course one has to justify passing the limit through the integral. The convergence is not uniform, but it is uniform on a compact set containing the path of integration. Since the Epstein zeta function has one simple pole, there is only one Siegel zero close to $s = 1$. □

In fact, you can say even more about the location of this zero:

Exercise. *Use* Mathematica *to compute the Laurent expansion*

$$T(s) + T(1-s) = \frac{c_{-1}}{s-1} + c_0 + O(s-1)$$

around $s = 1$. Neglecting the higher order terms and the $U(s)$ contribution, show the zero is at $1 - \beta$ for

$$\beta \sim \frac{6a}{\pi\sqrt{d}} \qquad \text{as} \qquad \sqrt{d}/a \to \infty. \qquad \square$$

Still under the hypothesis $h(-d) = 1$ and $d \gg 1$,

Theorem. *[Deuring] Except for the Siegel zero, all other zeros of $\Lambda_Q(s)$ in $0 < \sigma < 1$, $0 \leq t < \sqrt{d}$ have real part $\sigma = 1/2$. This is the "Local GRH".*

Outline of Proof: Write

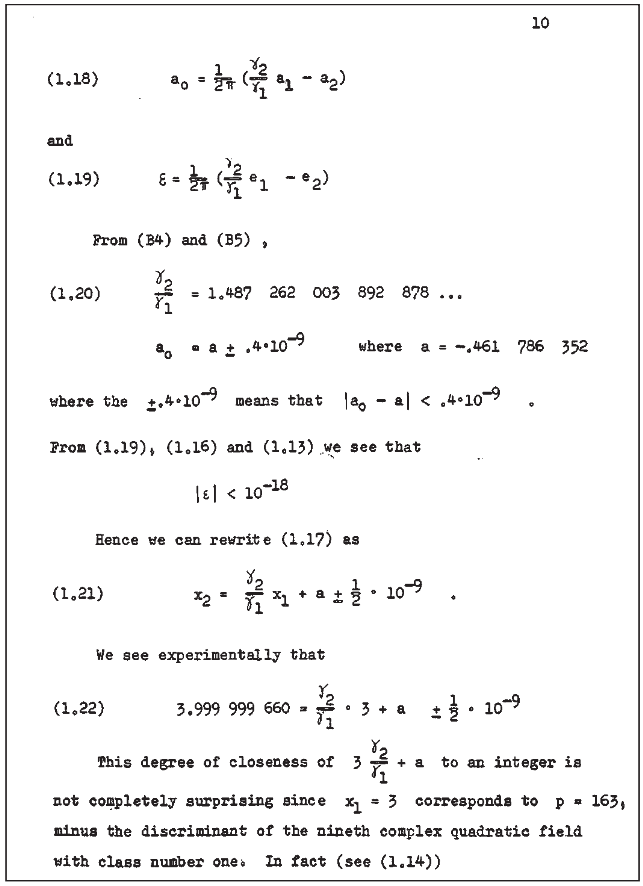
$$T(s) + T(1-s) = T(s)\left(1 + \frac{T(1-s)}{T(s)}\right).$$

The term $|T(s)|$ is never zero for $\sigma \geq 1/2$: $\Gamma(s)$ has no zeros, and the fact that $\zeta(s)$ is nonzero for

$\sigma \geq 1$ is equivalent to the Prime Number Theorem.

1. Deuring shows $|T(1-s)/T(s)| < 1$ as long as $\sigma > 1/2$ and $|s-1| > 1/4$. This depends on known estimates for $\Gamma(s)$ and $\zeta(2s)$ but basically works because for d large relative to t, the $(\sqrt{d}/2\pi)^{s-1/2}$ term dominates, as long as we stay away from the pole of $T(1-s)$ at $s=1$. Thus
$$T(s) + T(1-s) = 0 \quad \Rightarrow \quad \sigma = 1/2.$$
2. Next, he shows these zeros are simple as follows: Write $f(s)$ for $T(1-s)/T(s)$, so the zeros occur at $\rho = 1/2 + it$ such that $f(\rho) = -1$. Then
$$\frac{d}{ds} T(s)(1 + f(s))\bigg|_{s=\rho} = T(\rho) f'(\rho).$$
For
$$f'(\rho) = -\frac{f'(\rho)}{f(\rho)} = \frac{\frac{d}{ds}T(s)}{T(s)} - \frac{\frac{d}{ds}T(1-s)}{T(1-s)}\bigg|_{s=\rho}$$
he can get a lower bound $f'(\rho) \gg \log(d)$ from known estimates for the logarithmic derivative of the Riemann zeta function, as long as t is small enough relative to d. Similarly, known estimates give $T(\rho) \gg 1/\log(d)$.
3. He then shows that
$$|T(s) + T(1-s)| > |U(s)|,$$
and so by Rouché's Theorem $\Lambda_Q(s)$ has the same number of zeros in the box $0 < \sigma < 1$, $0 \leq t < \sqrt{d}$ as does $T(s) + T(1-s)$. Here he needs $t < \sqrt{d}$, since by Stirling's formula, $\Gamma(s)$ also has exponential decay as t increases.
4. Around each zero ρ of $T(s) + T(1-s)$, Deuring puts a circle of radius $\exp(-\pi\sqrt{d})$ and uses the Cauchy Integral Formula to get upper bounds on the Taylor series coefficients
$$T(s) + T(1-s) = c_1(s-\rho) + \sum_{n=2}^{\infty} c_n (s-\rho)^n$$
of the form $c_n \ll K^n$. By (2) above he already has $c_1 \gg 1$. The triangle inequality and summing a geometric series gives $T(s) + T(1-s) \gg \exp(-\pi\sqrt{d})$ on the circle. He can then apply Rouché's Theorem again to see $\Lambda_Q(s)$ has one zero in that circle. Since any zeros off the line would come in symmetric pairs (by the functional equation $s \to 1-s$), that zero is on the line. □

Deuring's theorem is quite strong. Since the zeros of $\zeta(s)$ are a subset of the zeros of $\zeta_Q(s)$ whenever $h(-d) = 1$ by (5), as a Corollary we get that *either* there are only finitely many d with $h(-d) = 1$, *or* the Riemann hypothesis for $\zeta(s)$ is true! Of course it is now known that the former is true, but Deuring's theorem was an essential first step in solving the problem.[1]



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(1.18) $a_0 = \frac{1}{2\pi}(\frac{\gamma_2}{\gamma_1} a_1 - a_2)$

and

(1.19) $\varepsilon = \frac{1}{2\pi}(\frac{\gamma_2}{\gamma_1} e_1 - e_2)$

From (B4) and (B5),

(1.20) $\frac{\gamma_2}{\gamma_1} = 1.487\ 262\ 003\ 892\ 878 \ldots$

$a_0 = a \pm .4 \cdot 10^{-9}$ where $a = -.461\ 786\ 352$

where the $\pm .4 \cdot 10^{-9}$ means that $|a_0 - a| < .4 \cdot 10^{-9}$.

From (1.19), (1.16) and (1.13) we see that

$|\varepsilon| < 10^{-18}$

Hence we can rewrite (1.17) as

(1.21) $x_2 = \frac{\gamma_2}{\gamma_1} x_1 + a \pm \frac{1}{2} \cdot 10^{-9}$.

We see experimentally that

(1.22) $3.999\ 999\ 660 = \frac{\gamma_2}{\gamma_1} \cdot 3 + a \pm \frac{1}{2} \cdot 10^{-9}$

This degree of closeness of $3\frac{\gamma_2}{\gamma_1} + a$ to an integer is not completely surprising since $x_1 = 3$ corresponds to $p = 163$, minus the discriminant of the nineth complex quadratic field with class number one. In fact (see (1.14))

Figure 2. A page from Stark's thesis.

Folklore Theorem. *[Deuring, Heilbronn] In the presence of a Siegel zero, the low-lying zeros $s = 1/2 + it$ of $L(s, \chi_{-d})$ are very regularly spaced:*

$$(11) \qquad t \sim \frac{\pi}{\log(\sqrt{d}/2\pi)} \cdot n, \quad \text{for integer } n.$$

Idea of proof: We make use of the fact that for $s = 1/2 + it$, $1 - s = \overline{s}$. The zeros of $L(s, \chi_{-d})$ are zeros of $\Lambda_Q(s)$, which we have seen are very near the zeros of

$$\begin{aligned} &T(s) + T(1-s) = \\ &T(s) + T(\overline{s}) = T(s) + \overline{T(s)} \\ =&2\mathrm{Re}(T(s)) \\ =&2\,|T(s)| \cos\left(\arg\left((\sqrt{d}/2\pi)^{it}\Gamma(s)\zeta(2s)\right)\right), \end{aligned}$$

We saw above that the term $|T(s)|$ is never zero on the line $\sigma = 1/2$. Meanwhile, $\Gamma(s)$ is very near to real for $t \ll 1$, so does not contribute much to the argument. And

$$\zeta(2s) = \frac{1/2}{s - 1/2} + \gamma + O(s - 1/2)$$

[1] *An editorial comment: this result made me have a lot more respect for Rouché's Theorem, which previously I thought existed only to provide problems for analysis qualifying exams.*

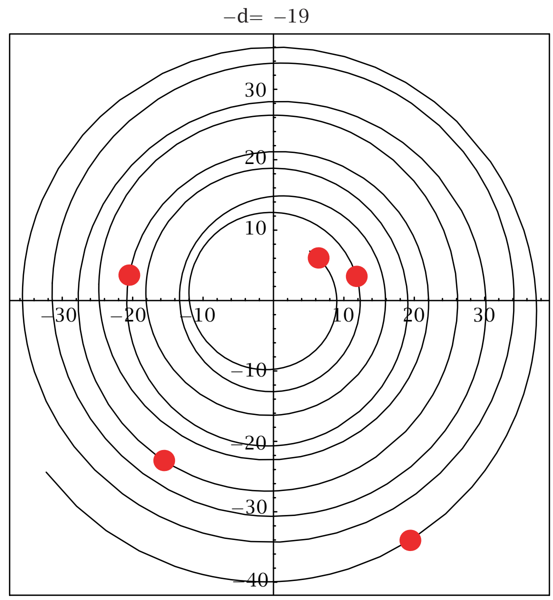

Figure 3. $d = 19$

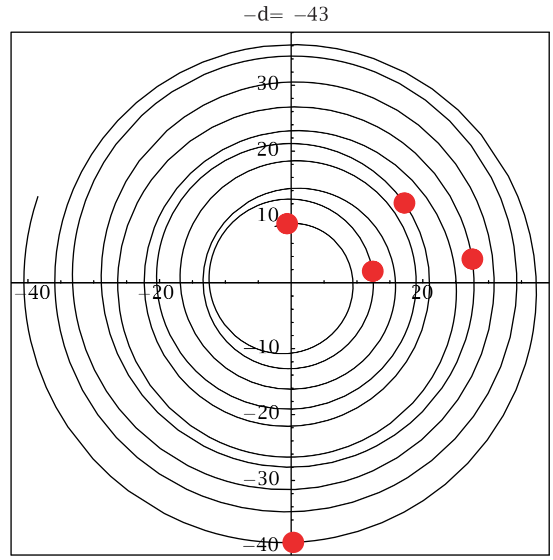

Figure 4. $d = 43$

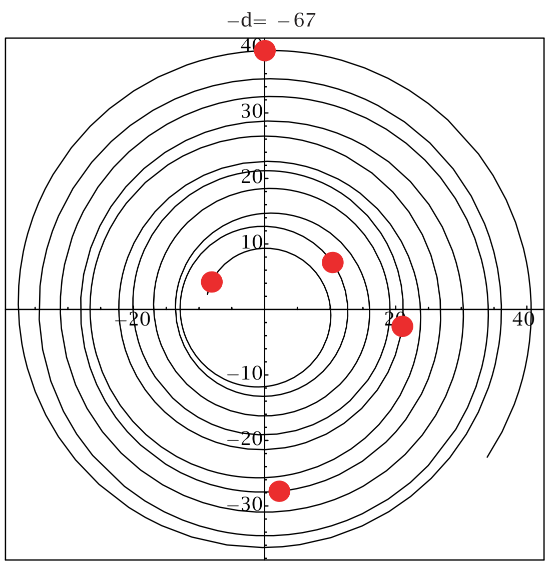

Figure 5. $d = 67$

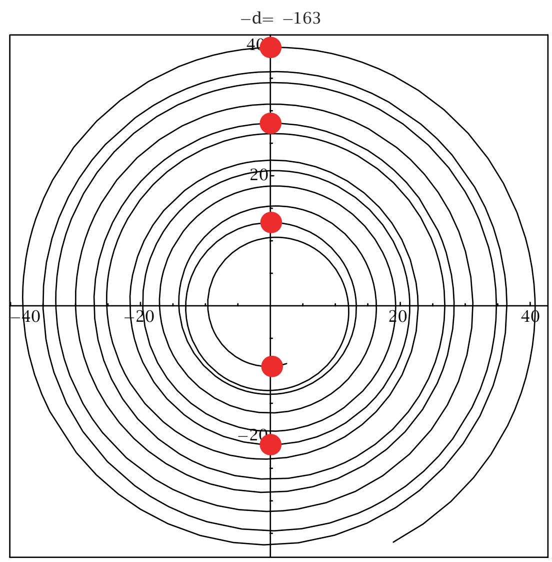

Figure 6. $d = 163$

so $\arg(\zeta(2s)) \sim -\pi/2$ for $s = 1/2 + it \sim 1/2$. Finally

$$\arg\left(\sqrt{d}/2\pi\right)^{it} = t\log\left(\sqrt{d}/2\pi\right),$$

and

$$\cos\left(t\log\left(\sqrt{d}/2\pi\right) - \pi/2\right) = \sin\left(t\log\left(\sqrt{d}/2\pi\right)\right).$$

So

$$t\log\left(\sqrt{d}/2\pi\right) \sim n\pi.$$

(Actually, it is not enough above that $|T(s)|$ is nonzero. To make this argument precise, we need to estimate a lower bound.) To the extent that one can bound the tail of the Fourier expansion, every n above corresponds to a zero by Rouché's Theorem. □

The other factor $\zeta(s)$ of $\zeta_Q(s)$ has no low-lying zeros, but the same analysis shows that the zeros $\rho = 1/2 + i\gamma$ of $\zeta(s)$ with $\gamma \ll \sqrt{d}$ make $T(\rho)$ nearly pure imaginary.

Experimental Observations I

Even in the absence of a Siegel zero, can one see this effect for a class number extremely small relative to its discriminant? Stark, in his 1964 Ph.D. thesis, was the first to investigate this numerically; see Figure 2 for a cryptic comment.

For a graphical interpretation, one can use *Mathematica* to plot

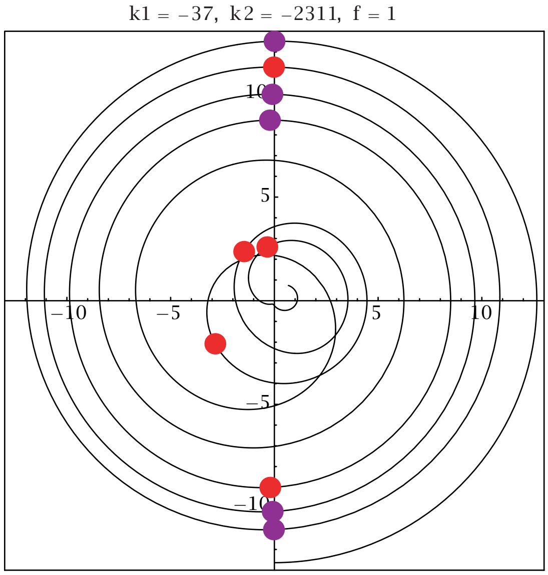

Figure 7. $d = 85507 = 2311 \cdot 37$

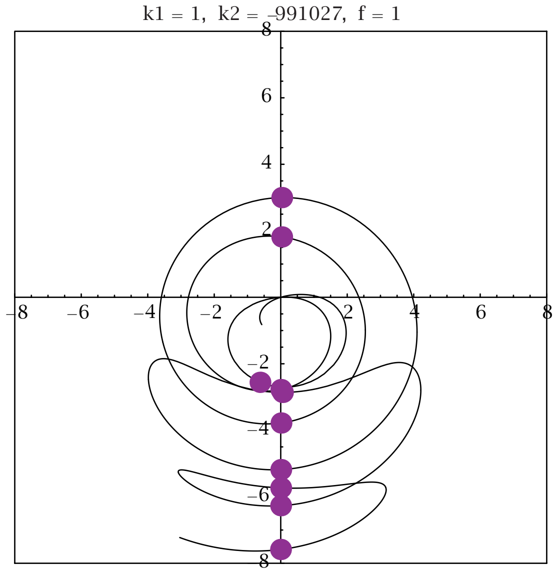

Figure 8. $d = 991027$

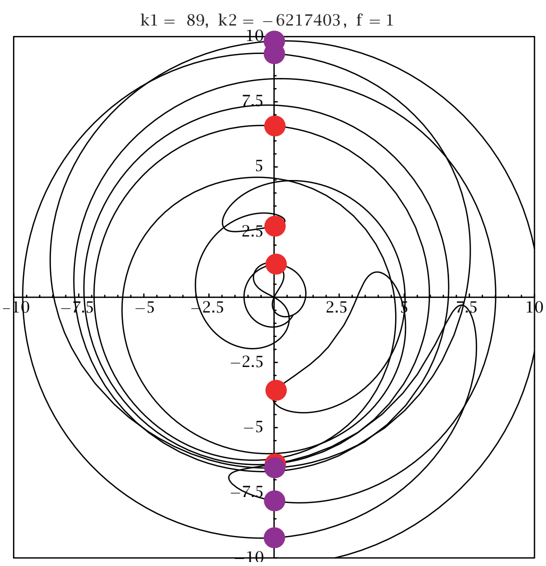

Figure 9. $d = 553348867 = 5507 \cdot 89 \cdot 1129$

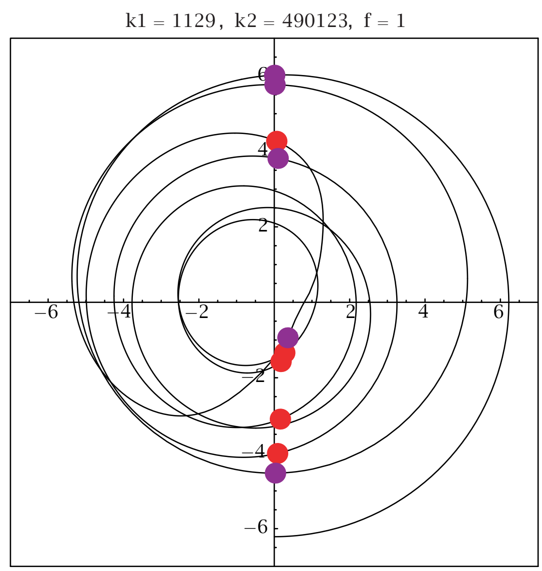

Figure 10. $d = 553348867 = 5507 \cdot 89 \cdot 1129$

$$T(s) = \left(\frac{\sqrt{d}}{2\pi}\right)^{s-1/2} \Gamma(s)\zeta(2s)$$

on the critical line $s = 1/2 + it$ for various values of d that have $h(-d) = 1$. How close to the imaginary axis is $T(\rho)$ for ρ that are known zeros of $\zeta(s)$? Figures 3–6 show several examples for various d, with the same five lowest zeros ρ of $\zeta(s)$ indicated in red on each. Since Stirling's Formula for $\Gamma(s)$ makes $|T(s)|$ decay exponentially as t increases, one does not see the function "wrap around" the origin, so I have renormalized the absolute value by taking the logarithm, without changing the argument. Increasing t corresponds to spiraling in counterclockwise.

Figure 6 shows graphically what Stark was referring to: the zeros of $\zeta(s)$ can "see" the extremal discriminant -163. This is, as Stark later showed, the largest d with $h(-d) = 1$.

L-function Magic

Since the class number problem $h(-d) = 1$ is already solved, we want to think about lower bounds in general. In this case the Dedekind zeta function

$$\sum_{[Q]} \zeta_Q(s) = \zeta(s)L(s, \chi_{-d})$$

is a sum over all classes of Epstein zetas. Whenever $d/a^2 > 200$, the corresponding $\zeta_Q(s)$ has a zero in $(1/2, 1)$ by Chowla-Selberg. This does not contradict any GRH, since the individual Epstein zetas do

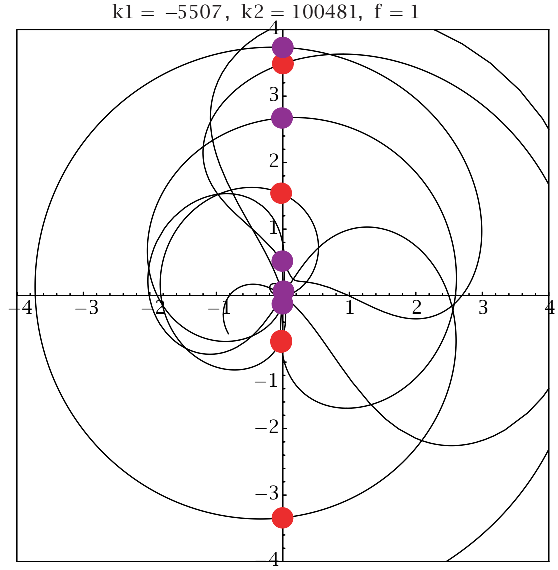

Figure 11. $d = 553348867 = 5507 \cdot 89 \cdot 1129$

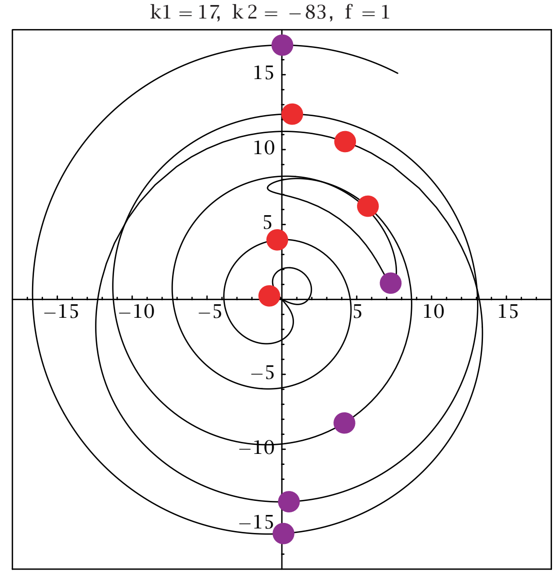

Figure 12. $d = 1411 = 83 \cdot 17$

not have Euler products over primes, only the sum does.

The reduced representatives of the forms can have a as big as $\sqrt{d/3}$, which means the exponential bound on the tail $U(s)$ in the Fourier expansion can be as big as $\exp(-\pi\sqrt{3}) \approx .0044$ for each of the $h(-d)$ terms. We still get the benefit of the $d^{1/4}$ term in the denominator, though, so if $h(-d)$ is very small relative to $\sqrt{d}$ (i.e., much smaller than $d^{1/4}$), then the sum of the tails is still small. Under this hypothesis $L(s, \chi_{-d})$ has a Siegel zero, and the Deuring-Heilbronn Phenomenon reappears as before: we get the Local GRH, and the low-lying zeros are uniformly spaced just as in (11). But more is true.

Given a homomorphism on the class group

$$\psi : C(-d) \to \mathbb{C}^\times$$

we can form an L-function

$$L(s, \psi) = \sum_{[Q]} \psi(Q)\zeta_Q(s).$$

More generally, for an auxiliary discriminant f we can "twist" the Epstein zeta function

$$\zeta_Q(s, \chi_f) = \sum_{n=1}^{\infty} \chi_f(n) r_Q(n) n^{-s}$$

and form

$$L(s, \psi \cdot \chi_f) = \sum_{[Q]} \psi(Q)\zeta_Q(s, \chi_f).$$

For *any* odd fundamental discriminant k_1, we can make the Dirichlet L-function $L(s, \chi_{k_1})$ appear as a factor of such an expression, by careful choice of ψ and f: Factor $-d = D_1 \cdot D_2$ as a product of fundamental discriminants, in such a way that $D_1 = \gcd(k_1, d)$. By a theorem of Kronecker, this factorization corresponds to a genus character ψ with

$$L(s, \psi) = L(s, \chi_{D_1}) L(s, \chi_{D_2}), \tag{12}$$

generalizing (5). In fact the genera of quadratic forms mentioned above are exactly the cosets of the class group $C(-d)$ modulo the subgroup $C(-d)^2$ of squares of classes. The genus group is therefore a product of copies of $\mathbb{Z}/2$, and Gauss showed the number of terms is $g - 1$, where g is the number of prime factors of d. The corresponding genus characters are exactly those taking only the values ± 1.

By a comparison of the corresponding Dirichlet series in (12)

$$\sum_{[Q]} \psi(Q) \sum_{n>0} r_Q(n) n^{-s} = \sum_{n>0} \left\{ \sum_{c|n} \chi_{D_1}(c)\chi_{D_2}(n/c) \right\} n^{-s},$$

and the uniqueness of Dirichlet series coefficients, we deduce that for all n,

$$\sum_{[Q]} \psi(Q) r_Q(n) = \sum_{c|n} \chi_{D_1}(c)\chi_{D_2}(n/c). \tag{13}$$

This is a generalization of (4).

Theorem. *Let $f = k_1/D_1$, $k_2 = fD_2$. Then*

$$L(s, \chi_{k_1}) L(s, \chi_{k_2}) = L(s, \psi \cdot \chi_f). \tag{14}$$

Proof: The idea goes back to Heilbronn:

$$\begin{aligned} L(s, \psi \cdot \chi_f) &= \sum_{[Q]} \psi(Q)\zeta_Q(s, \chi_f) \\ &= \sum_{[Q]} \psi(Q) \sum_{n>0} \chi_f(n) r_Q(n) n^{-s} \\ &= \sum_{n>0} \chi_f(n) \left\{ \sum_{[Q]} \psi(Q) r_Q(n) \right\} n^{-s}. \end{aligned}$$

By (13) we get

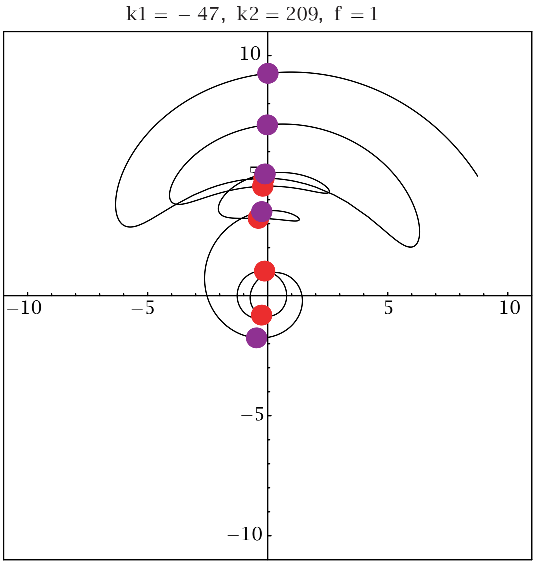

Figure 13. $d = 9823 = 11 \cdot 19 \cdot 47$

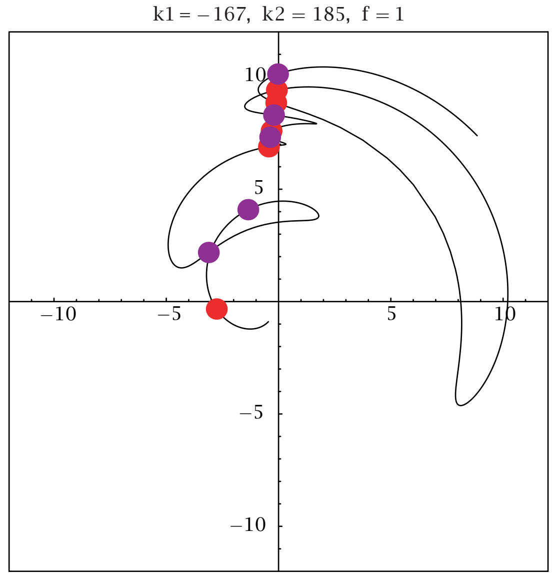

Figure 14. $d = 30895 = 167 \cdot 5 \cdot 37$

$$\begin{aligned} L(s,\psi\cdot\chi_f) &= \sum_{n>0}\chi_f(n)\left\{\sum_{c|n}\chi_{D_1}(c)\chi_{D_2}(n/c)\right\}n^{-s}\\ &= \sum_{n>0}\left\{\sum_{c|n}\chi_f(c)\chi_{D_1}(c)\chi_f(n/c)\chi_{D_2}(n/c)\right\}n^{-s}\\ &= \sum_{n>0}\left\{\sum_{c|n}\chi_{k_1}(c)\chi_{k_2}(n/c)\right\}n^{-s}\\ &= L(s,\chi_{k_1})L(s,\chi_{k_2}). \end{aligned}$$

□

From the right side of (14), we have a Fourier expansion

$$\left(\frac{|f|\sqrt{d}}{2\pi}\right)^{s-1/2}\Gamma(s)L(s,\chi_{k_1})L(s,\chi_{k_2}) = T(s)+T(1-s)+U(s)$$

similar to the previous one, with now

$$T(s) = \left(\frac{|f|\sqrt{d}}{2\pi}\right)^{s-1/2}\Gamma(s)\zeta(2s)P_f(s)A(s),$$

and

$$P_f(s) = \prod_{p|f}(1-p^{-2s}), \qquad A(s) = \sum_{[Q]}\psi(Q)\chi_f(a)a^{-s}.$$

In fact in the examples below, we take $f = 1$. In this case we have a linear combination of the Chowla-Selberg Fourier expansions; the coefficients are merely the character values $\psi(Q)$.

If the class number $h(-d)$ is too small, or $L(s,\chi_{-d})$ has a Siegel zero, the Deuring-Heilbronn phenomenon (11) appears for $L(s,\chi_{k_1})$ as well, as long as f is not too big.

Experimental Observations II

As in the case of $h(-d) = 1$, we can plot $T(s)$ on the line $s = 1/2 + it$ and see where the zeros of $L(s,\chi_{k_1})$ and $L(s,\chi_{k_2})$ end up. For "extreme" values of $-d$, will they tend towards the imaginary axis? Figures 7–11 show some examples with small class number. The lowest five zeros of $L(s,\chi_{k_1})$ are shown in red, while the lowest five zeros of $L(s,\chi_{k_2})$ are shown in blue. (In some cases not all five are visible if they lie nearly on top of each other.)

1. The discriminant $-85507 = -2311 \cdot 37$ has class group isomorphic to $\mathbb{Z}/22$, so $h(-d)/\sqrt{d} \approx .075$. There is one nontrivial genus character.
2. The discriminant -991027 has class group isomorphic to $\mathbb{Z}/63$; there is only the trivial genus character. This is the famous example of Shanks; $h(-d)/\sqrt{d} \approx .063$ which minimizes this ratio for all $d < 10^8$. (Because the zeros of $\zeta(s) = L(s,\chi_{k_1})$ are so high up, we show instead 10 zeros of $L(s,\chi_{-d}) = L(s,\chi_{k_2})$ in Figure 8.)
3. The discriminant $-553348867 = -5507 \cdot 89 \cdot 1129$ has class group isomorphic to $\mathbb{Z}/732 \times \mathbb{Z}/2$, so $h(-d)/\sqrt{d} \approx .062$. There are three nontrivial genus characters.

Observe that in each of the examples, the parameter f, which in some sense measures the correlation between $-d$ and the auxiliary discriminant k_1, is as small as possible: $f = 1$ or in other words $k_1|d$. Even so, the correlation between the zeros is by no means trivial. There is no obvious relation between the class groups $C(k_1)$ and $C(-d)$; one is not a direct factor of the other.

Of course, these examples are hand picked to show off this tendency towards the Deuring-Heilbronn phenomenon; in general one sees nothing like this. In the next section below we discuss contemporary conjectures about the distribution of zeros.

It is also interesting to look at some examples of discriminants $-d$ that are famous for $L(s,\chi_{-d})$ having a very low-lying zero. Figures 13–17 show

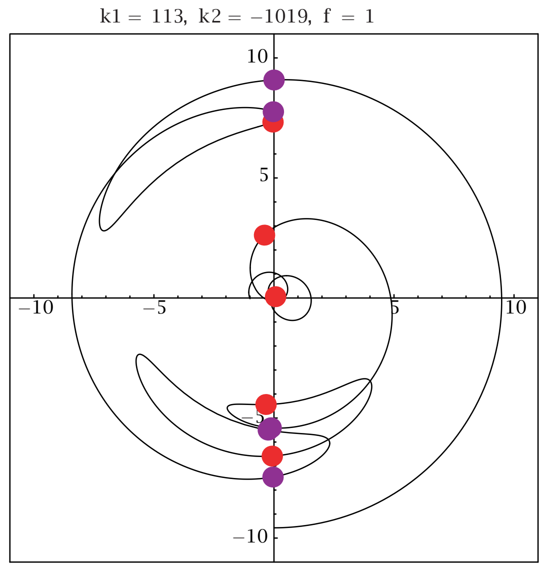

Figure 15. $d = 115147 = 1019 \cdot 113$

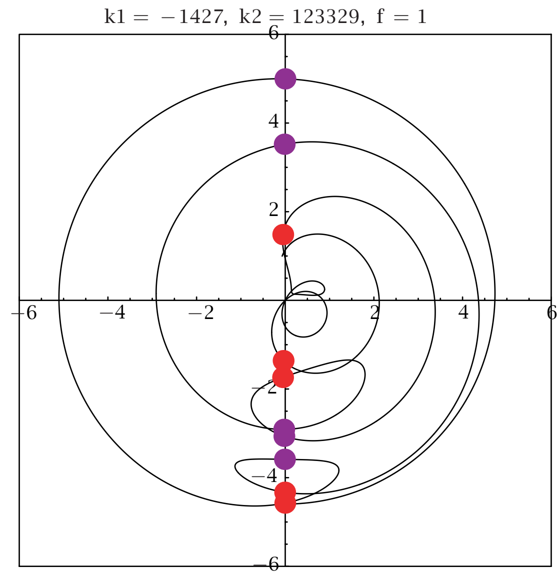

Figure 16. $d = 175990483 = 19 \cdot 1427 \cdot 6491$

some examples. The low-lying zero does not appear in the figures, rather it is "causing" the Deuring-Heilbronn phenomenon.

1. The L-function $L(s, \chi_{-1411})$ has a zero at $s = 1/2 + i \cdot 0.077967 \ldots$ The discriminant $-1411 = -83 \cdot 17$ has one nontrivial genus character.
2. The L-function $L(s, \chi_{-9823})$ has a zero at $s = 1/2 + i \cdot 0.058725 \ldots$ The discriminant $-9823 = -11 \cdot -19 \cdot -47$ has three nontrivial genus characters.
3. The L-function $L(s, \chi_{-30895})$ has a zero at $s = 1/2 + i \cdot 0.018494 \ldots$ The discriminant $-30895 = -167 \cdot 5 \cdot 37$ also has three nontrivial genus characters.
4. The L-function $L(s, \chi_{-115147})$ has a zero at $s = 1/2 + i \cdot 0.003158 \ldots$ The discriminant $-115147 = -1019 \cdot 113$ has one nontrivial genus character.
5. $L(s, \chi_{-175990483})$ has a zero at $s = 1/2 + i \cdot 0.000475 \ldots$ The discriminant $-d = -175990483 = -19 \cdot -1427 \cdot -6491$ has three nontrivial genus characters.

In both sets of examples, $T(\rho)$ is very nearly pure imaginary for zeros ρ of $L(s, \chi_{k_1})$ or $L(s, \chi_{k_2})$, and so $T(\rho) + T(1 - \rho)$ is also very near 0. Necessarily this means that the tail of the Fourier expansion $U(\rho)$ is also very near 0, much smaller than our estimate $O\left(h(-d)d^{-1/4}\right)$.

It would be nice to have examples where the zeros ρ not only forced $T(s)$ to be nearly purely imaginary, but also were restricted to near integer multiples of $\pi / \log(\sqrt{d}/2\pi)$ as in (11). This would require $\arg(\Gamma(s)\zeta(2s))$ to be very near its limiting value $-\pi/2$, and thus k_1 very large in order that $L(s, \chi_{k_1})$ have several zeros so low. But this may allow $f > 1$ as well.

Can You Hear the Class Number?

In 1966 Mark Kac posed the question, "Can you hear the shape of a drum?" In fact what one hears are solutions to the wave equation, i.e., eigenvalues of the Laplace operator. The mathematical meaning of Kac's question is, what does this spectrum determine about the geometry? In the very useful analogy between spectral geometry and number theory, eigenvalues of the Laplacian correspond to zeros of L-functions, while geometric properties correspond to properties of primes. It is very interesting that the Deuring-Heilbronn phenomenon (11), if it occurs, corresponds in this analogy to frequencies in harmony.

Above I mentioned Stark's Ph.D. thesis, in which he used precise values of zeros of the Riemann zeta function to show that a certain range of discriminants did not have $h(-d) = 1$. He later extended this to the problem of $h(-d) = 2$. Montgomery and Weinberger used low-lying zeros of auxiliary $L(s, \chi_{k_1})$ to attack $h(-d) = 2$ and 3 in [6]. This work led Montgomery to investigate the question, "If GRH is true and there are no Siegel zeros and no Deuring-Heilbronn phenomenon, what *is* the vertical distribution of the zeros on the critical line?" Remarkably, he discovered [7] that the "pair correlation" of the zeros is the same as that for the eigenvalues of random unitary matrices, the Gaussian Unitary Ensemble (GUE). Montgomery's proof works only for a restricted range of test functions, not in general, but the GUE hypothesis is also supported by the statistics of 10 billion zeros of the zeta function computed by Odlyzko [8]. This suggests the zeros of L-functions may indeed have a spectral interpretation, as conjectured by Hilbert and Pólya.

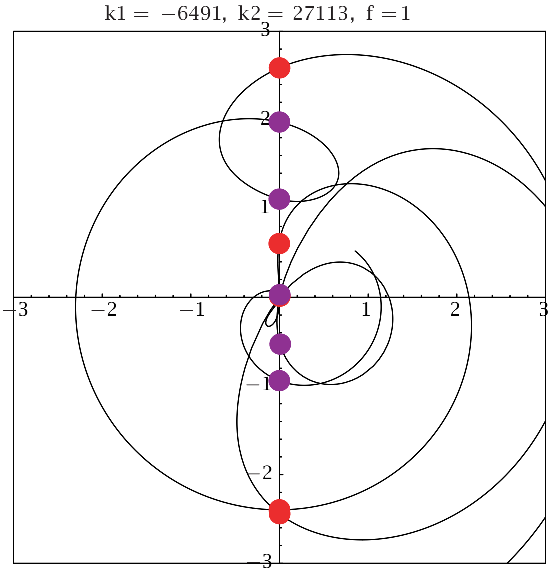

Figure 17. $d = 175990483 = 19 \cdot 1427 \cdot 6491$

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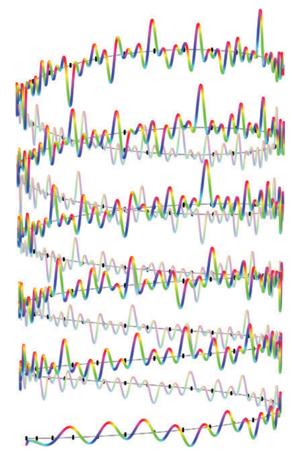
About the Cover

Z(t) on the critical line

Two articles in this issue are concerned with ζ-functions, the review of Dan Rockmore's book and the article by Jeffrey Stopple. This month's cover displays the behavior of $\zeta(s)$ on the critical line $\Re(s) = 1/2$. The function

$$\xi(s) = \Gamma(s/2)\pi^{-s/2}\zeta(s)$$

satisfies the functional equation $\xi(s) = \xi(1-s)$ and therefore takes real values on the critical line. If $\vartheta(t)$ is the argument of $\Gamma(1/4 + it/2)\pi^{-it/2}$ then $Z(t) = e^{i\vartheta(t)}\zeta(1/2 + it)$ also takes real values, and this is what is graphed along the helix. Lengths of the natural unit 2π are marked. The colors display the angle $\vartheta(t)$.



The behavior of $Z(t)$ encodes, in principle, the mysterious distribution of prime numbers, and it is hard to look at its graph without trying to read a message from it. But then humans are always trying to read meaning into random patterns.

—Bill Casselman,
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(notices-covers@ams.org)