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1. Vinogradov’s ‘less than less than’

Although you might not notice it, all of differential calculus is about a single idea: complicated functions can often be approximated, on a small scale anyway, by straight lines. Almost everything that happens in calculus is an application of this idea. In order to make sense of this idea of ‘approximation’, we need a language to compare functions. Generally we will want to replace complicated functions by simpler ones.

In comparing two functions \( f(x) \) and \( h(x) \) we define a new relation \( \ll \) (less than less than). We say

\[
f(x) \ll h(x) \quad \text{as} \quad x \to a
\]

if there is some positive constant \( C \) and some interval around \( a \) so that

\[
|f(x)| \leq C|h(x)|
\]

when \( x \) is in the interval. If \( h(x) \) happens to be always positive, a useful special case, this is equivalent to

\[
-Ch(x) \leq f(x) \leq Ch(x).
\]

For now we will use what we know about calculus to prove inequalities.

For example, it is true that

\[
x^3 + x^2 \ll x^2 \quad \text{as} \quad x \to 0,
\]

since \( x^3 + x^2 = x^2(x + 1) \), and \( C = 2 \) will work:

\[
|x^2(x + 1)| \leq 2|x^2| \quad \text{exactly when} \quad |x + 1| \leq 2
\]

The inequality on the right holds if \( |x| < 1 \).

The point of \( \ll \) notation is to simplify complicated expressions, and suppress constants we don’t care about. For that reason, there’s no point in worrying about the smallest choice of \( C \).

**Exercise 1.** Show the following relations for \( x \to 0 \). In each case, give an explicit value for the constant \( C \), as well as the interval around the base point \( a = 0 \).

\[
2x^2 \ll x
\]

\[
1 - x^2 \ll 1 + x
\]

\[
x^5 + 2x^3 \ll x^3
\]
Exercise 2. Show that
\[ \sin(x) \ll x \quad \text{as} \quad x \to 0. \]

Hint: Think about tangent lines and concavity.

Here’s a useful trick for dealing with inequalities: If \( F(x) \) is an increasing function, then \( a \leq b \) is true exactly when \( F(a) \leq F(b) \) is true. This is just the definition of ‘increasing’.

Exercise 3. Show that
\[ \sqrt{1 - x^2} \ll \sqrt{1 + x} \quad \text{as} \quad x \to 0. \]

2. Landau’s ‘Big Oh’

We need a new concept for two functions \( f(x) \) and \( g(x) \) that are about the same size, up to some error term or fudge factor of size \( h(x) \). We say

\[ f(x) = g(x) + O(h(x)) \quad \text{if} \quad f(x) - g(x) \ll h(x) \quad \text{as} \quad x \to a. \]

This is pronounced ‘\( f(x) \) is \( g(x) \) plus Big Oh of \( h(x) \)’.

For example, we will show
\[ \exp(x) = 1 + O(x) \quad \text{as} \quad x \to 0. \]

From the definition we need to show that for some \( C, \) \(|\exp(x) - 1| \leq C|x|\) on some interval around \( x = 0 \). From the definition of absolute values, this is equivalent to

\[ -C|x| \leq \exp(x) - 1 \leq C|x|. \]

It will turn out that \( C = 2 \) works again.

First consider the case of \( x \geq 0 \). Since \( \exp(x) \) is increasing and \( e^0 = 1, \exp(x) - 1 \) is certainly \( \geq 0 \) for \( x \geq 0 \), so the first inequality: \(-2x \leq \exp(x) - 1 \) is trivial. The other inequality is \( \exp(x) - 1 \leq 2x \), which is the same as \( \exp(x) \leq 2x + 1 \), which must now be proved.

At \( x = 0 \) it is true that \( 1 = e^0 \leq 2 \cdot 0 + 1 = 1 \). By taking the derivative of \( \exp(x) \) at \( x = 0 \) we find the slope of the tangent line is 1, less that that of the line \( 2x + 1 \). So \( \exp(x) \) lies under the line \( 2x + 1 \) at least for a little way.

Exercise 4. Show that, for \( x \leq 0 \), (where \(|x| = -x|\))
\[ 2x \leq \exp(x) - 1 \leq -2x. \]

This finishes the example.
For another example, we will show that
\[
\cos(x) = 1 + O(x^2) \quad \text{as } x \to 0.
\]
This is saying that for some \( C \), \(|\cos(x) - 1| \leq C|x^2|\), or, since \( x^2 \geq 0 \) always we can write \( x^2 \) instead. We must show
\[
-Cx^2 \leq \cos(x) - 1 \leq Cx^2,
\]
or, multiplying through by \(-1\),
\[
-Cx^2 \leq 1 - \cos(x) \leq Cx^2
\]
for some \( C \) in an interval around \( x = 0 \). Since everything in sight is an even function, we need only consider \( x \geq 0 \). Since \( 1 - \cos(x) \) is never less than \( 0 \), the inequality
\[
-Cx^2 \leq 1 - \cos(x)
\]
is trivial for any positive \( C \). The other can be shown with, for example, \( C = 1 \). At \( x = 0 \) the inequality reduces to \( 0 \leq 0 \) which is true. We are done if we can show that \( 1 - \cos(x) \) increases more slowly than \( x^2 \). Taking derivatives, this reduces to showing
\[
\sin(x) \leq 2x \quad \text{for } x \geq 0.
\]
But this follows from the calculation you did in exercise 2.

**Exercise 5.** Show the following relations as \( x \to 0 \). You may want to use calculus to maximize some function on some interval.
\[
\frac{1}{1-x} = 1 + O(x)
\]
\[
\frac{1}{1-x} = 1 + x + O(x^2)
\]
and, for any \( n \),
\[
\frac{1}{1-x} = 1 + x + x^2 + \ldots + x^{n-1} + O(x^n)
\]

**Exercise 6.** Show that for a fixed choice of \( h(x) \), the relation \( f(x) = g(x) + O(h(x)) \) really is an equivalence relation.

If we have a pair of functions that satisfy \( f(x) \ll h(x) \), then from the definitions it is certainly true that \( f(x) = 0 + O(h(x)) \). Since adding 0 never changes anything we might write
\[
f(x) = O(h(x)) \quad \text{if} \quad f(x) \ll h(x).
\]
Many books do this, but it can be confusing for beginners, because ‘\( =~ O(\cdot) \)’ is not an equivalence.
3. CONTINUITY AND DERIVATIVE

This is a lot to digest all at once, so let’s consider a much simpler case when \( g(x) \) is just a constant function equal to some number \( L \) and \( h(x) \) is the function \( x - a \). What does it mean to say that

\[
f(x) = L + O(x - a) \quad \text{as} \quad x \to a?
\]

It means there is some number \( C \) and an interval around \( a \), so that

\[
|f(x) - L| < C|x - a|
\]

for every value of \( x \) in the interval. This means that we can get \( f(x) \) to be as close to \( L \) as we need by taking values of \( x \) sufficiently close to \( a \).

**Exercise 7.** Suppose \( x \geq a \).

(i) Show that (4) is equivalent to

\[
L - C(x - a) \leq f(x) \leq L + C(x - a).
\]

(ii) Interpret this geometrically. What do the graphs of \( y = L + C(x - a) \) and \( y = L - C(x - a) \) look like? What happens at the point \((a, L)\) in the \( x - y \) plane?

(iii) Draw a diagram that illustrates (5).

(The case \( x \leq a \) is similar.)

**Exercise 8.** Suppose (4) is true, and I tell you some small amount \( \epsilon \) of error I am willing to tolerate. Tell me how close \( x \) must be to \( a \) so that the error is less than \( \epsilon \). In other words, tell me a number \( \delta \) (in terms of the parameters \( C, L, a, \) and \( \epsilon \)) so that if \( |x - a| < \delta \), then \( |f(x) - L| < \epsilon \).

We say the limit as \( x \) approaches \( a \) is \( L \) if, for every \( \epsilon > 0 \) there is a \( \delta > 0 \) so that whenever \( |x - a| < \delta \), it is true that

\[
|f(x) - L| < \epsilon.
\]

We write this as

\[
\lim_{x \to a} f(x) = L.
\]

If in addition, \( L \) is the actual value of the function at \( a \), that is, \( L = f(a) \) then then we say \( f(x) \) is continuous at \( a \), and

\[
f(x) = f(a) + O(x - a) \quad \text{as} \quad x \to a
\]

is true.

Equation (6) implies that \( f(x) \) is continuous at \( a \), but this is not equivalent to continuity; the Big Oh statement has more information.
because it specifies how fast the function is tending to the limit. For
example, $\sqrt{x}$ is continuous at 0, but it is not true that
$$\sqrt{x} = 0 + O(x) \quad \text{as} \quad x \to 0.$$  

Here’s why. For a linear error $O(x)$, we can interpret the unknown
constant $C$ as the slope of the line $y = Cx$. Then no matter what
slope $C$ we pick, eventually the graph of $y = \sqrt{x}$ is above the line
$y = Cx$.

**Exercise 9.** (i) Sketch the graph of $y = \sqrt{x}$, and also the lines
$y = x$ and $y = 2x$.

(ii) Suppose you try $C = 1$. How small does $x$ have to be so that
$$\sqrt{x} \leq Cx$$
fails to be true?

(iii) Suppose you try $C = 2$. How small does $x$ have to be so that
$$\sqrt{x} \leq Cx$$
fails to be true?

(iv) For an arbitrary $C$, how small does $x$ have to be so that
$$\sqrt{x} \leq Cx$$
fails to be true? Of course your answer will be a function of $C$.

Examples like this are pathological; $\sqrt{x}$ has no derivative at $x = 0$. 
For the nice functions we are interested in, it is convenient to do
everything with Big Oh notation. The beauty of this is that we can
use it to make sense of derivatives, too.

**Exercise 10.** Suppose there is some number (denoted $f'(a)$) so that
$$f(x) = f(a) + f'(a)(x - a) + O((x - a)^2) \quad \text{as} \quad x \to a$$

Show that
$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a).$$

In other words, $f(x)$ is differentiable at $x = a$ with derivative the
number $f'(a)$.

The point of (7) is that $y = f(x)$ can be a complicated graph, while
$y = f(a) + f'(a)(x - a)$ is the equation of a line through the point
$(a, f(a))$, with slope $f'(a)$. The Big Oh notation lets us replace com-
plicated expressions $f(x)$ by simpler ones, like equations of lines, as
long as we are willing to tolerate a small error. When $|x - a|$ is less
than 1, powers like $(x - a)^2$ are even smaller. The closer $x$ gets to $a$,
the smaller the error is.
Exercise 11.  

(i) Show that

\[ x^2 = 9 + 6(x - 3) + O((x - 3)^2) \quad \text{as } x \to 3 \]

which implies the derivative of \( x^2 \) at 3 is the number 6.

(ii) Show that

\[ x^3 = 0 + 0 \cdot (x - 0) + O((x - 0)^2) \quad \text{as } x \to 0 \]

which implies the derivative of \( x^3 \) at 0 is the number 0.

(iii) Show that, with \( f(x) = 7x + 4 \),

\[ f(x) = 11 + 7(x - 1) + O((x - 1)^2) \quad \text{as } x \to 1 \]

which implies the derivative of \( f(x) \) at 1 is the number 7.

(iv) Show that

\[ \frac{1}{t} = 1 - (t - 1) + O((t - 1)^2) \quad \text{as } t \to 1. \]

(This can be done with a simple substitution in Exercise ??.)

Interpret this as the computation of a derivative.

Mostly in calculus you look at very nice functions, those which do have derivatives at every point \( a \). So the rule that assigns to each point \( a \) the number \( f'(a) \) defines a new function, which we denote \( f'(x) \).

4. Fundamental Theorem of Calculus

Integral calculus as well as differential calculus is really all about a single idea. When you took the course, you got a lot of practice at ‘antiderivatives’, that is, undoing the operation of derivative. For example, you write

\[ \int x^2 \, dx = \frac{x^3}{3} + C \]

to mean that a function whose derivative is \( x^2 \) must be of the form \( x^3/3 \) plus some constant \( C \). This is an indefinite integral; it is a collection of functions. You also learned about the definite integral; this is a number which measures the area under a curve between two points. For example

\[ \int_0^1 x^2 \, dx \]

is the area under the parabola \( y = x^2 \) between \( x = 0 \) and \( x = 1 \). The symbols

\[ \int x^2 \, dx \quad \text{and} \quad \int_0^1 x^2 \, dx \]
mean two very different things, even though they look very similar. But why are these two things connected? What does the operation of 
un-doing derivatives have to do with area? After all, the geometric 
interpretation of derivative is the slope of the tangent line, nothing 
to do with area.

To explain this we need some basic facts about area, geometric 
properties that have nothing to do with calculus. In each of these, 
keep in mind that $\int_a^b f(t) \, dt$ just denotes the area under $y = f(t)$ 
between $a$ and $b$, nothing more.

(i) First of all, the area of a rectangle is the height times the width. 
So if we have a constant function $f(t) = c$ for all $t$, then by ge-
ometry

$$\int_a^b c \, dt = c \cdot (b - a).$$

(ii) Next, if we scale the function by some constant $c$ to change its 
height, the area under the graph changes by that same scalar 
factor. Figure 1 shows an example with $c = 2$. So

$$\int_a^b c \cdot f(t) \, dt = c \cdot \int_a^b f(t) \, dt.$$
(iii) This next one is a little trickier. If we add two functions \( f(t) \) and \( g(t) \) together, the area under the new function is the sum of the areas under each one. One way to convince yourself of this is to imagine approximating the area by lots of little rectangles. The height of a rectangle under the graph of \( f + g \) is just that of a rectangle under \( f \) and another under \( g \). So

\[
\int_a^b f(t) + g(t) \, dt = \int_a^b f(t) \, dt + \int_a^b g(t) \, dt.
\]

(iv) Finally, if we have three points \( a, b, \) and \( c \) on the \( t \)-axis, the area from \( a \) to \( c \) is just the area from \( a \) to \( b \) plus the area from \( b \) to \( c \). Figure 1 is convincing. In equations

\[
\int_a^c f(t) \, dt = \int_a^b f(t) \, dt + \int_b^c f(t) \, dt.
\]

That is all we need for now, but three other properties will be useful later on.

(v) First of all, it is clear from Figure 2 that if \( f(t) \leq g(t) \) then

\[
\int_a^b f(t) \, dt \leq \int_a^b g(t) \, dt.
\]

I lied above when I said that \( \int_a^b f(t) \, dt \) just denotes the area under \( y = f(t) \) between \( a \) and \( b \). As you remember from calculus, if any portion of the graph dips below the horizontal axis, that area is counted negatively by the definite integral. Comparing the top and
middle of Figure 3, it is clear that

\[ \int_a^b f(t)dt \leq \int_a^b |f(t)|dt; \]

they are equal exactly when \( f(t) \) is always positive, otherwise the definite integral of \( f(t) \) has a ‘negative’ chunk that the integral of \( |f(t)| \) does not.

**Exercise 12.** Maybe it isn’t clear. Looking at Figure 3 again, use property (iv) above, twice, and property (ii) with \( c = -1 \), to compare the two integrals.

Since \(-f(t)\) is just another function, it is similarly true that

\[ \int_a^b -f(t)dt \leq \int_a^b | -f(t)|dt. \]
The left side is \(-\int_a^b f(t) \, dt\) by property (ii), while the right side is just \(\int_a^b |f(t)| \, dt\), since \(|-f(t)| = |f(t)|\). This is just the comparison of the bottom and middle of Figure 3.

(vi) Because both \(\int_a^b f(t) \, dt\) and \(-\int_a^b f(t) \, dt\) are less than or equal to \(\int_a^b |f(t)| \, dt\), we deduce that

\[ | \int_a^b f(t) \, dt | \leq \int_a^b |f(t)| \, dt. \]

**Exercise 13.** (the M-L inequality) Use the preceding properties to show that if \(|f(t)| \leq M\) for \(a \leq t \leq b\), then

\[ \left| \int_a^b f(t) \, dt \right| \leq M(b-a). \]

(The L in the name refers to the length \(L = b - a\) of the interval.)

To prove the Fundamental Theorem of Calculus, we also need an important definition. Suppose \(f(x)\) is some function, nice enough so that (6) is true at each point \(a\). We can make a new function \(F(x)\), by assigning to each number \(x\) the area under \(f\) between 0 and \(x\). You should think of this as a definite integral. It can be computed to any degree of accuracy by approximating by rectangles (the so-called Riemann sums) without yet making any reference to antiderivatives. So

\[ F(x) = \int_0^x f(t) \, dt. \]

(We rename the variable of \(f\) to be \(t\) to avoid confusion.) Figure 4 shows an example. The height of the line on the upper graph represents the shaded area on the lower graph. As we vary the point \(x\), the amount of area to the left of the point changes, and this is the height on the graph of \(F(x)\). When \(f(x)\) is negative the increment of area is negative, so \(F(x)\) decreases. It crucial to recognize the role of the variable \(x\) in the function \(F\). It is to determine a portion of the horizontal axis; the area under \(f\) above that portion of the axis is the number \(F(x)\).

Now that we can define a function in terms of area, and compute it by Riemann sums, we can state the following

**Exercise 14** (Fundamental Theorem of Calculus, part I). Suppose \(f(x)\) satisfies (6); that is,

\[ f(x) = f(a) + O(x-a) \quad \text{as} \quad x \to a \]
is true. Show that the function $F(x)$ defined by

$$F(x) = \int_0^x f(t)dt$$

is differentiable, and $F'(a) = f(a)$ for every point $a$. In other words, show that

$$F(x) = F(a) + f(a)(x-a) + O((x-a)^2)$$

is true. Hint: It all reduces to properties of area.

How can we make use of this theorem? The function $F(x)$ computes the area by Riemann sums, which we prefer not to deal with if possible. The answer is
Exercise 15 (Fundamental Theorem of Calculus, part II). If $G(x)$ is any antiderivative for $f(x)$, that is $G'(x) = f(x)$, then

$$\int_a^b f(t)dt = G(b) - G(a).$$

Hint: What can you say about the derivative of $F(x) - G(x)$?

So we don’t need to compute $F(x)$ as long as we can guess some antiderivative.

Of course, there is nothing special about the base point $x = 0$, we can start measuring the area relative to any base point $x = a$; we get an antiderivative whose value differs from the one above by the constant

$$C = \int_0^a f(t)dt.$$

So far our discussion of differential and integral calculus has been purely geometric, and application-free. But even for pure mathematicians, it is necessary to think about the meaning of derivative and integral when the function involved has some physical interpretation. As we generalize these concepts to higher dimensions, there will be too many dimensions to visualize purely geometrically. The only common-sense interpretation we will have is the physical one.

The way we understand the physical meaning is by DIMENSIONAL ANALYSIS, i.e. keeping track of the units involved. For example, if $t$ represent time (in seconds) and $y = f(t)$ represents distance (in meters), then the slope of a line in the $t - y$ plane is measured in units of meters per second, because slope is measured in $y$ units divided by $t$ units. The derivative is the slope of the tangent line, so the units are meters per second. This should not be unfamiliar to you.

Exercise 16. (i) Suppose $t$ is measured in seconds, and $y = f(t)$ measures liters of fluid in a tank at time $t$. What are the units of $f'(t)$?

(ii) Suppose $a$ is measured in square inches, and $y = F(a)$ is measured in pounds. What are the units of $F'(a)$?

(iii) Suppose $g$ is measured in gallons, and $y = m(g)$ is measured in miles. What are the units of $m'(g)$?

(iv) Suppose $s$ is measured in feet, and $y = w(s)$ is measured in foot pounds (a unit of work). What are the units of $w'(s)$?

Just as the slope of a line is computed by division, the area of a rectangle is computed by multiplication. Because area generally is approximated by rectangles, the same holds true generally. Area under a curve can represent some quantity of interest. For example if
$t$ is time in seconds, and $y = f(t)$ represents velocity in meters per second, then the area of a rectangle in the $t - y$ plane represents a distance. If this is confusing, keep in mind the distinction between the geometry (i.e., the rectangle in the plane) and the physics (distance) which is not pictured.

**Exercise 17.** (i) Suppose $t$ is measured in seconds, and $y = f(t)$ is measured in liters per second. What units does area under the graph of $y = f(t)$ represent?

(ii) Suppose $s$ is measured in feet, and $y = f(s)$ is measured in pounds of force. What units does area under the graph of $y = f(s)$ represent?

(iii) Suppose $a$ is measured in square inches, and $y = P(a)$ is measured in pressure units, that is, pounds per square inch. What units does area under the graph of $y = P(a)$ represent?

With this point of view, we can give a very simple statement of the Fundamental Theorem of Calculus, part II:

‘If some function $f(t)$ represents a rate of change, then the area under its graph between two points represents total change.’

This answers the question on page 8 of what the operation of undoing derivatives has to do with area. Slopes are computed by a division; undoing division is multiplication, which computes area.

You may be asking the question, “If we don’t care about area, but only what it represents, then why talk about it at all?” If so, congratulations; it is a very good question. One answer is that the mind is inherently geometric; to solve a problem it always helps to have a diagram or picture to refer to.

A more subtle answer is that area is something that can be computed when all else fails. For example, suppose you need to find a function whose derivative is $f(x) = \exp(-x^2)$. This the graph of this particular function is the ‘bell shaped curve’; it arises in probability and statistics. No method you learn in calculus will find an antiderivative in this case. In fact, there is a theorem which says that no combination of elementary functions (polynomial, trigonometric, exponential, etc.) has a derivative which is $\exp(-x^2)$. But some function exists whose derivative is $\exp(-x^2)$. In fact, it is called the ‘Error function’ Erf($x$); it is related to the probability that a random variable will take on a value $\leq x$. By the Fundamental Theorem of Calculus,
another way to write this is

\[ \text{Erf}(x) = \int_0^x \exp(-t^2) \, dt \]

So Erf(x) can be computed, to any degree of accuracy we like, by approximating the area under the curve with rectangles. These approximations are Riemann sums. For another example, it is perfectly reasonable to define the logarithm of x by

\[ \log(x) = \int_1^x \frac{1}{t} \, dt. \]

The Fundamental Theorem of Calculus says the derivative is 1/x, positive for x > 0. So this function is always increasing and thus has an inverse which we can define to be exp(x). All the properties you know and love can be derived this way.

**Exercise 18.** Taking the definition of \( \log(x) \) to be

\[ \log(x) = \int_1^x \frac{1}{t} \, dt, \]

show that \( \log(xy) = \log(x) + \log(y) \) is still true. (Hint: property (iv) and change the variables.)

In summary, the Fundamental Theorem of Calculus tells us that if we know an antiderivative we can use it to compute area easily. If we don’t already know an antiderivative we can use it to define one by computing the area directly.

### 5. Vectors

I assume you’ve already seen vectors in the plane. This section will fill in some details. We can add vectors \( \vec{v}_1 = (x_1, y_1) \) and \( \vec{v}_2 = (x_2, y_2) \) by

\[ \vec{v}_1 + \vec{v}_2 = (x_1 + x_2, y_1 + y_2) \]

and multiply \( \vec{v} \) by a scalar \( c \)

\[ c \cdot \vec{v} = (cx, cy). \]

Every vector \( \vec{v} = (x, y) \) has a polar form \([R, \theta]\) with

\[
\begin{align*}
R &= \sqrt{x^2 + y^2} \\
\theta &= \arctan(y/x) \\
x &= R \cos(\theta) \\
y &= R \sin(\theta)
\end{align*}
\]

The key tool here is the **inner product** (or dot product)

\[ \vec{v}_1 \cdot \vec{v}_2 = x_1 x_2 + y_1 y_2. \]
Exercise 19. Which of the following are true? Give proofs.

(1) \((\vec{v}_1 + \vec{v}_2) \cdot \vec{v}_3 = \vec{v}_1 \cdot \vec{v}_3 + \vec{v}_2 \cdot \vec{v}_3\)

(2) \((c \cdot \vec{v}_1) \cdot \vec{v}_2 = c(\vec{v}_1 \cdot \vec{v}_2)\)

(3) \(\vec{v}_1 \cdot \vec{v}_2 = \vec{v}_2 \cdot \vec{v}_1\)

(4) If \(\vec{v} \cdot \vec{v} = 0\), then \(\vec{v} = \vec{0}\).

(5) \(R = \sqrt{\vec{v} \cdot \vec{v}}\)

The length \(R\) of a vector \(\vec{v}\) is also denoted \(\|\vec{v}\|\).

Exercise 20. Show that

\[\vec{v}_1 \cdot \vec{v}_2 = \|\vec{v}_1\|\|\vec{v}_2\| \cos(\phi)\]

where \(\phi\) is the angle between the vectors. Hint: The formula has lengths and angles in it.

This exercise is important because the left side is defined purely in terms of algebra, while the right side is defined purely in terms of geometry. In particular it says that

\[\vec{v}_1 \cdot \vec{v}_2 = 0 \quad \Leftrightarrow \quad \phi = \pm \pi/2\]

We say that two vectors \(\vec{T}\) and \(\vec{N}\) are ORTHONORMAL if

\[\|\vec{T}\| = 1 = \|\vec{N}\| \quad \text{and} \quad \vec{T} \cdot \vec{N} = 0.\]

Exercise 21. Show that if \(\vec{T} = (a, b)\) has length 1, then \(\vec{N} = (-b, a)\) and \(\vec{T}\) are orthonormal. Show that for any vector \(\vec{v} = (x, y)\),

\[\vec{v} = (\vec{v} \cdot \vec{T})\vec{T} + (\vec{v} \cdot \vec{N})\vec{N}\]

is true.

6. ARCLength

Suppose we have some graph \(y = f(t)\) and we want not the area under the graph, but the arc length of the curve; that is the distance we would travel if we lay the graph down flat and follow it from the beginning \((a, f(a))\) to the end \((b, f(b))\). One way to approach the problem is to approximate the length by the sum of the lengths of little straight line segments which approximate the path. (This is similar to approximating area by a sum of areas of rectangles.) Figure
Exercise 22. This exercise will connect arc length with definite integrals.

(1) What is the exact slope of the hypotenuse in terms of $\Delta t$ and $\Delta y$?

(2) Observe that this slope is approximately the slope $f'(t)$ of the tangent line.

(3) Combine these two facts to get an approximate formula for $\Delta y$ in terms of $\Delta t$ and $f'(t)$.

(4) Use the Pythagorean Theorem to get an approximate formula for the length of the hypotenuse. Factor out a $\Delta t$.

(5) This step requires only a new point of view. Interpret your last answer as the width $\Delta t$ of some rectangle times the height of a new function. What is the new function?

(6) Width times height is area of a rectangle. So adding the approximate lengths of all the hypotenuses is the same as adding the areas of the rectangles. The exact length of the curve should correspond to an exact area under this new graph. Write this down as a definite integral. It is the formula for arc length.

Exercise 23. Here is a specific example.

(1) Let $y = f(t) = \sqrt{1-t^2}$; the graph is an arc of a circle. Work out the formula for arclength, not between points $a$ and $b$, but between points 0 and $x$ instead. Simplify as much as possible (common denominators.)

(2) In fact, the definition of what ‘radians’are is that angle $\theta$ is the length of the arc of the unit circle. (This is why the entire circle is $2\pi$ radians). Write $x$ in terms of $\theta$ and a trig function. Now write $\theta$ in terms of $x$ and an inverse trig function.

(3) The two previous parts give two answers for the arclength. Set them equal to each other to get a theorem: an inverse trig function is a definite integral. By the Fundamental Theorem of Calculus, Part I, you have proved a theorem about the derivative of an inverse trig function. What is it?

To go from ‘scalar calculus’ to vector calculus, we need to generalize all the concepts. For example, instead of an interval $a \leq t \leq b$ on the $t$-axis, we need to consider a curve in the $x$-$y$ plane. It will be defined by some function $\vec{\sigma}(t) = (x(t), y(t))$, for $a \leq t \leq b$. It may help
to think about $t$ as ‘time’, then $x(t)$ and $y(t)$ represent position in the plane (of some particle perhaps) at that time $t$. We will restrict our attention to curves that are PIECEWISE SMOOTH, that is, except for a finite number of possible bad points, the functions $x(t)$ and $y(t)$ will be differentiable in the sense of equation (7). For example, a square in the plane can be traced out by a particle with only four bad points, the corners.

If $x(t)$ and $y(t)$ denote position, in say meters, and $t$ is time, in say seconds, then $x'(t)$ and $y'(t)$ denote rates of change in meters per second, as usual. We let $\vec{\sigma}'(t) = (x'(t), y'(t))$, and interpret it as a velocity vector.

A curve is said to be CLOSED if the beginning and end points in the plane are the same, that is $\vec{\sigma}(a) = \vec{\sigma}(b)$. A curve is SIMPLE if it does not cross itself, except possibly to link the beginning and end at $t = a$ and $t = b$.

**Exercise 24.** Draw a picture of a curve which is

1. Simple, not closed.
2. Closed, not simple.
3. Simple and closed.
4. Neither simple nor closed.

Answers on p. 16 of the text, Figure 1.13.

If we rescale $\vec{\sigma}'$, dividing out its length, we get a tangent vector which has length 1. This UNIT TANGENT VECTOR is denoted $\vec{T}$. (It is a function of the independent variable, but this may be suppressed.) If $\vec{T} = (x, y)$, then the vectors $(y, -x) \text{ and } (-y, x)$ are both unit vectors perpendicular to $\vec{T}$ (since the dot product is 0). If $\vec{\sigma}$ is a simple closed curve we choose the sign so the vector points outside the region enclosed by the curve, and denote it $\vec{N}$, the UNIT NORMAL VECTOR. If the path is traveled in the canonical counterclockwise direction, $\vec{N} = (y, -x)$ when $\vec{T} = (x, y)$. The choice of sign can be justified by imagining a $\vec{T}$ with, for example, both $x$ and $y$ positive. $\vec{T}$ points in the direction of the first quadrant. Since we’re going around counterclockwise, the inside is above and to the left (draw a picture), so $\vec{N}$ should point to the fourth quadrant, with positive first coordinate and negative second coordinate. This means $(y, -x)$ not $(-y, x)$.

We need to understand arclength in this more general setting, but it is easy from what we did before. Each of $x(t)$ and $y(t)$ have their own graph as a function of $t$, in the $x$-$t$ plane and in the $y$-$t$ plane respectively. Imagine we are at some time $t$, and consider a small increment of time $\Delta t$. We approximate the arclength of the curve
between $\vec{\sigma}(t)$ and $\vec{\sigma}(t + \Delta t)$ by the length of a straight line, which is the hypotenuse of a right triangle. The triangle has sides $\Delta x$ and $\Delta y$.

**Exercise 25.**

1. $\Delta y$ is the exact change in $y$ over the time interval $\Delta t$. How does this relate (approximately) to $\Delta t$ and $y'(t)$?

2. What definite integral do you think gives the exact answer for arc length (distance traveled) between $t = a$ and $t = b$?

3. The units involved should all make sense. Explain why this is so.

We can use the Fundamental Theorem of Calculus idea here, making a function $F(\tau)$ which computes distance traveled between the start $t = a$ and a variable time $t = \tau$. By the previous exercise it is a definite integral with the variable $\tau$ in the upper limit of integration:

$$F(\tau) = \int_a^\tau \|\vec{\sigma}'(t)\| dt.$$  

By the Fundamental Theorem of Calculus, part I, we can compute $F'(\tau)$ and we see it is always positive. So $F(\tau)$ is always increasing. (This makes physical sense, $F(\tau)$ computes distance traveled like the odometer on your car. Even if you turn around and retrace your path, the odometer only increases.)

An increasing function has an inverse function which ‘undoes’ the original. Here we mean inverse in the sense that $\log(x)$ is the inverse to $e^x$. We do not mean reciprocal. The reason the inverse exists is that if we flip the axes in the graph of $F(\tau)$, we still have something that satisfies the definition of a function. (It may help to look at the graphs of $\log(x)$ and $e^x$. On the other hand, $t^2$ is not increasing between $-1$ and $1$, nor does it have an inverse in this range.) Let $E(s)$ denote the inverse function, so that

$$E(F(\tau)) = \tau \quad \text{and} \quad F(E(s)) = s.$$  

We need to distinguish between the curve, which is a set of points in the plane, and the parametrization $\vec{\sigma}(t)$ which traces it out. The same point set can be traced out in many different ways. For example, the unit circle is given by $(\sin(t), \cos(t))$ for $0 \leq t \leq 2\pi$, but also by $(\sin(2t), \cos(2t))$ for $0 \leq t \leq \pi$. Of special interest to us will be the PARAMETRIZATION BY ARCLength which gives position, not as a function of time $t$, but rather as a function of distance.
traveled $s$ from the starting point. This can always be done: given any parametrization $\vec{\sigma}(t)$ for $a \leq t \leq b$, we compute first $F(\tau)$, then its inverse $E(s)$. This latter function gives time as a function of distance traveled. The composition $\vec{\sigma}(E(s))$ is the parametrization by arclength, for $0 \leq s \leq S$, where $S = s(b)$ is the total distance traveled.

**Exercise 26.** Consider the path given by $\vec{\sigma}(t) = (t, \cosh(t))$ for $0 \leq t \leq 1$, where $\cosh(t) = (e^t + e^{-t})/2$. Compute $F(\tau)$. Find $E(s)$ (Hint: quadratic formula.) Find the parametrization by arclength. This problem is messy if you screw it up, but not bad if you are very careful.

**Exercise 27.** Let $\vec{\sigma}(t)$ be the circle of radius 2 parametrized by

$$\vec{\sigma}(t) = (2 \cos(t), 2 \sin(t)), \quad 0 \leq t \leq 2\pi.$$

Find the parametrization by arclength. This comes out nicely if you are careful. Compute the velocity vector, and its length (speed) for the parametrization by arclength.

**Exercise 28.** This exercise proves that what you observed in the special case above, always happens.

Suppose $\vec{\sigma}(t) = (x(t), y(t))$ is any parametrization of a path, and $\Gamma(s)$ is the parametrization by arclength, so

$$\Gamma(s) = (x(E(s)), y(E(s)))$$

in the notation above.

1. Compute derivatives with respect to $s$ on both sides of the equation

$$F(E(s)) = s.$$

Solve for $E'(s)$ in terms of $F'(E(s))$.

2. With $E(s)$ a time $\tau$, what does the Fundamental Theorem of Calculus, part I say about $F'(E(s)) = F'(\tau)$?

3. Use the chain rule in each coordinate to compute the `velocity vector' $\Gamma'(s)$. With $E(s) = \tau$, what can you compute about the length of the vector $\Gamma'(s)$?

**Exercise 29.** Suppose $\Gamma$ is a simple closed curve, $\vec{a}$ is a point on $\Gamma$, $\vec{T}$ and $\vec{N}$ are the unit tangent vector and unit outward normal vector at $\vec{a}$. Let $\vec{v}$ be any vector in the plane, viewed as originating from the point $\vec{a}$. Interpret geometrically (with a diagram)

1. $\vec{v} \cdot \vec{T} > 0$, $\vec{v} \cdot \vec{N} = 0$.

2. $\vec{v} \cdot \vec{T} < 0$, $\vec{v} \cdot \vec{N} = 0$. 
7. Derivatives again

Suppose $u : \mathbb{R}^2 \to \mathbb{R}$ is a scalar valued function of a vector variable $\vec{v}$. This is the opposite of the function $\vec{\sigma}$ we considered above. The graph of $u$ is a surface in three dimensional space $\mathbb{R}^3$. We want to make sense of the derivative of such a function at a point $\vec{a}$ in the plane. Geometrically, just as before, the graph should be approximated by something flat; in this case, by a plane. We will say that $u$ is differentiable at $\vec{a}$ with derivative $M$ if $M$ is a $1 \times 2$ matrix so that

$$u(\vec{v}) = u(\vec{a}) + M(\vec{v} - \vec{a}) + O(|\vec{v} - \vec{a}|^2).$$

Alternately we could view $M = (M_1, M_2)$ as a vector and interpret $M(\vec{v} - \vec{a})$ as a dot product. Just as before, this equation says the function $u(\vec{v})$ is well approximated by something linear, in this case the equation of a plane $u(\vec{a}) + M(\vec{v} - \vec{a})$ up to an error $O(|\vec{v} - \vec{a}|^2)$.

Let’s consider a couple of special cases. Suppose $\vec{a} = (a_1, a_2)$ and the variable $\vec{v}$ is restricted to lie on the same horizontal line as $\vec{a}$. So $\vec{v} = (a_1 + h, a_2)$ for some $h$. Then $\vec{v} - \vec{a} = (h, 0)$ and $M \cdot (h, 0) = M_1 h$.

Equation (9) says that

$$u(a_1 + h, a_2) = u(a_1, a_2) + M_1 h + O(h^2),$$

and this implies that $M_1$ is the derivative of $u$ if we fix the $y$-coordinate and think of it as a function of the first coordinate, $h$, only. In geometrical terms, there are lots of tangent lines to the surface $z = u(\vec{v})$ at $\vec{a}$, a whole plane’s worth of them. One tangent line is parallel to the $x$ axis, and $M_1$ is the slope of this line, more commonly denoted $\partial u / \partial x$. Similarly $M_2 = \partial u / \partial y$, and $M$ is usually written

$$\nabla u = (\partial u / \partial x, \partial u / \partial y),$$

the gradient of $u$.

The mere existence of partial derivatives at a point is not enough to guarantee that a function is differentiable at that point. For example, one can take

$$u(x, y) = \begin{cases} 
2xy/(x^2 + y^2) & \text{if } (x, y) \neq (0, 0) \\
0, & \text{if } (x, y) = (0, 0).
\end{cases}$$
See Figure 5. In polar coordinates, \( u[R, \theta] \) simplifies to \( 2 \sin(\theta) \cos(\theta) = \sin(2\theta) \). The height of the surface is constant along rays \( \theta = \text{const.} \). Thus \( u \) is not continuous; approaching the origin along the line \( y = mx \) (rectangular coordinates again) the function is constant \( u(x, mx) = 2m/(1 + m^2) \). No value for \( u(0, 0) \) will make a continuous function. Since differentiable implies continuous, we are done once you do the following

**Exercise 30.** With \( u \) as above, show that

\[
  u_x(0, 0) = 0 = u_y(0, 0).
\]

If we want the slope of the tangent line to the surface, oriented in some other direction than parallel to the \( x \) or \( y \) axis, we introduce a direction vector \( \vec{z} = (\cos(\theta), \sin(\theta)) \) of length 1, and consider a line through \( \vec{a} \) in the direction \( \vec{z} \), given by \( \vec{a} + h\vec{z} \) for real \( h \). Then equation (9) reduces to

\[
  u(\vec{a} + h\vec{z}) = u(\vec{a}) + h\nabla u \cdot \vec{z} + O(h^2).
\]

which again looks like the derivative of a function of one variable. The constant \( \nabla u \cdot \vec{z} \) (relative to the variable \( h \) anyway) is the **Directional Derivative** of \( u \) in the direction of the unit vector \( \vec{z} \).
Exercise 31. The directional derivative is computed by a dot product, and exercise 20 relates dot products to angles (directions). For what direction \( \vec{z} = (\cos(\theta), \sin(\theta)) \) is the directional derivative largest? Smallest? Imagine for concreteness that \( u(x, y) \) represents temperature at the point \((x, y)\). You are at some point \( \vec{a} = (a_x, a_y) \) in the plane. What direction should you go to make temperature increase as quickly as possible? What direction should you go to make temperature decrease as rapidly as possible?

For example, if we have a simple closed curve, with unit tangent vector \( \vec{T} \) and unit outward normal vector \( \vec{N} \), and also a function \( u \), the outward normal derivative \( \partial u / \partial \eta \) is defined to be the directional derivative of \( u \) in the direction \( \vec{N} \), computed via \( \nabla u \cdot \vec{N} \).

Figure 6 shows two examples. In the left column we have \( u(x, y) = x^2 - y^2 \). The top picture shows level curves for the function: the curves in the plane of the form \( u(x, y) = \) a constant. In this example they are all hyperbolas. The middle picture shows the gradient vector field. At each point in the plane \((x, y)\) we have the vector \( \nabla u = (2x, -2y) \). The bottom shows the two together. The right column is the same for the function \( u(x, y) = x^2 + y^2 \).

Exercise 32. Let \( u = x - x^2 + 3y^2 \) and consider the point \( \vec{a} = (1, 2) \). Show using algebra that

\[
u(x, y) = 12 - (x - 1) + 12(y - 2) + O((x - 1)^2 + (y - 2)^2)
\]

is true. What choice of constant \( C \) works? Hint: substitute

\[
x = (x - 1) + 1 \quad \text{and} \quad y = (y - 2) + 2
\]
on the left hand side.

To understand in physical terms all the calculus going on, the key object will be the Laplacian \( \Delta u = u_{xx} + u_{yy} \) of \( u \). The function \( u \) is said to be harmonic if \( \Delta u = 0 \) at all points in the domain. This is a partial differential equation (PDE).

Exercise 33. Which functions are harmonic?

1. \( 3x - 2y \)
2. \( x^2 + y^2 \)
3. \( x^2 - y^2 \)
4. \( 2xy \)
5. \( e^x \cos(y) \)
6. \( e^x \cos(x) \)
7. \( \log \| (x, y) \| \)
(8) \( \text{arg}(x, y) \), the angle \((x, y)\) makes with the horizontal axis in polar coordinates.
We will need a version of the chain rule for vector valued functions, and functions of a vector variable. The proof is straightforward, requiring only confidence in manipulating the Big Oh notation. Suppose we have a function \( u \), and a path \( \vec{\sigma}(t) \). We will be careful and write the vectors as columns. So
\[
\vec{\sigma}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad u(\vec{\sigma}(t)) = u \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.
\]
And
\[
\begin{pmatrix} x(t + h) \\ y(t + h) \end{pmatrix} = \begin{pmatrix} x(t) + x'(t)h + O(h^2) \\ y(t) + y'(t)h + O(h^2) \end{pmatrix}
\]
by using the Big Oh idea on \( x(t) \) and \( y(t) \) separately. Let
\[
\vec{H} = \begin{pmatrix} x'(t)h + O(h^2) \\ y'(t)h + O(h^2) \end{pmatrix},
\]
where we interpret the \( O(h^2) \) symbols as referring to unknown functions, the errors, which are bounded by a constant times \( h^2 \). From equation (9) we know
\[
u \begin{pmatrix} x(t) + \vec{H} \\ y(t) \end{pmatrix} = u \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + \nabla u \cdot \vec{H} + O(\|\vec{H}\|^2).
\]

The key fact here is that viewing \( u \) as a function of \( t \), if we increment \( t \) by some small amount \( h \) we have an equation
\[
u(t + h) = u(t) + \nabla u \cdot \vec{\sigma}'(t)h + O(h^2)
\]
which means that
\[
\frac{du}{dt} = \nabla u \cdot \vec{\sigma}'(t) = u_x x'(t) + u_y y'(t).
\]

8. LINE INTEGRALS

There are two ‘ingredients’ to a line integral. One is a VECTOR FIELD \( \vec{V} = (p, q) \), a function from \( \mathbb{R}^2 \) to \( \mathbb{R}^2 \); that is, \( p(x, y) \) and \( q(x, y) \) are each functions of two variables. Geometrically \( \vec{V} \) associates a
vector \((p, q)\) to each point \((x, y)\) in the plane. Gradients are examples of vector fields, see Figure 6. But most vector fields are not the gradient field of some function.

**Exercise 34.** Find a vector field \(\vec{V} = (p, q)\) which cannot be of the form \(\nabla u\) for any \(u\). Hint: For a function \(u(x, y)\), the mixed partials are equal, that is \(u_{xy} = u_{yx}\).

From the vector field we make a **one form** \(p \, dx + q \, dy\). This is just notation, but useful for keeping track of dimensions and variables. In 3B language we have functions \(f(t)\) and 1-forms \(f(t) \, dt\).

The other ingredient is a path \(\Gamma\) in the plane, parametrized as a function \(\vec{\sigma}(t)\) for \(a \leq t \leq b\). The two ingredients are combined much the way we combined a function (or rather 1-form \(f(t) \, dt\) and an interval \([a, b]\) on the \(t\) axis. View

\[
\begin{align*}
    dx & \quad \text{as meaning} \quad x'(t) \, dt \\
    dy & \quad \text{as meaning} \quad y'(t) \, dt
\end{align*}
\]

then

\[
\int_{\Gamma} p \, dx + q \, dy \quad \text{means} \quad \int_{a}^{b} (p(x(t), y(t))x'(t) + q(x(t), y(t))y'(t)) \, dt.
\]

What we have done is use the output vector of the path \(\vec{\sigma}(t)\) as the input vector of the vector field \(\vec{V}\). Since this output is still a vector, we take dot product with the velocity vector \(\vec{\sigma}'(t)\). This is now a scalar valued function of \(t\), and we can integrate if we make a 1-form by adding the \(dt\). So we can also write the line integral as

\[
\int_{\Gamma} p \, dx + q \, dy = \int_{a}^{b} \vec{V}(\vec{\sigma}(t)) \cdot \vec{\sigma}'(t) \, dt.
\]

It is hard to see what this is computing; we will think about this below. But whatever it is, it should be determined by the vectors \(\vec{V}\) in the vector field and the points on the curve \(\Gamma\), and not on the parametrization \(\vec{\sigma}(t)\) of the curve. In other words, viewing \(t\) as time and \(\vec{\sigma}(t)\) as the position of a particle at time \(t\) is a useful fiction, but we want our integral to be ‘time independent’. Luckily our calculations above about parametrization by arc length imply that this is so: Suppose \(\vec{\sigma}(t) = (x(t), y(t))\) is any parametrization of a path, and \(\Gamma(s)\) is the parametrization by arclength, so

\[
\Gamma(s) = (x(E(s)), y(E(s)))
\]

in the notation above. Then a long computation involving the Chain Rule, and the \(u\)-substitution trick of Math 3B will show that the line
integrals for the two different parametrizations give the same answer. Since the parametrization by arc length is unique, any two parametrizations give the same answer. In exercise 28 above you computed that \( \Gamma'(s) = \vec{T} \), the unit tangent vector, so another convenient way to write the line integral is

\[
\int_\Gamma p\,dx + q\,dy = \int_\Gamma \vec{V} \cdot \vec{T}\,ds
\]

Exercise 35. Figure 7 shows a circular segment of path \( \Gamma \), oriented counterclockwise, with four different vector fields \( \vec{V} \) on top of it. Decide in each case if \( \int_\Gamma \vec{V} \cdot \vec{T}\,ds \) is positive, zero, or negative. Hint: exercise 21.

Exercise 36. Compute some line integrals:

1. Let \( \vec{V} = (x + y, y) \) and \( \Gamma \) be the portion of the unit circle in the first quadrant, oriented counterclockwise.
2. Let \( \vec{V} = (x, y) \) and \( \Gamma \) be a straight line from \((1, 0)\) to \((0, 2)\).
3. Same \( \vec{V} \), and \( \Gamma \) the segment of the parabola \( y = 2 - 2x^2 \) connecting the points \((1, 0)\) to \((0, 2)\).
4. Same \( \vec{V} \), and \( \Gamma \) the triangle connecting the points \((1, 0)\) to \((0, 2)\) to \((0, 0)\) to \((1, 0)\) again.
5. \( \vec{V} = (y, x) \) and \( \Gamma \) the upper half of the unit circle, followed by the straight line from \((-1, 0)\) to \((1, 0)\).

Exercise 37. (the M-L inequality) Suppose that at all points along the path \( \Gamma \), the vectors in the vector field have length bounded by some constant \( M \), that is

\[ ||\vec{V}|| \leq M, \]

and the arclength of the path \( \Gamma \) is \( L \). Show that

\[ \left| \int_\Gamma \vec{V} \cdot \vec{T}\,ds \right| \leq ML. \]

Hint: write this out as a single variable integral, use exercise 20 and the properties (i)-(vi) of area. This exercise generalizes exercise 13 to line integrals.

Exercise 38. Suppose the vector field \( \vec{V} \) represents force vectors, in units of (pounds,pounds), and \( \Gamma'(s) \) is the parametrization by arc length, with distance \( s \) as well as the \( x \) and \( y \) coordinates of position in feet.

1. What are the units of the coordinates of \( \vec{T} = \Gamma'(s) \)? This answer is related to the answer to exercise 28.
(2) What are the units of $\vec{V} \cdot \vec{T}$?
(3) Suppose the path travels a small, approximately straight segment of length $\Delta s$. What are the units of $\vec{V} \cdot \vec{T} \Delta s$?
(4) What is $\vec{V} \cdot \vec{T} \Delta s$ measuring? This is more subtle than the last question; you need to see exercise 21 and exercise 29
(5) With $\vec{V}$ as a force field, what does the line integral $\int_{\Gamma} \vec{V} \cdot \vec{T} \, ds$ compute?

We say a vector field $\vec{V} = (p, q)$ is CONSERVATIVE, and the corresponding 1-form $p \, dx + q \, dy$ is EXACT if there is a function $u$ so that
\vec{V} = \nabla u; \text{ that is } p = u_x \text{ and } q = u_y. \text{ The function } u, \text{ if it exists, is called the } \textbf{potential}. \text{ The notation for } 1\text{-forms that } du \text{ means } p \, dx + q \, dy \text{ is convenient. We will need to be careful about the domain } \Omega \text{ where functions are defined. If we are going to apply our theorems such as FTC I and II we want } u \text{ to be not only continuous, but moreover to have partial derivatives } u_x \text{ and } u_y \text{ which are continuous. The notation for this is that } u \text{ is } C^1(\Omega).

**Exercise 39** (FTC II for line integrals). Suppose that the vector field \vec{V} has potential \( u \), and \( \Gamma \) is a path in the domain \( \Omega \) of \( u \) connecting points \( \vec{a} \) and \( \vec{b} \). Show that

\[
\int_{\Gamma} \vec{V} \cdot \vec{T} \, ds = u(\vec{b}) - u(\vec{a}).
\]

**Exercise 40** (FTC II for line integrals). Suppose that the vector field \( \vec{V} \) has potential \( u \), and \( \Gamma \) is a closed path in the domain \( \Omega \) of \( u \). What can you deduce about

\[
\int_{\Gamma} \vec{V} \cdot \vec{T} \, ds?
\]

**Exercise 41.** Figure 8 shows a vector field \( \vec{V} \) and two points \( \vec{a} \) below and \( \vec{b} \) above.

1. Find three different paths \( \Gamma \) from \( \vec{a} \) to \( \vec{b} \) that make the line integral \( \int_{\Gamma} \vec{V} \cdot \vec{T} \, ds \) alternately positive, zero, and negative.

2. Is \( \vec{V} \) conservative?

So far we’ve introduced two differential operators, the gradient \( \nabla \) which turns a scalar valued function into a vector field, and the Laplacian which turns one scalar valued function into another. We need two more. The **scalar curl** converts a vector field into a scalar valued function: for \( \vec{V} = (p, q) \) we define

\[
\nabla \times \vec{V} = q_x - p_y.
\]

Similarly, the **divergence** of \( \vec{V} \) is defined to be

\[
\nabla \cdot \vec{V} = (p_x + q_y).
\]

Vector fields whose scalar curl is 0 are called **irrotational**. If, on the other hand the divergence of \( \vec{V} \) is 0 we say \( \vec{V} \) is **incompressible**. This terminology comes from applications to fluid flow.

**Exercise 42.** Suppose \( \vec{V} \) is conservative in \( \Omega \), with potential \( u \).

1. Show that \( \nabla \times \vec{V} = 0 \). Thus conservative fields are irrotational.
(2) Show that \( \Delta u = \nabla \cdot (\vec{V}) \). In particular, the gradient fields of harmonic functions are incompressible.

Another useful operation we can perform on vector fields is to rotate all the vectors 90° counterclockwise around their starting point (not around the origin). In coordinates this is

\[
\vec{V}^\perp = (-q, p) \quad \text{when} \quad \vec{V} = (p, q).
\]

**Exercise 43.** In the last exercise you showed that for any function \( u \), \( \nabla \times (\nabla u) = 0 \). Now show that \( u \) is a harmonic function if and only if \( \nabla \times (\nabla u^\perp) = 0 \). The vector field \( \nabla u^\perp \) is called the HAMILTONIAN vector field.

**Exercise 44 (FTC I for line integrals).** Suppose \( \vec{V} = (p, q) \) is a vector field in \( \Omega \) with the properties

(1) For each \( \vec{z} \) in \( \Omega \), we have\(^1\)

\[
\vec{V}(\vec{z} + \vec{h}) = \vec{V}(\vec{z}) + O(\vec{h}).
\]

\(^1\)The definition of Big Oh statements involving vectors is exactly the same as for scalars, except we replace absolute values \( | | \) by lengths of vectors \( \| \| \).
(2) For every closed path \( \Gamma \) in \( \Omega \),
\[
\int_{\Gamma} \vec{V} \cdot \vec{T} \, ds = 0.
\]
Then \( \vec{V} \) is conservative. This exercise proves it.

Fix any base point \( \vec{z}_0 \) in \( \Omega \), and define a function
\[
u(\vec{z}) = \int_{\Lambda} \vec{V} \cdot \vec{T} \, ds,
\]
where \( \Lambda \) connects \( \vec{z}_0 \) to \( \vec{z} \).

(1) Prove that \( \nu \) is ‘well defined’, that is, if \( \Lambda_1 \) and \( \Lambda_2 \) are two different paths connecting \( \vec{z}_0 \) to \( \vec{z} \), then
\[
\int_{\Lambda_1} \vec{V} \cdot \vec{T} \, ds = \int_{\Lambda_2} \vec{V} \cdot \vec{T} \, ds.
\]
Hint: How did you get exercise 40 from 39? The function \( \nu(\vec{z}) \) will turn out to be the potential. (Compare FTC, I).

(2) Use the triangle inequality to show that the Big Oh hypothesis implies that
\[
p(\vec{z} + \vec{h}) = p(\vec{z}) + O(\|\vec{h}\|).
\]
(3) For any point \( \vec{z} \) in \( \Omega \), let \( h \) be small enough so that the disk of radius \( h \) around \( \vec{z} \) is still contained in \( \Omega \). Let \( \vec{h} = (h, 0) \) a ‘horizontal’ vector. Show that
\[
p(x + h, y) = p(x, y) + O(h).
\]
This is the hypothesis you will need to apply the (scalar) version of FTC, I, below.

(4) Show that for \((a_1, a_2)\) in \( \Omega \),
\[
u(a_1 + h, a_2) - \nu(a_1, a_2) = \int_{t=0}^{t=h} p(a_1 + t, a_2) \, dt
\]
Hint: Consider a horizontal path segment
\[
\tilde{\sigma}(t) = (a_1 + t, a_2) \quad \text{for} \quad 0 \leq t \leq h.
\]
What is true about \( dy \) on this segment?

---

This means that \( \Omega \) needs to have the topological property of being ‘connected’; you can get from any point to any other without leaving \( \Omega \). Draw a picture of a set which is connected, and another which is not. See Figure 1.7 on p. 9 in the text.

This means the domain \( \Omega \) needs to have the topological property of being ‘open’. The set of vectors such that \( x^2 + y^2 < 1 \) is open, but \( x^2 + y^2 \leq 1 \) is not. Convince yourself of this.
(5) Use the (scalar) version of FTC,1 to deduce that
\[ u(a_1 + h, a_2) = u(a_1, a_2) + p(a_1, a_2)h + O(h^2), \]
and thus \( u_x = p \). An analogous argument shows \( u_y = q \).

**Exercise 45.** Let
\[ \vec{V} = \left( \frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right). \]

(1) Compute, carefully, \( \nabla \times \vec{V} \).
(2) Compute \( \int_{\Gamma} \vec{V} \cdot \vec{T} \, ds \) for \( \Gamma \) the unit circle.
(3) Is \( \vec{V} \) conservative?

**Exercise 46.** Let \( \text{arg}(x, y) = \arctan(y/x) \) be the angle the vector \((x, y)\) makes with the \( x \) axis. To be precise, since tangent is periodic we need to specify: \(-\pi < \text{arg}(x, y) \leq \pi\). Compute, carefully, \( \nabla \text{arg} \). Compare this answer with your answers to the previous exercise.

**Exercise 47.** The point of this exercise is that the answers you get depend not just on the functions \( u \) or vector fields \( \vec{V} \), but on the topology of the underlying domain \( \Omega \). This resolves the paradox between exercises 45 and 46. Figure 9 shows the vector field
\[ \vec{V} = \left( \frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right). \]

(1) The largest domain \( \Omega \) on which \( \vec{V} \) is defined is \( \mathbb{R}^2 \setminus \{(0, 0)\} \). In exercise 45 you computed a nonzero line integral around a closed path in \( \Omega \). By exercise 44, \( \vec{V} \) is not conservative in \( \Omega \). Is \( \text{arg}(x, y) \) a continuous function on \( \Omega \)? Is \( \text{arg}(x, y) \) a \( C^1(\Omega) \) function? (See page 29 for the definition of \( C^1(\Omega) \).)
(2) Suppose you further restrict the domain of the function to
\[ \Omega' = \mathbb{R}^2 \setminus \{(x, 0) | x \leq 0\}. \]
This is the plane ‘cut’ along the negative real axis. Consider several closed paths \( \Gamma \) which stay inside \( \Omega' \), and try to visually estimate the line integral \( \int_{\Gamma} \vec{V} \cdot \vec{T} \, ds \). Does it look like they should be positive, negative, or zero?
(3) Is \( \text{arg}(x, y) \) a \( C^1(\Omega') \) function? Is \( \vec{V} \) conservative on \( \Omega' \)?

9. **DOUBLE INTEGRALS**

You are already familiar with a second kind of integration, the double integral. The double integral is easier to understand than the line integral; it is usually introduced first. Instead of having to
deal with a vector field $\vec{V} = (p(x, y), q(x, y))$, we need only a single, scalar valued function $u(x, y)$. The graph of the function defines a surface $z = u(x, y)$ above the $x - y$ plane, and we want to measure the volume under the surface. This is an easy generalization of the concept of area under a graph.

Of course we don’t really care about volume any more than we care about area; it is just a convenient fiction. As before, the units involved tells us what area represents. For example, if the dependent variable $z$ is pressure, measured in pounds per square inch, and $x$ and $y$ are distances in inches, then volume, the product of the three is measured in pounds and so represents force.

In mathematics you get insight by finding more than one way to think about an idea. We will squeeze as much insight as possible, both physical and theoretical, by comparing different kinds of integrals.

10. **Divergence Theorem**

The line integral has a physical interpretation as work, if the vectors in the vector field represent force vectors. This was easy, but not very deep or very useful. In the next two problems we will study a
different, more useful physical interpretation. Our point of view is 'kitchen stove physics', i.e. our common sense intuition.

We’ll let $\vec{V} = (p(x, y), q(x, y))$ be a vector field in the plane. Think of velocity vectors of a fluid flowing over the plane, so $x$ and $y$ are in meters, and $p$ and $q$ are in meters/second. We’ll assume the flow is 'steady state' so even though fluid is moving over time, the way it moves does not change, i.e. $p$ and $q$ don’t depend on a time variable. The fluid may be expanding at some points, and perhaps being compressed at others.

**Exercise 48.** We have an operation on vector fields called the divergence: $\nabla \cdot \vec{V} = p_x + q_y$, a scalar valued function. The domain of $\vec{V}$ is a set $\Omega$. This exercise studies the physical meaning of

$$\int \int_{\Omega} \nabla \cdot \vec{V} \, dA.$$

(1) We can write a vector as a sum of a horizontal and vertical vector, e.g. $(5, 7) = (5, 0) + (0, 7)$. Write the vector field $\vec{V}$ as a sum of vector fields $\vec{U} + \vec{W}$, where $\vec{U}$ has only horizontal flow and $\vec{W}$ only vertical.

(2) Horizontal flow: Suppose we have a vector field $\vec{U}$ flowing horizontally. If the function $p(x, y)$ is positive at a point $(x_0, y_0)$, is the flow right-to-left there or left-to-right?

(3) Suppose $p_x$ is positive at $(x_0, y_0)$. Are the vectors getting longer or shorter as you look from left-to-right? Draw a diagram of several 'consecutive' vectors. Suppose we stick a short segment of pipe into the fluid at $(x_0, y_0)$, oriented in the $x$-direction. Still assuming $p_x$ is positive at $(x_0, y_0)$, write something about how much fluid goes in vs. how much comes out. You may use words like 'source' or 'sink.' Carry out the analogous argument if $p_x < 0$, and draw a diagram. Source or sink?

(4) Imagine small rectangle with sides $\Delta x$ and $\Delta y$ around $(x_0, y_0)$. What units is $\Delta x \Delta y$ measured in? What are the units of $p_x(x_0, y_0)$? (tricky, remember $p_x$ comes from a difference quotient of $p$’s divided by $x$’s.) What about $p_x(x_0, y_0) \Delta x \Delta y$? The double integral

$$\int \int_{\Omega} \nabla \cdot \vec{U} \, dA.$$

is a (limit of a) sum of such numbers. What is this integral measuring?
(5) Vertical flow: Carry out the analogous arguments for the ‘other’ vector field $\vec{W}$.

(6) The double integral - general case: We want to know how the double integrals for $\vec{U}$ and for $\vec{W}$ are related to the double integral for $\vec{V} = \vec{U} + \vec{W}$.

(a) How does $\nabla \cdot (\vec{U} + \vec{W})$ relate to $\nabla \cdot \vec{U}$ and $\nabla \cdot \vec{W}$?

(b) What is $\int \int_\Omega \nabla \cdot \vec{V} dA$ in terms of $\int \int_\Omega \nabla \cdot \vec{U} dA$ and $\int \int_\Omega \nabla \cdot \vec{W} dA$?

(c) What does $\int \int_\Omega \nabla \cdot \vec{V} dA$ measure?

Next we will look at the line integral $\int_\Gamma \vec{V} \cdot \vec{N} ds = \int_\Gamma qdx - pdy$ when the vector field $\vec{V} = (p, q)$ again represents velocity vectors of a fluid flow.

**Exercise 49.** In this case it doesn’t help to split up the vector field, we’ll just write $\vec{V} = (p, q)$. We have a parametrization of the boundary $\partial \Omega$ by arclength $\Gamma(s) = (x(s), y(s))$, with $s$ in meters. We have the unit tangent vector $\vec{T} = (x'(s), y'(s))$ and unit normal vector $\vec{N} = (y'(s), -x'(s))$.

(1) Make a diagram that shows a piece of the boundary. Sketch a pair of perpendicular axes at a point on the boundary, pointing in the $\vec{N}$ and $\vec{T}$ directions. We know from a previous exercise that we can write the flow vector $\vec{V}$ as

$$\vec{V} = (\vec{V} \cdot \vec{N})\vec{N} + (\vec{V} \cdot \vec{T})\vec{T}.$$

(2) What are the units of the coordinates of $\vec{N}$?

(3) Imagine a little segment of the boundary of length $\Delta s$, a window through which fluid flows. What are the units of $(\vec{V} \cdot \vec{N})$? What does this measure?

(4) What units are $\vec{V} \cdot \vec{N} \Delta s$ measured in? What does it measure?

(5) The line integral

$$\int_\Gamma \vec{V} \cdot \vec{N} ds = \int_\Gamma qdx - pdy$$

is a (limit of a) sum of such numbers. What is this integral measuring?
**Exercise 50.** (Gauss Divergence Theorem) Comparing the physical meaning of the double integral in exercise 48 and the line integral in exercise 49, can you draw any conclusion?

Recall that we have four differential operators:

1. The gradient $\nabla$ which turns a scalar valued function into a vector field,
   $$\nabla u = (u_x, u_y).$$
2. The Laplacian which turns one scalar valued function into another,
   $$\Delta u = u_{xx} + u_{yy}.$$
3. The scalar curl which converts a vector field into a scalar valued function,
   $$\nabla \times \vec{V} = q_x - p_y.$$
4. The divergence similarly converts a vector field into a scalar valued function,
   $$\nabla \cdot \vec{V} = p_x + q_y.$$

In fact the scalar curl and the divergence are closely connected. An operation we looked at earlier (exercise 43) is to rotate all the vectors $90^\circ$ counterclockwise around their starting point (not around the origin). In coordinates this is

$$\vec{V}^\perp = (-q, p) \quad \text{when} \quad \vec{V} = (p, q).$$

**Exercise 51.** This exercise shows the connection, and leads you to conjecture another theorem, equivalent to the Divergence Theorem.

1. Compute the divergence $\nabla \cdot (\vec{V}^\perp)$ of the rotated field. What differential operator is this in terms of the original vector field $\vec{V}$?
2. Suppose we have a path $\Gamma = (x(s), y(s))$, with unit tangent vector $\vec{T} = (x', y')$ and unit normal vector $\vec{N} = (y', -x')$. Compute $\vec{V}^\perp \cdot \vec{N}$; how does this relate to the original field $\vec{V}$ and tangent vector $\vec{T}$?
3. You have conjectured that for any vector field $\vec{Z}$,
   $$\iint_{\Omega} \nabla \cdot \vec{Z} \, dA = \int_{\partial \Omega} \vec{Z} \cdot \vec{N} \, ds.$$
   Apply this to the vector field $\vec{Z} = \vec{V}^\perp$ and use the above to get an alternative version of the Divergence Theorem. (The statement is a little cleaner if you cancel a minus sign from
both sides of the equation.) In this notation it is called Green’s Theorem.

The physical meaning of the line integral in Green’s theorem is the CIRCULATION, the net amount of flow around the boundary of \( \Omega \) (just as the Divergence Theorem measured the FLUX across it.) The units are still \( m^2/\text{sec} \). The physical meaning of the double integral is the net rotation within \( \Omega \) of the fluid, in the same units. They are physically equal, although I have a harder time seeing this, since it involves adding rotations together at different points.

**Exercise 52.** We will prove Green’s Theorem first in the easy case that \( \Omega \) is a rectangle in the plane, defined by \( a \leq x \leq b, c \leq y \leq d \).

1. To make the problem even easier, we repeat a trick that worked at the beginning of exercise 48. Use the properties of integrals to break your conjecture up into two separate conjectures; one involving only \( p \) and the other involving only \( q \).
2. We will prove the part involving \( q \); the other case is similar. Write the double integral as an iterated integral (with the \( x \) variable on the inside) and apply the Fundamental Theorem of Calculus.
3. To conclude, we consider the line integral and try to show it is equal to what FTC gave us above. The boundary \( \partial \Omega \) of the rectangle is four straight lines. For two of them, the line integral is easy. Which two, and why?
4. For the other two, write a parametrization of the line as usual, and see the line integral is what we want. Where does the minus sign come from?

To prove the general case, we need to think about the topological properties of the region \( \Omega \).

We will say that a region in the plane is NICE if it has three sides, two of which are lines (parallel to the \( x \) and \( y \) axes), and the third side is a curve, either always increasing or always decreasing. A curve which is increasing or decreasing can be written as the graph of a function two ways, both \( y = f(x) \), and also \( x = g(y) \).

**Exercise 53.** There are four cases for a nice region, depending on where the region lies relative to the straight line sides. Draw an example of each one.

**Exercise 54.** This exercise proves Green’s theorem in the case that \( \Omega \) is a nice region. Suppose there are three sides:

1. The line \( y = c \) for \( a \leq x \leq b \)
(2) The line \( x = b \) for \( c \leq y \leq d \)
(3) The curve connecting \((a, c)\) to \((b, d)\) which can be written \( y = f(x) \) for \( a \leq x \leq b \) and also as \( x = g(y) \) for \( c \leq y \leq d \).

(The other three cases are exactly the same.)

Imitate what you did in exercise 52 to prove Green’s Theorem for this more general case. For variety, you might consider the part of the vector field involving \( p \) (instead of \( q \)) this time.

**Exercise 55.** Suppose the region \( \Omega \) can be cut up into pieces, each of which are nice.

(1) How is the double integral for \( \Omega \) related to the double integrals for each of the pieces? (Easy.)

(2) How is the line integral around the boundary \( \partial \Omega \) related to the line integrals around each of the pieces? (A little harder.)

(3) What do you conclude when \( \Omega \) can be decomposed into nice regions?

**Exercise 56.** The most general case of Green’s Theorem will require some topological handwaving: Any region\(^4\) in the plane can be decomposed into a finite number of nice regions. We will be content with proof by example. Suppose \( \Omega \) is the donut shaped region between two concentric circles. Draw a picture which shows how to decompose it into nice regions. (The mathematical name for a donut shaped region is an ANNULUS.)

The Divergence Theorem is a true generalization of the Fundamental Theorem of Calculus to the plane. In other words, consider a ‘one dimensional Divergence Theorem’. We start with a function of one variable, \( F(x) \). Corresponding to \( \Omega \) we use an interval \([a, b]\). The divergence operator is just \( \frac{d}{dx} \) so corresponding to the double integral of the divergence we have \( \int_a^b F'(x)dx \). The ‘boundary’ of \([a, b]\) is just the two points \( \{a\} \) and \( \{b\} \). A ‘point integral’ of \( F \) over this finite set can only be defined by evaluating \( F \), with the outward orientation relative to the interval \([a, b]\), that is, +1 at \( b \) and −1 at \( a \). So we get

\[
\int_a^b F'(x)dx = +F(b) - F(a),
\]

the Fundamental Theorem of Calculus.

\(^4\)Actually, this requires some hypothesis about the curve \( \Gamma = \partial \Omega \). The functions \( (x(s), y(s)) \) should have derivatives but we’ve secretly been assuming that all along.
Everything in complex analysis is a Corollary to Green’s Theorem (or, equivalently, the Gauss Divergence Theorem.)

We still have the unresolved question of which vectors fields $\vec{V} = (p, q)$ are conservative, or equivalently, which one forms $pdx + qdy$ are exact. We know that if this is true, then $\nabla \times \vec{V} = 0$, but not necessarily conversely.

**Exercise 57.** Suppose that $\Gamma$ is a simple closed curve, and at every point inside $\Gamma$, $\nabla \times \vec{V} = 0$. Show that

$$\int_{\Gamma} \vec{V} \cdot \vec{T} \, ds = 0.$$

We will now define a topological property of set $\Omega$, that will be relevant in deciding which vector fields are conservative. We say that $\Omega$ is **simply connected** if, for every simple closed curve lying in $\Omega$, the inside of $\Gamma$ is a subset of $\Omega$. For example, a set $\Omega$ whose boundary $\partial \Omega$ is itself a simple closed curve has this property. On the other hand, the punctured plane $\mathbb{R}^2 \setminus (0, 0)$ is not simply connected, because the inside of the unit circle contains the missing point $(0, 0)$.

**Exercise 58.** Show that if $\Omega$ is simply connected and $\nabla \times \vec{V} = 0$ at all points $(x, y)$ in $\Omega$, then $\vec{V}$ is conservative. Hint: In exercise 44 you showed that if $\vec{V} = (p, q)$ is a vector field in $\Omega$ with the property that for every closed path $\Gamma$ in $\Omega$,

$$\int_{\Gamma} \vec{V} \cdot \vec{T} \, ds = 0,$$

then $\vec{V}$ is conservative.

**Exercise 59.** Suppose our region $\Omega$ has a boundary which consists of two simple closed curves $\Gamma_1$ and $\Gamma_2$, for example the outer and inner circles which bound an annulus. In order that the right hand normal $\vec{N} = (y', -x')$ point outwards from $\Omega$, we think of $\partial \Omega$ as $\Gamma_1 - \Gamma_2$. That is, as a curve on its own it is oriented counterclockwise as usual. But when we consider it part of the boundary we travel it clockwise.

(1) Suppose $\nabla \times \vec{V} = 0$ in the region between $\Gamma_1$ and $\Gamma_2$. What can you prove about the relationship between the lines integrals

$$\int_{\Gamma_1} \vec{V} \cdot \vec{T} \, ds \quad \text{and} \quad \int_{\Gamma_2} \vec{V} \cdot \vec{T} \, ds?$$
As an application of this, compute the line integral of our favorite vector field

$$\vec{V} = \left( \frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right)$$

from exercise 45 around the curve

$$\Gamma = \{(\cos^3(t), \sin^3(t)) \mid 0 \leq t \leq 2\pi\}$$

see Figure 10.

Any set $\Omega$ in the plane whose boundary consists of $k + 1$ simple closed curves (i.e. Jordan curves) will be called a $k$ CONNECTED JORDAN DOMAIN, or more simply just a Jordan domain. For example, the previous exercise referred to a 1-connected Jordan domain. The boundary curves are oriented so the domain is on the left and the outward normal vector $\vec{V}$ points to the right, out of $\Omega$.

Recall we defined a particular directional derivative of a function $u$ relative to a curve $\Gamma$, the outward normal derivative

$$\frac{\partial u}{\partial \eta} = \nabla u \cdot \vec{N}.$$  

This measures the rate of change of $u$ across $\Gamma$. Suppose, for example, $u$ represents temperature at a point. $\Delta u$ will point in the hottest
direction, and so heat will tend to flow in the direction $-\Delta u$. Except for the minus sign, $\partial u/\partial \eta$ measures the tendency of heat to flow across $\Gamma$.

**Exercise 60** (The Inside-Outside Theorem). Show that when $\Omega$ is a $k$-connected Jordan domain, and $\Gamma = \partial \Omega$,

$$\int_{\Gamma} \frac{\partial u}{\partial \eta} \, ds = \int_{\Omega} \Delta u \, dA,$$

where, of course $\Delta u$ is the Laplacian of $u$.

The name comes from the two sides of the formula, relating what the outward normal is doing outside $\Omega$ to what the Laplacian is doing inside $\Omega$.

11. **HARMONIC FUNCTIONS**

**Exercise 61.** Suppose $u$ is harmonic on a $k$-connected Jordan domain $\Omega$. Show that, for every simple closed curve $\Gamma$ in $\Omega$, whose interior is also contained in $\Omega$, we have

$$\int_{\Gamma} \frac{\partial u}{\partial \eta} \, ds = 0.$$

**Exercise 62.** Conversely, suppose that, for every simple closed curve $\Gamma$ in $\Omega$, whose interior is also contained in $\Omega$, we have

$$\int_{\Gamma} \frac{\partial u}{\partial \eta} \, ds = 0.$$

Show that $u$ is harmonic in $\Omega$. Hint: Suppose there is some point $\vec{z}$ where $\Delta u(\vec{z}) \neq 0$. Without loss of generality, $\Delta u(\vec{z}) > 0$. Then by continuity there is some small disk around $\vec{z}$ where $\Delta u > 0$. (This is the ‘bump principle,’ p. 76 in the text. See also Figure 1.27 on p. 51 of the text.) Take $\Gamma$ to be the boundary of this disk.

Exercises 61 and 62 together give a characterization of harmonic functions in terms of the outward normal line integral. This is remarkable; it relates a differential equation to (a collection of) integral equations! It is also important; it gives the following physical interpretation of what it means for a function to be harmonic:

Suppose $\Omega$ is a Jordan domain whose boundary $\partial \Omega$ is a simple closed curve. We assume that $\Omega$ is made of some heat conducting substance (a metal plate, for example.) Further suppose that at each point $\vec{z}$ of $\partial \Omega$, heat is either being fed into the plate, or perhaps drawn out by cooling. We will denote $u(\vec{z})$ the temperature at each point $\vec{z}$ in $\Omega$. This is a real number, positive, negative, or zero - but will not
be referred to on any scale such as Celsius. We suppose that as $\vec{z}$ moves around $\partial \Omega$, the temperature $u(\vec{z})$ varies continuously.

Heat is flowing in the entire plate, from hotter spots to cooler ones. Thus the temperature $u(\vec{z})$ will vary from one point to another in $\Omega$. But we make the additional very important assumption that at any fixed point $\vec{z}$ the temperature $u(\vec{z})$ does not vary with time. In this case we say that steady state conditions prevail. (This is the analog for heat flow of what we assumed about fluid flow earlier. The two physical problems have similar mathematical formulations.)

Figure 2.1 on p. 71 of the text shows a circular plate being heated on the left and right sides, and cooled on the top and bottom. The vectors show the heat flow. The solid curves are level curves for the function $u(\vec{z})$, that is, curves where the temperature in the plate is constant.

Our goal is to show that $u(\vec{z})$ is a steady state temperature exactly when the function $u(\vec{z})$ is harmonic.

(1) First of all, if $u(\vec{z})$ is a steady state temperature, then the amount of heat (an idea we do not define!) flowing into any small region is the same as the amount flowing out. Otherwise that region would heat up over time (or cool off); the temperature would change over time contradicting the steady state hypothesis. And conversely if the amount of heat flowing into any small region is the same as the amount of heat flowing out, then the temperature does not change over time and we have a steady state.

(2) The mathematical formulation of the physical condition ‘the amount of heat flowing into any small region is the same as the amount of heat flowing out’ is that the net flux of $u$ across any small loop is 0: For all closed curves $\Gamma$ in $\Omega$,

$$\int_{\Gamma} \frac{\partial u}{\partial \eta} \, ds = 0.$$  

(3) By exercises 61 and 62 this happens if and only if $\Delta u(\vec{z}) = 0$ for all $\vec{z}$ in $\Omega$, that is, if and only if $u$ is harmonic.

The two mathematical questions that follow from this characterization of steady state temperatures are existence, and uniqueness.

Existence: Given a temperature $u(\vec{z})$ on the boundary $\partial \Omega$, perhaps positive at some points being heated, perhaps negative at other points being cooled, is there a steady state temperature inside $\Omega$ with this boundary behavior? The answer is ‘Yes’, $u(\vec{z})$ is

\footnote{It is also the cover of the text!}
defined inside $\Omega$ by a line integral around the boundary $\partial \Omega$. Specifically with $\vec{w}$ denoting a point in $\Omega$, $\vec{z} = \Gamma(s)$ the boundary parametrized by arc length, and $r = |\vec{w} - \vec{z}|$, then

$$u(\vec{w}) = -\frac{1}{2\pi} \int_\Gamma \left( \log r \frac{\partial u}{\partial \eta}(\vec{z}) - u(\vec{z}) \frac{\partial \log r}{\partial \eta} \right) ds.$$ 

This follows from something called Green’s Identity III, or you can wait until §17 below.

**Uniqueness:** If we specify the temperature $u(\vec{z})$ on the boundary $\partial \Omega$, can there be more than one steady state temperature $u(\vec{z})$ inside $\Omega$? Your physical intuition may tell you ‘Obviously not,’ (or maybe it won’t). In any case we will prove mathematically that there is at most one solution. (Existence above is the statement that there is at least one solution.

**Exercise 63.** We want to know that if $u_1$ and $u_2$ are harmonic functions with $u_1(\vec{z}) = u_2(\vec{z})$ for $\vec{z}$ on $\partial \Omega$, then $u_1(\vec{w}) = u_2(\vec{w})$ for all $\vec{w}$ in $\Omega$. Show that the following theorem implies this:

‘If $u$ is a harmonic function with $u(\vec{z}) = 0$ for all $\vec{z}$ in $\partial \Omega$, then $u(\vec{w}) = 0$ for all $\vec{w}$ in $\Omega$.’

So it suffices to prove the theorem in quotes above.

**Exercise 64.** Suppose $u$ is a harmonic function on $\Omega$, let $\vec{V}$ denote the vector field $u \nabla u = (u \cdot u_x, u \cdot u_y)$.

1. Show that $\nabla \cdot \vec{V} = u_x^2 + u_y^2$.

2. Suppose that $u = 0$ on $\partial \Omega$. Show that

$$\int \int_\Omega (u_x^2 + u_y^2) dA = 0.$$ 

**Exercise 65.** With the hypotheses of the previous exercise,

1. What does the ‘bump principle’ tell you must be true about $u_x^2 + u_y^2$ on $\Omega$? What do you deduce about $u_x$ and $u_y$?

2. From this, what do you deduce about $u$ on $\Omega$?

3. Since by hypothesis, $u = 0$ on $\partial \Omega$, what do you deduce about $u$ inside $\Omega$?

So what do harmonic functions look like? On a Jordan domain $\Omega$ with no holes (0 connected), we know that $u$ is harmonic with gradient vector field $\vec{V} = (u_x, u_y)$ if and only if $\vec{V}$ is irrotational and incompressible. We know we can rotate a vector field to get another, $\vec{V} \perp$. 
And by what we know about what \( \mathbf{\nabla} \cdot \) and \( \mathbf{\nabla} \times \), the vector field \( \mathbf{V} \perp \) is incompressible and irrotational. So, \( u \) is harmonic if and only if the vector field \( \mathbf{V} \perp \) is the gradient vector field of some other harmonic function \( v \). That is, if \( \mathbf{V} = (u_x, u_y) \) then there is another harmonic function \( v \), called the HARMONIC CONJUGATE, so that

\[
(v_x, v_y) = (-u_y, u_x).
\]

The vectors in the two fields are always perpendicular to each other. Since the gradient vector fields are perpendicular to the level curves for the functions, the level curves for \( u \) and \( v \) are perpendicular as well. These are examples of orthogonal trajectories. In physical terms, if you think of \( u \) as the potential, the level curves for the conjugate \( v \) will be the STREAMLINES along which the fluid, or heat, or whatever flows.

Here’s an example: \( u = x^2 - y^2 \) is harmonic, since \( \Delta u = u_{xx} + u_{yy} = 2 - 2 = 0 \) for any \( x \) and \( y \). We seek to find \( v \) so that \( v_x = -u_y = 2y \) and also \( v_y = u_x = 2x \). Integrating in \( x \) we have

\[
v = \int 2y \, dx = 2xy + C(y),
\]

since the derivative of any function \( C(y) \) with respect to \( x \) is 0. Also we want

\[
2x = v_y = 2x + C'(y) \quad \text{so} \quad C'(y) = 0.
\]

In this case we can take \( C(y) \) to be a constant \( C \), for example \( C = 0 \). (The harmonic conjugate is unique only up to an additive constant.) So with \( u = x^2 - y^2 \) we have \( v = 2xy \). The level curves for this example are shown on the upper left in Figure 11. The level curves \( x^2 - y^2 = \text{const.} \) are hyperbolas with asymptote lines \( y = \pm x \) (shown as solid lines). The level curves for \( 2xy = \text{const.} \) are also hyperbolas, with asymptotes the \( x \) and \( y \) axes (shown as dotted lines). The dotted lines here are the streamlines; compare this with Figure 2.1 on p. 71 of the text.

**Exercise 66.** For each function \( u \) given below, verify that it is harmonic, and find the harmonic conjugate \( v \). Try if possible to identify the corresponding level curves in Figures 12-16.

1. \( x^3 - 3xy^2 \)
2. \( e^x \cos(y) \)
3. \( \log(x^2 + y^2)/2 \)
4. \( x/(x^2 + y^2) \)
We have seen that harmonic functions are the same as steady state temperatures. We will end this section by thinking about the more general, non-steady state case. For functions $u$ of both a position $\vec{z} = (x, y)$ in the plane, and also of time $t$, the heat equation is

$$\Delta u = \frac{\partial u}{\partial t}.$$ 

In the steady state case, $\partial u / \partial t = 0$, so this just says that $u$ is then harmonic. Similarly, the wave equation is

$$\Delta u = \frac{\partial^2 u}{\partial t^2}.$$ 

The idea of a steady state wave doesn't make much sense, so we haven't mentioned this before. The wave equation describes the position of a vibrating surface, such as a drum.

The way one finds a solution to either equation, given an initial temperature or position $U(x, y, 0)$, is to first assume that we can separate the variables $u(x, y, t) = f(x, y)h(t)$. (This is an outrageous
assumption, but we will fix it later.) The heat equation for example then becomes
\[(\Delta f)h = fh'.\]
This means that
\[(\Delta f)/f = \lambda = h'/h\]
for some constant \(\lambda\), since the left side is a function of \((x, y)\) and the right side is a function of \(t\) only. Therefore,
\[\Delta f = \lambda f,\]
which means that \(f\) is an eigenvector of the differential operator \(\Delta\), with eigenvalue \(\lambda\). Once \(\lambda\) is known, the function \(h\) is easy: we have
\[h' = \lambda h,\]
which means that \(h(t) = \exp(\lambda t)\) is an exponential function.

Once the eigenvalues \(\lambda\) and eigenvectors \(f_\lambda\) are known, we can remove the separation of variables hypothesis by trying to write the
general solution $u(x, y, t)$ as a linear combination

$$u(x, y, t) = \sum_{\lambda} c_{\lambda} f_\lambda(x, y) \exp(\lambda t).$$

The initial condition

$$U(x, y, 0) = \sum_{\lambda} c_{\lambda} f_\lambda(x, y)$$

should tell us the coefficients $c_{\lambda}$. (The wave equation is treated similarly, but leads to $h(t)$ equals a trig function.)

To compute the coefficients $c_{\lambda}$, we need the structure of an inner product space. Let

$$\mathcal{N} = \{ \text{smooth real valued functions } f \text{ on } \Omega \mid \partial f / \partial \eta = 0 \text{ on } \partial \Omega \},$$

where ‘smooth’ means that $\Delta f$ makes sense and is also in $\mathcal{N}$. Define the inner product

$$\langle f, g \rangle = \iint_{\Omega} fg \, dA.$$
The condition that the outward normal derivative $\partial f/\partial \eta = 0$ at all points on $\partial \Omega$ neither implies that $f$ is harmonic, nor is implied by that condition. Exercises 61 and 62 require only that $\partial f/\partial \eta = 0$ on average when integrated around closed curves, but for all closed curves $\Gamma$ in $\Omega$, not just $\partial \Omega$.

**Exercise 67.** Show that for arbitrary $f, g$ (i.e., without the hypothesis that $\partial / \partial \eta = 0$ on $\partial \Omega$)

$$
\int_{\partial \Omega} f \frac{\partial g}{\partial \eta} ds = \iint_{\Omega} \nabla f \cdot \nabla g + f \Delta g \, dA.
$$

This is known as Green’s Identity I. Hint: use a big theorem.

Green’s Identity I is some kind of two dimensional analog of integration by parts. To see this, rearrange the terms as

$$
\iint_{\Omega} \nabla f \cdot \nabla g = \iint_{\partial \Omega} f \frac{\partial g}{\partial \eta} ds - \iint_{\Omega} f \Delta g \, dA,
$$
and imagine \( u \) to be \( \nabla g \), and \( dv \) to be \( \nabla f \). Then \( v = f \), and \( du = \nabla \cdot \nabla g = \Delta g \). In this two dimensional version, evaluating \( uv \) at the endpoints is replaced by the line integral.

Exercise 68. Now show that for \( f, g \in \mathcal{N} \), the extra hypothesis implies that the Laplacian \( \Delta \) is self adjoint, that is

\[
\int_{\Omega} \Delta f \ g \ dA = \int_{\Omega} f \ \Delta g \ dA.
\]

Exercise 69. Suppose that \( f(x, y) \) denotes a temperature in \( \Omega \). What is the physical meaning of the condition \( \partial f / \partial \eta = 0 \) on \( \partial \Omega \) defining the vector space \( \mathcal{N} \)? The letter \( \mathcal{N} \) denotes what are called NEUMANN BOUNDARY CONDITIONS. If, instead, we are dealing with a vibrating membrane and the wave equation \( \Delta u = \partial^2 u / \partial t^2 \), the theory is pretty much the same, except, as mentioned above, the relevant functions are trigonometric not exponential, and we consider instead of \( \mathcal{N} \), the vector space

\[
\mathcal{D} = \{ \text{smooth real valued functions } f \text{ on } \Omega \mid f = 0 \text{ on } \partial \Omega \}
\]
defined by the DIRICHLET BOUNDARY CONDITION. What is the physical relevance of the condition \( f = 0 \) on \( \partial \Omega \) for the vibrating membrane problem? What (very minor) changes need to be made in the solution of exercise 68 to see that the Laplacian \( \Delta \) is also self adjoint on the vector space \( D \)?

Basic Math 108B stuff then says the eigenvalues \( \lambda \) are real numbers, and the eigenvectors \( f_\lambda \) are orthogonal, and thus also linearly independent. The coefficients \( c_\lambda \) are then determined by the projection of \( U \) onto the space spanned by the eigenvectors, i.e.;

\[
c_\lambda = \langle U, f_\lambda \rangle / \langle f_\lambda, f_\lambda \rangle
\]

The Spectral Theorem of Math 108B would say the eigenvectors form an orthogonal basis for \( \mathcal{F} \), except for two complications. First, \( \mathcal{F} \) is an infinite dimensional vector space. Second, differential operators such as \( \Delta \) are rather badly behaved; they are not continuous (in the topology on the vector space \( \mathcal{F} \) coming from the norm \( \| f \| = \)
A more general version of the Spectral Theorem still holds, however.

12. COMPLEX NUMBERS, FINALLY

You know that for a real number $x$, $x^2 \geq 0$, so no negative number had a real square root. The terminology ‘real’ and ‘imaginary’ is misleading. A complex number $z = (x, y)$ is just a vector in the plane; it is as simple as that. Complex numbers are no more dubious than any other mathematical object you are used to dealing with. As vectors they add together in the usual way, that is,

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2).$$

Any vector $(x, y)$ can be written in polar coordinates; just specify the length $R$ of the vector and the angle $\theta$ it makes with the $x$-axis. By trigonometry we get that

$$x = R \cos(\theta) \quad \text{or} \quad R = \sqrt{x^2 + y^2}$$
$$y = R \sin(\theta) \quad \text{or} \quad \theta = \arctan(y/x)$$

We will use square brackets $z = [R, \theta]$ when we mean that a pair of numbers is to be interpreted as polar coordinates of a vector. We can easily define a multiplication of vectors in polar coordinates. The rule is that we multiply the lengths and add the angles. That is, if $z_1 = [R_1, \theta_1]$ and $z_2 = [R_2, \theta_2]$ then we define

$$z_1 \cdot z_2 = [R_1 R_2, \theta_1 + \theta_2].$$

In other words if we fix a $z = [R, \theta]$, and define a function from $\mathbb{R}^2$ to $\mathbb{R}^2$ (or equivalently from $\mathbb{C}$ to $\mathbb{C}$) via multiplication by $z$, then it is just DILATION by $R$ and ROTATION by the angle $\theta$.

**Exercise 70.** Write the vector $(0, 1)$ in polar coordinates. Compute $(0, 1) \cdot (0, 1)$, and convert your answer back to rectangular coordinates.

**Exercise 71.** Convert $(1, 0)$ to polar coordinates. What happens if you multiply this vector by any other vector $[R, \theta]$?

This definition is very geometric and very pretty, but it is tedious to convert back and forth between rectangular and polar coordinates. What is the formula for multiplication in rectangular coordinates? We have

$$z_1 = (x_1, y_1) = (R_1 \cos(\theta_1), R_1 \sin(\theta_1))$$
$$z_2 = (x_2, y_2) = (R_2 \cos(\theta_2), R_2 \sin(\theta_2)),$$
so by definition
\[ z_1 \cdot z_2 = (R_1 R_2 \cos(\theta_1 + \theta_2), R_1 R_2 \sin(\theta_1 + \theta_2)) \]
\[ = R_1 R_2 (\cos(\theta_1 + \theta_2), \sin(\theta_1 + \theta_2)). \]

By some trig identities
\[ \cos(\theta_1 + \theta_2) = \cos(\theta_1) \cos(\theta_2) - \sin(\theta_1) \sin(\theta_2), \]
\[ \sin(\theta_1 + \theta_2) = \cos(\theta_1) \sin(\theta_2) + \cos(\theta_2) \sin(\theta_1)). \]

Plug this in, and regroup the \( R \) terms with their matching \( \theta \)s to get
\[ z_1 \cdot z_2 = (R_1 R_2 \cos(\theta_1) \cos(\theta_2) - R_1 \sin(\theta_1) R_2 \sin(\theta_2), \]
\[ R_1 \cos(\theta_1) R_2 \sin(\theta_2) + R_2 \cos(\theta_2) R_1 \sin(\theta_1)). \]

But this just says that
\[ z_1 \cdot z_2 = (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1). \]

You computed above that \((0, 1)^2 = (-1, 0)\), and that the vector \((1, 0)\) acts like the number 1. From now on we will identify a real number \( x \) with the vector \((x, 0)\) on the horizontal axis. We use the special symbol \( i \) to denote the vector \((0, 1)\). So \( i^2 = -1 \) in this notation. A typical complex number can now be written
\[ z = (x, y) = x(1, 0) + y(0, 1) = x \cdot 1 + y \cdot i = x + yi. \]

This is the traditional way to write complex numbers, but you should never forget that they are just vectors in the plane. In this notation
\[ z_1 \cdot z_2 = (x_1 x_2 - y_1 y_2) + (x_1 y_2 + x_2 y_1)i \]

Here’s some terminology that is useful. For \( z = x + yi \), we say that \( x \) is the REAL PART, \( x = \text{Re}(z) \), and \( y \) is the IMAGINARY PART, \( y = \text{Im}(z) \). This is a little confusing; the imaginary part is still a real number. The COMPLEX CONJUGATE of \( z = x + yi \), denoted \( \bar{z} \), is just the complex number \( x - yi \). Geometrically this is flipping the vector across the horizontal axis. The polar coordinates \( R \) and \( \theta \) are often referred to as \( R = |z| = \sqrt{x^2 + y^2} \) is the MODULUS of \( z \) and \( \theta = \tan^{-1}(y/x) \) is the ARGUMENT of \( z \). We always need to be careful and remember to specify some choice such as \(-\pi < \arg(z) \leq \pi\).

**Exercise 72.** Compute the real and imaginary parts of \( z^2 \) in terms of \( x \) and \( y \).

**Exercise 73.** Show that
\[ z \bar{z} = x^2 + y^2 + 0i = R^2. \]

So the length of the vector \( z \) can be computed by \( |z| = R = \sqrt{z \bar{z}}. \)
Since we can multiply, we also want to divide. It is enough to compute $1/z$ for any $z \neq 0$, then $z_1/z_2$ is just $z_1 \cdot 1/z_2$. But
\[
\frac{1}{z} = \frac{1}{x+yi} = \frac{1}{x+yi} \cdot \frac{x-iy}{x-iy} = \frac{x-iy}{x^2+y^2} = \frac{x}{x^2+y^2} - \frac{y}{x^2+y^2}i.
\]

**Exercise 74.** Compute $1/i$.

Finally we need to define some of the basic transcendental functions, like cosine, sine, and the exponential function. Certainly we want
\[
\exp(x+yi) = \exp(x) \exp(yi)
\]
to be true, since the analogous identity is true for real numbers. To understand $\exp(yi)$, we use the series expansion
\[
\exp(yi) = \sum_{n=0}^{\infty} \frac{(iy)^n}{n!} = \sum_{n=0}^{\infty} i^n \frac{y^n}{n!}.
\]
We split the sum up into even $n = 2k$ and odd $n = 2k+1$ terms
\[
= \sum_{k=0}^{\infty} i^{2k} \frac{y^{2k}}{2k!} + \sum_{k=0}^{\infty} i^{2k+1} \frac{y^{2k+1}}{2k+1!} = \sum_{k=0}^{\infty} (-1)^k \frac{y^{2k}}{2k!} + i \sum_{k=0}^{\infty} (-1)^k \frac{y^{2k+1}}{2k+1!} = \cos(y) + \sin(y)i,
\]
by the series expansions for sine and cosine. To summarize
\[
(10) \quad \exp(x+yi) = \exp(x)(\cos(y) + \sin(y)i).
\]

Logarithms of complex numbers are tricky; with $w = u + iv$ we want
\[
x + iy = \log(w)
\]
to mean
\[
(11) \quad w = \exp(x+iy) = \exp(x)(\cos(y) + i \sin(y)).
\]
So certainly taking absolute values in (11) gives
\[
|w| = |\exp(x)(\cos(y) + i \sin(y))| = \exp(x),
\]
since $|\cos(y) + i \sin(y)| = 1$. Thus
\[
(12) \quad \log |w| = x.
\]
Meanwhile if we instead compute the argument in (11)
\[
(13) \quad \arg(w) = \arg(\exp(x)(\cos(y) + i \sin(y))) = \arg((\cos(y) + i \sin(y)) = y,
\]
since scaling the complex number \(\cos(y) + i \sin(y)\) by the real number \(\exp(x)\) does not change its argument. Combining (12) and (13) we get that

\[
(14) \quad \log |w| + i \arg(w) = \log(w)
\]

should be the right definition. The problem is, as we well know by now, that defining the argument \(\arg(w)\) of a vector in the plane requires us to make some choice, such as \(-\pi < \arg(w) \leq \pi\), and the function is not continuous along the negative real axis (with this choice.) We shouldn’t be too surprised, we couldn’t take logarithms of negative real numbers before this, either.

**Exercise 75.** What is \(e^{\pi i}\)?

From the special cases

\[
\exp(yi) = \cos(y) + \sin(y)i \\
\exp(-yi) = \cos(y) - \sin(y)i
\]

we can solve algebraically for \(\cos(y)\) or \(\sin(y)\) to get

\[
\cos(y) = \frac{\exp(yi) + \exp(-yi)}{2} \\
\sin(y) = \frac{\exp(yi) - \exp(-yi)}{2i}
\]

Since this identity relates cosine and sine to exponentials for real variables, it makes sense to define the complex cosine and sine so that this still holds true. So we define

\[
\cos(z) = \frac{\exp(zi) + \exp(-zi)}{2} \quad \sin(z) = \frac{\exp(zi) - \exp(-zi)}{2i}
\]

**13. Visual Complex Analysis**

The concept of graph of a function is a more difficult for complex variables. In the real variables case, the graph of a function like \(y = x^2\) is the set of points of the form \((x, x^2)\) in the two dimensional plane \(\mathbb{R}^2\). In contrast, a complex function has two real inputs \(x\) and \(y\) and two real outputs, the real and imaginary parts of \(f(z)\). With \(f(z) = z^2\) as in the exercise above, the real part is \(x^2 - y^2\) and the imaginary part is \(2xy\). So the graph of this function is the set of points of the form \((x, y, x^2 - y^2, 2xy)\) inside four dimensional space \(\mathbb{R}^4\). Since we can’t see \(\mathbb{R}^4\), we can’t view the graph directly.
One possibility is to think of the function \( f(z) \) as defining a vector field \( \vec{V} = (u(x, y), v(x, y)) \). We still can’t graph this for the reasons described above, but we can and have been making diagrams of vector fields by plotting some of the vectors, see for example Figure 6.

An alternate approach is to think of \( u = \text{Re}(f(z)) \) and \( v = \text{Im}(f(z)) \) separately as functions of two variables. Each then defines a surface in \( \mathbb{R}^3 \). To visualize both surfaces at the same time, we can draw level curves for the surfaces on the \( x-y \) plane. For example, the function \( f(z) = z^2 \) has \( u = x^2 - y^2 \) and \( v = 2xy \). The level curves are shown in Figure 11. The function \( f(z) = 1/z \) has real part \( u = x/(x^2 + y^2) \) and imaginary part \( v = -y/(x^2 + y^2) \). These level curves are shown in Figure 13.

**Exercise 76.** Compute the real and imaginary parts of \( z^3 = (x + iy)^3 \), and of \( \exp(z) = \exp(x + iy) \). Compare your answers to the computations you did in exercise 66 and look at Figures 12 and 15 again. Make a conjecture.

A different way to see what’s going on comes from looking at the dependent variable \( w = f(z) \) in polar coordinates. We can view the positive real number \( |f(z)| \) as defining the height of a surface above the \( z = (x, y) \) plane; this is nothing new. The other polar parameter \( \theta = \arg(f(z)) \) can be interpreted as a color on the color wheel. Each point in the \( z \) plane can be colored with the color associated to \( \arg(f(z)) \). Figure 23(a) shows the colors associated to each point in the complex plane (i.e.; \( f(z) = z \)).

Figure 23(b) shows \( \arg(z^2) \). Since this function doubles angles (as well as squaring lengths), it wraps the color wheel around the plane twice. Figure 23(c) shows \( \arg(z^3) \). This function multiplies angles by 3; it wraps the color wheel around the plane three times.

**Exercise 77.** Figure 23(d) shows \( \arg(1/z) \). Compare this to Figure 23(a). In both figures, each color appears only once as you go around the unit circle counterclockwise. None the less, there is a difference between the two; what is it? Justify this based on the formula for \( 1/z \).

Figure 23(e) shows the argument of

\[
f(z) = (z - 2)^2(z + 2) = z^3 - 2z^2 - 4z + 8.
\]

This cubic polynomial has a single zero at \( z = -2 \), and a ‘double zero’ at \( z = 2 \). You can see this clearly; the zeros are the points where the colors all come together. If you go around the zero on the left, at \(-2\) in a small circle, you see every color once. If, on the other hand,
you go around 2 in a small circle, you see every color twice. If you
go in a larger circle that encompasses both −2 and 2, you see every
color three times. This is as it should be: for \( |z| \) large, \( f(z) \approx z^3 \).

**Exercise 78.** Show that

\[
\begin{align*}
f(z) &= 16(z + 2) + O(z + 2)^2 \quad \text{as } z \to -2 \\
f(z) &= 4(z - 2)^2 + O(z - 2)^3 \quad \text{as } z \to 2.
\end{align*}
\]

Figure 23(f) shows the argument of \( g(z) = (z + 1)/(z - 1) \). This
function has a simple zero at \( z = -1 \), and a simple ‘pole’ (i.e. like
\( 1/z \)) at \( z = 1 \).

14. **Complex derivative**

A function \( w = f(z) \) of a complex variable can be though of as
a vector field \( \vec{w} = (u, v) \) where \( u = \text{Re}(w) \) and \( v = \text{Im}(w) \), or as
a function from \( \mathbb{R}^2 \) to \( \mathbb{R}^2 \) via \( (x, y) \to (u, v) \). We say that \( f \) is \( \mathbb{R}^2 \)-differentiable
at a point \( \vec{c} = (a, b) \) if there is a \( 2 \times 2 \) matrix \( M \) so
that

\[
\vec{w}(\vec{z}) = \vec{w}(\vec{c}) + M (\vec{z} - \vec{c}) + O(\|\vec{z} - \vec{c}\|^2),
\]

or equivalently,

\[
\begin{bmatrix}
u(x, y) \\
v(x, y)
\end{bmatrix} = \begin{bmatrix}
u(a, b) \\
v(a, b)
\end{bmatrix} + M \begin{bmatrix}x - a \\y - b\end{bmatrix} + O\left(\left\| \begin{bmatrix}x - a \\y - b\end{bmatrix} \right\|^2\right).
\]

We are thinking of all the variables as vectors in \( \mathbb{R}^2 \). Compare this
to the definitions in (7) and (9). It is exactly the same idea in higher
dimensions: the derivative at a point is the linear map which best
approximates the function.\(^6\)

**Exercise 79.** By considering the two special cases first \( y = b \), and
then \( x = a \), show that the entries of the matrix \( M \) are the four partial
derivatives

\[
M = \begin{bmatrix}
u_x(a, b) & u_y(a, b) \\
v_x(a, b) & v_y(a, b)
\end{bmatrix}.
\]

\( M \) is called the jacobian matrix.

The mere existence of partial derivatives at a point is not enough to
guarantee that a function is \( \mathbb{R}^2 \) differentiable at that point. (We saw
a counterexample in the case already of functions from \( \mathbb{R}^2 \to \mathbb{R} \)
in exercise 30. If you must have a counterexample \( \mathbb{R}^2 \to \mathbb{R}^2 \), just take

\(^6\)The text makes a comment on p.111 which indicates the author has completely
missed this point.
There is however a very nice theorem in real analysis which says that if the partials \(u_x, u_y, v_x, v_y\) all exist and are continuous not just at \((a, b)\) but also in some small disk around that point, then \(f\) is \(\mathbb{R}^2\) differentiable.

A \(2 \times 2\) matrix has four real parameters; there's no way that can be a complex number. We will restrict our attention to functions where multiplying by the derivative matrix \(M\) is really just multiplying by a complex number. We know every complex number has a polar form \(R \exp(i\theta)\), and that multiplication by this complex number just scales a vector in the plane by \(R\), and rotates it by the angle \(\theta\). In matrix language, this means that \(M\) has the form

\[
M = \begin{bmatrix}
R \cos(\theta) & -R \sin(\theta) \\
R \sin(\theta) & R \cos(\theta)
\end{bmatrix}.
\]

We say \(f\) is \(\mathbb{C}\)-DIFFERENTIABLE if (18) is true, and identify the derivative of \(f\) at \(c\) with the complex number \(R \exp(i\theta)\), denoted \(f'(c)\).

**Exercise 80.** Equations (17) and (18) implies some extremely important identities between the partial derivatives \(u_x, u_y, v_x, v_y\). What are they? These are called the CAUCHY RIEMANN EQUATIONS, abbreviated CRE.

**Exercise 81.** Use your answer to exercise 80 to show that

\[
M \begin{bmatrix} s \\ t \end{bmatrix} = (u_x + iv_x)(s + it)
\]

when we make the usual identification of vectors in the plane and complex numbers. This shows that multiplication by the Jacobian matrix \(M\) is the same as multiplying by the complex number \(u_x + iv_x\), which we identify with \(f'(c)\). By the Cauchy Riemann equations, \(f'(c)\) also equals \(v_y - iv_y\).

Most graduate level complex analysis books use this as a definition, but we will make one further restriction. We will say a function \(w = f(z)\) is HOLOMORPHIC\(^7\) at a complex number \(c\) if it is \(\mathbb{C}\)-differentiable not just at \(c\), but also in some small disk around \(c\), and the derivative varies continuously in this disk. Our hypotheses will thus be a little stronger than the graduate texts, so our theorems will be a little bit weaker. The proofs will be infinitely easier however; we get to use Green’s Theorem. And in the end, exactly the same

\(^7\)The text (p. 113) says that ‘holomorphic’ and ‘analytic’ are synonyms. This is a huge mistake. The definition of analytic will come later. It is a deep theorem that \(f\) is holomorphic if and only if it is analytic.
functions turn out to be holomorphic. For those of you going to graduate school, the distinction is between Cauchy’s Theorem and the Cauchy-Goursat theorem, see p. 165 in the text.

**Exercise 82.** Let \( f(z) = |z|^2 \), that is, \( u = x^2 + y^2 \) and \( v = 0 \).
   
   (i) Show that \( f \) is \( \mathbb{R}^2 \)-differentiable for all \( z \).
   
   (ii) Show that \( f \) is \( \mathbb{C} \)-differentiable only at \( z = 0 \).

This is the canonical example showing that \( \mathbb{C} \)-differentiable at a point does not imply holomorphic.

**Exercise 83.** Show that \( f(z) = \bar{z} \) is \( \mathbb{R}^2 \)-differentiable for all \( z \), but not \( \mathbb{C} \)-differentiable for any \( z \).

**Exercise 84.** Suppose \( f = u + iv \) is holomorphic on \( \Omega \). Using the Cauchy-Riemann Equations you deduced in exercise 80, what can you deduce about the second partials \( u_{xx} \) and \( u_{yy} \). What about \( \Delta u = u_{xx} + u_{yy} \)? How are the vector fields \( \nabla u \) and \( \nabla v \) related?

Actually, the previous exercise requires that these second partials actually exist, and we don’t know that yet; we don’t know the function \( f'(z) \) is itself holomorphic.

**Exercise 85.** Let \( f(z) = \log |z| + i \arg(z) \) where as usual \( |z| = \sqrt{x^2 + y^2} \) and \( \arg(z) = \tan^{-1}(y/x) \).
   
   (i) Compute, carefully, the Jacobean matrix \( M \).
   
   (ii) You should be able to factor a term \( 1/|z| \) out of every entry in the matrix, and view what is left as a rotation by some angle \( \phi \).
   
   (iii) Multiplication by \( z \) is a dilation by \( |z| \) and rotation by \( \arg(z) \). Multiplication by \( M \) does not correspond to multiplying by \( z \), but rather what complex number?
   
   (iv) You have just proven a formula for \( f'(z) \); what is it? We saw \( f(z) \) in (14).

15. **COMPLEX LINE INTEGRAL**

For a function \( f(z) \) of a complex variable \( z \) which is at least continuous, and a piecewise smooth path \( \Gamma \) in the complex plane, we want to define the **LINE INTEGRAL**

\[
\int_{\Gamma} f(z) \, dz.
\]

It is natural to write \( f \) in terms of its real and imaginary parts \( f(z) = u + iv \). And it is reasonable to expect that \( dz = dx + idy \). If we multiply
out $f(z)dz$ we see that we want
\[
\int_{\Gamma} f(z)dz = \int_{\Gamma} (udx - vdy) + i \int_{\Gamma} (udy + vdx)
\]
to be true, and so we take this right hand side as the definition. The line integral is a complex number, whose real and imaginary parts are formed by ordinary line integrals of vector fields in the plane.

How do these vector fields relate to the ones we looked at before? What is the physical interpretation? Remembering that the independent variable $z$ and the dependent variable $w = f(z)$ are just vectors in the plane, we can view the function $f$ as a vector field $f = (u, v)$. To connect up with what we did before, we need to adjust $f$ slightly, so we introduce the corresponding Polya Vector Field
\[
\bar{f} = (u(x, y), -v(x, y)).
\]

**Exercise 86.** Interpret the real part of the integral, namely
\[
\int_{\Gamma} (udx - vdy)
\]
and the imaginary part
\[
\int_{\Gamma} (udy + vdx)
\]
in terms of line integrals involving the Polya vector field $\bar{f}$ and the unit tangent $\vec{T}$ and unit normal $\vec{N}$ to the path $\Gamma$. If we view the Polya vector field as velocity vectors of a fluid flow, what is the physical interpretation of the real part and of the imaginary part of the complex line integral?

It is a little disturbing that the physics involves the Polya vector field $\bar{f}$ rather than the vector field $f = (u, v)$ itself. For one thing, if $f(z)$ is holomorphic, then $\bar{f}(z)$ is not holomorphic because complex conjugation is not, by exercise 83.

**Exercise 87.** Let $\Omega = \mathbb{C}\{x \in \mathbb{R}, x \leq 0\}$ be the complex plane with the negative real axis removed. For $z$ in $\Omega$, we will define a function $F(z)$ by a complex line integral
\[
F(z) = \int_{\Gamma} \frac{1}{w} dw
\]
The path $\Gamma$ is defined to be the following two segments:

$$\gamma_1(t) = t \quad 1 \leq t \leq |z|$$

$$\gamma_2(t) = |z| \exp(it) = (|z| \cos(t), |z| \sin(t)) \quad 0 \leq t \leq \arg(z).$$

In words, $\gamma_1$ goes from 1 to $|z|$ along the real axis; then $\gamma_2$ goes in a circular arc of radius $|z|$ out to $z$. Remember here that $z$ is fixed.

Compute the line integral $F(z)$. Your answer will of course depend on $z$. Looking at the definition of $F(z)$, what do you expect to get? Compare your answer to (14).

If we have a path $\Gamma$ in the complex plane parametrized by a complex values function of a real variable $z(t) = x(t) + iy(t)$, $a \leq t \leq b$, then it is an easy consequence of the definitions that

$$\int_{\Gamma} f(z) \, dz = \int_{a}^{b} f(z(t)) z'(t) \, dt.$$

For example, take $f(z) = 1/z$ and $\Gamma$ the portion of the unit circle connecting 1 to $i$. So we can take $z(t) = \cos(t) + i \sin(t)$, for $0 \leq t \leq \pi/2$. Then one can verify that $f(z(t)) = \cos(t) - i \sin(t)$ so

$$\int_{\Gamma} f(z) \, dz = \int_{0}^{\pi/2} (\cos(t) - i \sin(t))(-\sin(t) + i \cos(t)) \, dt$$

$$= \int_{0}^{\pi/2} -\cos(t) \sin(t) + \sin(t) \cos(t) \, dt + i \int_{0}^{\pi/2} \sin^2(t) + \cos^2(t) \, dt$$

$$= i\pi/2.$$

If it is easier, one can also use polar notation. In the previous example, $z(t) = \exp(it)$, $0 \leq t \leq \pi/2$. Thus $z'(t) = i \exp(it)$, and $f(z(t)) = \exp(-it)$. So,

$$\int_{\Gamma} f(z) \, dz = \int_{0}^{\pi/2} \exp(-it)i \exp(it) \, dt = \int_{0}^{\pi/2} idt = i\pi/2.$$

**Exercise 88.** Let $a$ be any point in the complex plane, and $\Gamma_r$ a circle of radius $r$ centered at $a$. Compute

$$\int_{\Gamma_r} \frac{1}{z-a} \, dz.$$

**Exercise 89.** (the $M$-$L$ inequality) Suppose that there is some constant $M$ so that $|f(z)| \leq M$ for all $z$ on a path $\Gamma$ of length $L$. Show that

$$\left| \int_{\Gamma} f(z) \, dz \right| \leq ML.$$
Hint: We want to reduce this to exercise 37. But the complex line integral has a real and imaginary part, each of which is an ordinary line integral. To fix this, write
\[ \int_{\Gamma} f(z) \, dz = R \exp(i\theta), \]
so we want to show that \( R \leq ML \). Observe that now
\[ \int_{\Gamma} \exp(-i\theta) f(z) \, dz = R \]
is a real number, so the imaginary part of this complex line integral must be 0. Now use exercise 37.

16. ALL THE CAUCHY THEOREMS

**Exercise 90.** (Fundamental Theorem of Calculus, II) Suppose \( \Omega \) is some region in the complex plane, and \( F(z) = U + iV \) is holomorphic in \( \Omega \). Let \( \Gamma \) be a path in \( \Omega \) with endpoints \( a \) and \( b \) (possibly the same), and let \( f(z) = u + iv = F'(z) \).

Show that
\[ \int_{\Gamma} f(z) \, dz = F(b) - F(a). \]

Hint: You will need exercise 81 and of course exercise 39: the version of FTC for line integrals of vector fields.

**Exercise 91.** (Cauchy Integral Theorem) Suppose \( \Omega \) is some region in the complex plane, \( \Gamma \) is a simple closed curve in \( \Omega \), and \( f(z) \) is holomorphic on and inside of \( \Gamma \). Show that
\[ \int_{\Gamma} f(z) \, dz = 0. \]

Hint: Reduce everything to vector fields.

We say that two simple closed curves \( \Gamma_1 \) and \( \Gamma_2 \) in \( \Omega \) are **HOMOTOPI**C in \( \Omega \) if we can ‘stretch’ one into the other without leaving \( \Omega \). (This is an idea from topology, which one can make more precise at the expense of a lot more definitions. Homotopy is an equivalence relation on the set of simple closed curves.) For example, let \( \Omega \) be the punctured plane \( \mathbb{R}^2 \setminus \{(0,0)\} \) and let \( \Gamma_1 \) be the unit circle, and \( \Gamma_2 \) the star shaped curve of Figure 10.

Similarly we say \( \Gamma \) is homotopic to a point in \( \Omega \) if we can shrink it down to a point without leaving \( \Omega \). You may interpret the Cauchy Integral Theorem above as saying that if \( f \) is holomorphic in \( \Omega \) and \( \Gamma \) is homotopic to a point in \( \Omega \), then \( \int_{\Gamma} f(z) \, dz = 0. \)
Exercise 92. (Cauchy Integral Theorem) Suppose $\Omega$ is some region in the complex plane, $\Gamma_1$ and $\Gamma_2$ are simple closed curves homotopic in $\Omega$, and $f(z)$ is holomorphic on the region between them. Show that
\[ \int_{\Gamma_1} f(z)\,dz = \int_{\Gamma_2} f(z)\,dz. \]

We were already doing this back in exercise 59.

Exercise 93. (Cauchy Integral Theorem) More generally, if $\Omega$ is a $k$-connected Jordan domain, then the boundary $\partial \Omega$ consists of $k+1$ simple closed curves. If $f(z)$ is holomorphic in $\Omega$, then
\[ \int_{\partial \Omega} f(z)\,dz = 0. \]

We may interpret this as computing the integral of $f(z)$ over any one of the pieces of $\partial \Omega$ in terms of the remaining $k$ pieces.

Exercise 94. (Fundamental Theorem of Calculus, I) Fix a base point $z_0$ in $\Omega$. Suppose that either

1. $\Omega$ is simply connected and $f(z)$ is holomorphic in $\Omega$, or
2. $f(z)$ is continuous in $\Omega$, so that $f(z + h) = f(z) + O(h)$,

and
\[ F(z) = \int_{z_0}^{z} f(w)\,dw \]

is independent of path.

Then $F(z)$ is holomorphic in $\Omega$, and $F'(z) = f(z)$. Hint: Show that the hypothesis (1) also implies $F(z)$ is independent of path. Now use exercise 44.

Exercise 95. (Cauchy Integral Formula) Suppose $f(z)$ is holomorphic in $\Omega$, $\Gamma$ is a simple closed curve in $\Omega$ with the inside of $\Gamma$ a subset of $\Omega$, and $z$ a point of the inside of $\Gamma$. Then
\[ f(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(w)}{w - z}\,dw. \]

Hints:

1. Observe that $f(w)/(w - z)$ is holomorphic (as a function of $w$, for fixed $z$) in $\Omega \setminus z$. Change the curve to a circle of radius $r$ around $z$.
2. Do the ‘add and subtract stuff’ trick to make $f(z)/(2\pi i)$ appear.
(3) Use the M-L inequality to bound what’s left over. Now let $r$ go to 0.

The theorem says that the values of $f$ inside $\Gamma$ are completely determined by the values of $f$ on the curve $\Gamma$. Converting $f$ to real and imaginary parts, this gives the existence of solutions to boundary value problems for harmonic functions.

**Exercise 96.** Suppose $\Omega$ is a $k$-connected Jordan domain, and $f(z)$ is holomorphic in $\Omega$. Show that

$$f(z) = \frac{1}{2\pi i} \int_{\partial \Omega} \frac{f(w)}{w-z} dw.$$  

Hint: Let $\Gamma_r$ be a circle of radius $r$ for some small $r$, centered at $z$. Use the Cauchy Integral Formula to write $f(z)$ as an integral over $\Gamma_r$. Let $\Omega' = \Omega \setminus D_r$, where $D_r$ is the disk of radius $r$ centered at $z$. The boundary of $\Omega'$ consists of $\partial \Omega$ along with the circle $\Gamma_r^{-1}$. (reverse orientation when though of as a piece of $\partial \Omega'$.) As a function of the variable $w$, the integrand $f(w)/(w-z)$ above is holomorphic on $\Omega'$. Now use the Cauchy Integral Theorem.

We would like to take the Cauchy Integral Formula, which expresses $f(z)$ as an integral with parameter $z$, and differentiate ‘under the integral’ to get an expression for $f'(z)$ as an integral. The subtlety is that integration and differentiation are both limiting processes, and if we change the order the result may be different. We will give a careful proof that is it legal, in a series of relatively simple exercises.

**Exercise 97.** $(n$-th order Cauchy Integral Formula) Suppose that $f(z)$ is holomorphic in $\Omega$, $z$ is in $\Omega$, and $r > 0$ is small enough so that the disk $D_r(z)$ with boundary circle $\Gamma_r$ is a subset of $\Omega$.

(1) We will repeatedly need the following lemma: For $m$ and $n$ positive integers, use the M-L inequality to show that there exists a constant $C$ so that

$$\left| \frac{1}{2\pi i} \int_{\Gamma_r} \frac{f(w)}{(w-z)^m(w-(z+h))^n} dw \right| < C.$$  

Make sure it is clear that the constant $C$ does not depend on $h$. You may assume that $f(w)$ is bounded on $\Gamma_r$, and that $h$ is small enough so that $z+h$ is contained in the disk $D_{r/2}(z)$, which implies that the distance from $z+h$ to the circle $|w-z| = r$ is $\geq r/2$. 
(2) Show that
\[ f(z + h) = f(z) + \frac{h}{2\pi i} \int_{\Gamma_r} \frac{f(w)}{(w - z)(w - (z + h))} \, dw. \]

Hint: apply the Cauchy Integral Formula to each term of the difference \( f(z + h) - f(z) \).

(3) Show that
\[ f(z + h) = f(z) + \frac{h}{2\pi i} \int_{\Gamma_r} \frac{f(w)}{(w - z)^2} \, dw + O(h^2). \]

This already shows that
\[ f'(z) = \frac{1}{2\pi i} \int_{\Gamma_r} \frac{f(w)}{(w - z)^2} \, dw. \]

We now want to prove that \( f'(z) \) is itself holomorphic.

(4) Apply the previous equation to \( f'(z + h) \) as well as \( f'(z) \) to see that
\[ f'(z + h) - f'(z) = \frac{2h}{2\pi i} \int_{\Gamma_r} \frac{f(w)}{(w - z)(w - (z + h))^2} \, dw + O(h^2). \]

(5) Now do the ‘add and subtract’ trick to see that
\[ f'(z + h) - f'(z) = \frac{2h}{2\pi i} \int_{\Gamma_r} \frac{f(w)}{(w - z)^3} \, dw + O(h^2). \]

This shows that \( f'(z) \) is itself differentiable, and its derivative is given by
\[ f''(z) = \frac{2}{2\pi i} \int_{\Gamma_r} \frac{f(w)}{(w - z)^3} \, dw. \]

Furthermore, the expression for \( f''(z) \) is continuous in \( z \), by another application of the lemma (1). Thus \( f'(z) \) is actually holomorphic. We can now proceed by induction, and deduce that derivatives of all orders exist! (This is very different from the situation of real valued functions.) And, by applying the Cauchy Integral Theorem to the circle \( \Gamma_r \) we can say that for \( z \) in \( \Omega \) and \( \Gamma \) any simple closed curve around \( z \) with inside contained in \( \Omega \), then
\[ f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\Gamma} \frac{f(w)}{(w - z)^{n+1}} \, dw. \]

This theorem is remarkable in that it relates derivatives to integrals.
Exercise 98. (Morera’s Theorem) Recall the alternate hypothesis in exercise 94: Suppose \( f(z) \) is continuous in \( \Omega \), and

\[
F(z) = \int_{z_0}^{z} f(w)dw
\]

is independent of path. Or equivalently, suppose

\[
\int_{\Gamma} f(w)dw = 0
\]

for all simple closed curves \( \Gamma \) in \( \Omega \). Then \( f(z) \) is actually holomorphic in \( \Omega \). Hint: apply the previous theorem to \( F(z) \).

(From our treatment of the material, Morera’s Theorem is so easy one wonders why someone’s name is attached to it. There are alternative treatments where one must work much harder at this point. In any case, it will be a very useful theorem later on.)

Now that we know the derivative of a holomorphic function is holomorphic, we can plug a gap that was left in exercise 84. For \( f = u + iv \), the second partials \( u_{xx} \) and \( u_{yy} \) actually exist, so the Cauchy Riemann Equations imply that \( u \) is a harmonic function, and similarly for \( v \).

Exercise 99. (Cauchy Inequalities) Suppose \( f \) is holomorphic in \( \Omega \), \( z \) is a point in \( \Omega \), \( \Gamma_r \) a circle of radius \( r \) centered at \( z \) is such that the inside is contained in \( \Omega \). Let

\[
M(z, r) = \max_{w \text{ on } \Gamma_r} |f(w)|.
\]

Then

\[
|f^{(n)}(z)| \leq n! \frac{M(z, r)}{r^n}.
\]

Hint: Apply the M-L inequality to the \( n \)th order Cauchy Integral Formula. Observe that the distance \( |w - z| \) appearing in the denominator is exactly \( r \), as \( w \) is on the circle and \( z \) is the center.

We say a function \( f(z) \) is ENTIRE if it is holomorphic everywhere on the complex plane \( \mathbb{C} \).

Exercise 100. (Liouville’s Theorem). If \( f(z) \) is entire, and also bounded (i.e., there is some \( B \) so that \( |f(z)| \leq B \) for all \( z \)), then \( f(z) \) must be a constant. Hint: Use the \( n = 1 \) case of the Cauchy Inequalities.
17. Harmonic Functions, Again

In this section we will answer the existence question for the boundary value problem $\Delta u = 0$, when the region $\Omega$ is a disk. In other words, specifying values for $u(z)$ on the boundary of the disk determines the value of a harmonic function $u(z)$ inside the disk as well. Recall that we already answered the uniqueness question in exercises 63 and 64; there is at most one function that does this. Our point of view will be to use the fact that we already know this for holomorphic functions $f(z) = u + iv$, by the Cauchy Integral Formula. We just need to fool around with that theorem to extract the real part $u(z)$. We will start with $\Omega$ a disk of radius $R$ centered at the origin. A simple translation change of variable will give analogous results for a disk centered at any $z_0$ in $\mathbb{C}$.

Before we prove existence, we first need the following important formula for harmonic functions we already have:

Exercise 101. (Poisson Integral Formula) Suppose $u(z)$ is harmonic in some simply connected $\Omega$ containing the disk $|w| \leq R$, so is the real part of a holomorphic function $f(z) = u + iv$ there. Then for $z = r \exp(i\phi)$ with $r < R$, we have

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(R \exp(i\theta)) \frac{R^2 - r^2}{R^2 - 2rR \cos(\theta - \phi) + r^2} d\theta. \quad (19)$$

Here are the steps.

1. Use the polar coordinates for $z$ to show that the point $z^* = R^2/\bar{z}$ is outside the circle of radius $R$.
2. Justify both equalities in

$$0 = \int_{|w|=R} \frac{f(w)}{w - z^*} dw = \int_{|w|=R} \frac{\bar{z}f(w)}{w\bar{z} - R^2} dw.$$ 

3. Justify

$$f(z) = \frac{1}{2\pi i} \int_{|w|=R} f(w) \left\{ \frac{1}{w - z} + \frac{\bar{z}}{R^2 - w\bar{z}} \right\} dw = \frac{1}{2\pi i} \int_{|w|=R} f(w) \left\{ \frac{R^2 - z\bar{z}}{(w - z)(R^2 - w\bar{z})} \right\} dw.$$ 

4. Use the polar coordinates $w = R \exp(i\theta)$ to show that

$$\frac{R^2 - z\bar{z}}{(w - z)(R^2 - w\bar{z})} dw = \frac{id\theta}{R^2 - 2rR \cos(\theta - \phi) + r^2}.$$
(5) Plug (4) into (3), write \( f(z) = u(z) + iv(z), \) and
\[
f(w) = f(R \exp(i\theta)) = u(R \exp(i\theta)) + iv(R \exp(i\theta))
\]
Now take the real part of each side to get (19).

In fact if \( u(R \exp(i\theta)) \) is any continuous function define on the boundary circle of radius \( R \), then we can define a function \( u(z) \) inside the circle by using (19) as a definition. The function defined this way will be harmonic. The proof is a slightly tedious calculation:

(1) The Laplacian \( \Delta \) written in polar coordinates (for \( z = r \exp(i\phi) \)) is
\[u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\phi\phi} = 0.\]

(2) It is legal to differentiate under the integral in (19), as we did in the proof of the \( n \)th order Cauchy Integral Formula. That is,
\[
\Delta u(r \exp(i\phi)) = \frac{1}{2\pi} \int_0^{2\pi} u(R \exp(i\theta)) \Delta P(r, \phi) \, d\theta,
\]
where
\[
P(r, \phi) = \frac{R^2 - r^2}{R^2 - 2rR \cos(\theta - \phi) + r^2}
\]
as in the Poisson Integral Formula.

(3) Finally, a computation shows that \( P \) is harmonic in the variables \( r, \phi \), i.e.
\[
\Delta P(r, \phi) = 0.
\]
This proves the existence of a harmonic function with a given value on the boundary.

A translation change of variables in (19) shows that if \( z_0 \) is any point in the plane and \( z - z_0 = r \exp(i\phi) \) with \( r < R \), then
\[
u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + R \exp(i\theta)) \frac{R^2 - r^2}{R^2 - 2rR \cos(\theta - \phi) + r^2} \, d\theta.
\]

**Exercise 102.** (Circumferential Mean Value Theorem) Use the above equation to give an identity that holds at the center of the circle, i.e., for \( z = z_0 \). Hint: \( \phi \) is not defined, but what is the value of \( r? \)

**Exercise 103.** (Maximum Principle) A harmonic function \( u(z) \) defined on an open set \( \Omega \) has no strict local maximum. That is, for every \( z_0 \) in \( \Omega \), it is not true that there exists some \( R > 0 \) so that \( u(z_0) > u(z) \) for all \( z \) with \( |z - z_0| \leq R \). Hints: Contradiction, the previous exercise, and the M-L inequality.
This implies there is no global maximum, and, since $-u$ is also harmonic, no global minimum. Every critical point is some kind of saddle.

18. Homology

The idea of homotopy of curves lets us change one curve into another, for the purpose (hopefully) of making integrals easier to compute. As mentioned above, homotopy is an equivalence relation on closed curves in $\Omega$. In this section we will look at a weaker equivalence relation, homology.

To start with an example, let $\Omega$ be the complex plane with two small disks removed, say around the points $z = 1$ and $z = -1$. Consider the closed curve $\Gamma$ of Figure 17. (The removed disks are shown in red.)

Exercise 104.  (1) Convince yourself that $\Gamma$ is not homotopic to a point in $\Omega$. (This is a topological question, so we won’t ask for a proof.)

(2) We choose an orientation so that we are moving from left to right on the lower horizontal piece. Follow $\Gamma$ all the way around with this orientation, marking it on Figure 17 as you go.
(3) Convince yourself that by pinching the curve in the middle so it passes repeatedly through \( z = 0 \), we can homotopy \( \Gamma \) in \( \Omega \) to look like the curve in Figure 18.

(4) We view the curve in Figure 18 as four consecutive pieces:
   (a) A curve \( \Gamma_1 \) starting at \( z = 0 \) and going around \( z = 1 \) once counterclockwise (the larger piece), followed by
   (b) a curve \( \Gamma_2 \) starting at \( z = 0 \) and going around \( z = -1 \) once clockwise (the smaller), followed by
   (c) a curve \( \Gamma_3 \) starting at \( z = 0 \) and going around \( z = 1 \) once clockwise (the smaller piece), followed by
   (d) a curve \( \Gamma_4 \) starting at \( z = 0 \) and going around \( z = -1 \) once counterclockwise (the larger).

Convince yourself this is correct. Label the curves and the orientations on Figure 18.

(5) Convince yourself that the curve \( \Gamma_1 \) is homotopic to \( -\Gamma_3 \) in \( \Omega \).

Similarly, \( \Gamma_2 \) is homotopic to \( -\Gamma_4 \) in \( \Omega \).

(6) So, for \( f(z) \) holomorphic in \( \Omega \), even though \( \Gamma \) is not homotopic to a point,

\[
\int_{\Gamma} f(z) \, dz = \int_{\Gamma_1+\Gamma_2+\Gamma_3+\Gamma_4} f(z) \, dz = \\
\int_{\Gamma_1} f(z) \, dz + \int_{\Gamma_2} f(z) \, dz + \int_{\Gamma_3} f(z) \, dz + \int_{\Gamma_4} f(z) \, dz = \\
\int_{\Gamma_1} f(z) \, dz + \int_{\Gamma_2} f(z) \, dz - \int_{-\Gamma_3} f(z) \, dz - \int_{-\Gamma_4} f(z) \, dz = \\
\left( \int_{\Gamma_1} f(z) \, dz - \int_{-\Gamma_3} f(z) \, dz \right) + \left( \int_{\Gamma_2} f(z) \, dz - \int_{-\Gamma_4} f(z) \, dz \right) = 0
\]

Make sure you really believe this. For each equality above, give a reason why it is true.

This example indicates that homotopy is not quite the right idea for doing complex analysis. We will make a brief detour into topology and algebra. The idea is that if we fix a base point \( z_0 \) in \( \Omega \) (like the point \( z_0 = 0 \) in the example above), we can consider all the closed curves that start and end at the base point. The set of all such actually forms a semigroup in the sense of Math 111A, where the group operation \( \Gamma_1 \ast \Gamma_2 \) is simply the path that first does \( \Gamma_1 \), then \( \Gamma_2 \). The identity element \( e \) is the constant path that never leaves the base point \( z_0 \).

In fact, for purposes of complex analysis we might as well combine curves that are homotopic in \( \Omega \), and consider equivalence classes of curves... These next few paragraphs are optional, you may skim.
curves rather than the curves themselves. The operation described above is ‘well defined’ on homotopy classes of curves. (This must be proven.) In fact, now we actually have a group, not just a semigroup: the inverse of \( \Gamma \) is simply travelling around \( \Gamma \) backwards. The composition of these two is not the trivial curve \( e \), but it is homotopic to \( e \). (Since we are using a multiplicative notation \( * \) for the group operation, we will now write \( \Gamma^{-1} \) instead of \( -\Gamma \) for the inverse of \( \Gamma \).)

The group we get this way is called the \textit{Fundamental Group} of \( \Omega \), denoted \( \pi_1(\Omega) \). Let’s denote the homotopy class of a closed curve \( \Gamma \) by \( \{ \Gamma \} \), a typical element of \( \pi_1(\Omega) \).

Exercise 104 shows that the group \( \pi_1(\Omega) \) is in general not abelian: We had curves \( \Gamma_1 \), and \( \Gamma_2 \), and \( \Gamma_3 \) which is \( \equiv \Gamma_1^{-1} \), and \( \Gamma_4 \equiv \Gamma_2^{-1} \) (where \( \equiv \) means homotopic to.) In that exercise you deduced that

\[
\Gamma \equiv \Gamma_1 * \Gamma_2 * \Gamma_1^{-1} * \Gamma_2^{-1} \text{ is not } \equiv e.
\]

This means, multiplying on the right by \( \Gamma_2 * \Gamma_1 \), that

\[
\Gamma_1 * \Gamma_2 \text{ is not } \equiv \Gamma_2 * \Gamma_1.
\]

If we fix a function \( f(z) \), holomorphic in \( \Omega \), we have a map (i.e. a function) from the group to the complex numbers, determined by
integrating:
\[ \rho_f : \pi_1(\Omega) \to \mathbb{C} \]
\[ \{ \Gamma \} \to \int_{\Gamma} f(z)dz \]

The mapping \( \rho_f \) is well defined on equivalence classes because of Cauchy’s Integral Theorem. But more is true: The mapping \( \rho_f \) is actually a group homomorphism:
\[ \rho_f ([\Gamma_1] \ast [\Gamma_2]) = \rho_f ([\Gamma_1]) + \rho_f ([\Gamma_2]) \]

This is just a lot of fancy notation for the fact that
\[ \int_{\Gamma_1 \ast \Gamma_2} f(z)dz = \int_{\Gamma_1} f(z)dz + \int_{\Gamma_2} f(z)dz, \]
which is just a generalization of our axiom (iv) for integrals.\(^8\)

Here’s the point. Since we have a homomorphism into the abelian group \( \mathbb{C} \), it must evaluate to 0 on all the ‘commutators’ of the form
\[ \Gamma_1 \ast \Gamma_2 \ast \Gamma_1^{-1} \ast \Gamma_2^{-1}, \]
even when they are homotopically nontrivial. This was the content of part (6) of exercise 104. More generally, the value \( \rho_f(\Gamma) \) depends not on the homotopy class of \( \Gamma \), as Cauchy’s Integral Theorem told us, but merely on the coset of \( \Gamma \) modulo the subgroup of commutators. We call these cosets homology classes, and say that two closed curves \( \Gamma \) and \( \Gamma’ \) are HOMOLOGOUS if they are in the same coset modulo commutators, i.e. \( \Gamma’ \ast \Gamma^{-1} \) is in the subgroup generated by the commutators. We write \( \Gamma \sim \Gamma’ \) in this case. The quotient group
\[ \pi_1(\Omega)/\text{commutators} \]
is the abelianization \( \pi_1(\Omega)_{ab} \), the first homology group.

Suppose \( z \) is a point in the complex plane, and \( \Gamma \) is a closed curve (not necessarily simple), which does not pass through the point \( z \). We will need the following fact: the complex number
\[ n(\Gamma, z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{1}{w - z}dw \]
is actually an integer, which we define to be the WINDING NUMBER of \( \Gamma \) around \( z \). The proof is as follows: let \( \Gamma \) be parametrized by \( \sigma(t) \),

\(^8\)Again, we are using a multiplicative notation \( \ast \) now (on the left) where we used to write \( + \).
Define a function
\[ \phi(x) = \int_a^x \frac{\sigma'(t)}{\sigma(t) - z} \, dt \]
for \( a \leq x \leq b \). So \( \phi(a) = 0 \) and we want to show that \( \phi(b)/(2\pi i) \) is an integer. It suffices to show that
\[ \exp(\phi(b)) = 1. \]
The integrand defining \( \phi \) is continuous as a function of \( t \), because the curve \( \Gamma \) does not meet \( z \). So, by the Fundamental Theorem of Calculus, \( \phi \) is differentiable and
\[ \phi'(x) = \frac{\sigma'(x)}{\sigma(x) - z}. \]
(This is still true even though \( \phi \) is a function from \( \mathbb{R} \) to \( \mathbb{C} \). The proof goes through the same way.) Now let
\[ \Phi(x) = (\sigma(x) - z) \exp(-\phi(x)). \]

**Exercise 105.**

1. Compute, carefully, that \( \Phi'(x) = 0 \) for all \( x \) by using the chain rule.
2. Deduce that for all \( x \),
\[ (\sigma(x) - z) \exp(-\phi(x)) = \sigma(a) - z, \]
so in particular
\[ (\sigma(b) - z) \exp(-\phi(b)) = \sigma(a) - z. \]
3. Now use the fact that we have a closed curve.

From the terminology, we expect that this integral defining \( n(\Gamma, z) \) should somehow measure how many times the curve \( \Gamma \) ‘winds around’ the point \( z \). We will not make this topological idea precise, but offer the following justification.

If \( \Gamma \) is a simple closed curve, with the counterclockwise orientation, \( n(\Gamma, z) = 0 \) if \( z \) is outside \( \Gamma \), and \( n(\Gamma, z) = 1 \) if \( z \) is inside.

**Exercise 106.** Quote the relevant Cauchy Theorems which prove this.

More generally, if \( \Omega \) is a \( k \)-connected region, the boundary \( \partial \Omega \) consists of \( k + 1 \) simple closed curves. The complement \( \mathbb{C} \setminus \Omega \) consists of \( k + 1 \) connected components, exactly one of which is unbounded. Suppose \( \Gamma \) is a closed curve which lies inside \( \Omega \).

**Exercise 107.**

1. As a function of \( z \) in \( \mathbb{C} \setminus \Omega \), \( n(\Gamma, z) \) is continuous (since everything in sight is.) But it is also integer valued. What can you deduce about it?
(2) By the M-L inequality,

\[ |n(\Gamma, z)| \leq L \max_{w \text{ on } \Gamma} \frac{1}{|w - z|}, \]

where as usual $L$ is the length of $\Gamma$. What can you deduce about the value of $n(\Gamma, z)$ on the unbounded component of $\mathbb{C}\setminus\Omega$, where we can find points $z$ arbitrarily far away from $\Gamma$?

Let $\Omega$ be a Jordan domain (of any connectivity). We say two different closed curves $\Gamma_1$ and $\Gamma_2$ are HOMOLOGOUS in $\Omega$ if the winding numbers are equal:

\[ n(\Gamma_1, z) = n(\Gamma_2, z) \quad \text{for all } z \text{ in } \mathbb{C}\setminus\Omega. \]

We denote this $\Gamma_1 \sim \Gamma_2$. (Similarly, we say $\Gamma$ is homologous to 0 and write $\Gamma \sim 0$ if the winding number $n(\Gamma, z) = 0$ for all points $z$ in $\mathbb{C}\setminus\Omega$.) You may object that we already defined homologous differently in the paragraphs you skimmed above. In fact, the two different definitions can be proven to be equivalent.

**Exercise 108.** Convince yourself that the curve $\Gamma$ in exercise 104 winds 0 times around $z = 1$, and the same for $z = -1$. You may use your intuitive grasp of what ‘winding’ is supposed to mean, or try to do it more carefully by using the Cauchy Theorems to compute the relevant integrals.

**Exercise 109.** (Cauchy Integral Theorem, homology version) Suppose $f(z)$ is holomorphic in a Jordan domain $\Omega$, and $\Gamma$ is a closed curve in $\Omega$. If $\Gamma \sim 0$ then

\[ \int_{\Gamma} f(z) dz = 0. \]

If $\Gamma_1 \sim \Gamma_2$ are two closed curves in $\Omega$, then

\[ \int_{\Gamma_1} f(z) dz = \int_{\Gamma_2} f(z) dz. \]

This exercise proves the first statement, the second is similar. Suppose to start that we have a point $z$ which lies on the curve $\Gamma$.

(1) Use exercise 96 to get an expression for $f(z)$ in terms of an integral around $\partial \Omega$.

(2) Now use this expression to write $\int_{\Gamma} f(z) dz$ as a double integral, and change the order of integration.

(3) The parameter $w$ is on $\partial \Omega$ which is contained in $\mathbb{C}\setminus\Omega$; use the hypothesis.
19. Power series

In the next few sections we will think about functions which can be represented by a power series

\[ f(z) = a_0 + a_1(z - a) + a_2(z - a)^2 + \cdots + a_n(z - a)^n + \cdots \]

\[ = \sum_{n=0}^{\infty} a_n(z - a)^n \]

The power series is described by an infinite list of complex numbers, the \( a_n \), and the base point \( a \).

The relevant questions we need to address are

1. What does it mean to add infinitely many things?
2. On what domain does this operation actually define a function?
3. What are the analytic properties of the function (continuous? holomorphic?)
4. Conversely, given a function \( f(z) \), can it be represented this way?

We start by answering question (1), somewhat more generally. Suppose \( \{c_n\} \) is an infinite list of numbers (a sequence). When we write \( \sum_{n=0}^{\infty} c_n \) (an infinite series or just a series) we mean a new list of numbers \( \{s_n\} \) formed by taking partial sums

\[ s_0 = c_0 \]
\[ s_1 = c_0 + c_1 \]
\[ s_2 = c_0 + c_1 + c_2 \]
\[ \vdots \]

We will say in general a series converges if the sequence of partial sums \( s_n \) approach some limiting value \( S \), and we’ll say the series diverges otherwise. More precisely, the sequence of partial sums is said to converge, if, for every \( \epsilon > 0 \), there is some integer \( N \) (typically a function of \( \epsilon \)) so that

\[ |s_n - S| < \epsilon \quad \text{for all} \quad n > N. \]

In this case we say the series converges, and write \( \sum_{n=0}^{\infty} c_n = S \).
The Ur-series example is the geometric series. We fix a complex number \( z \) such that \(|z| < 1\), and we define \( c_n \) to be simply \( z^n \).

**Exercise 110.** Show that
\[
\sum_{n=0}^{\infty} z^n \text{ converges to } S = \frac{1}{1-z}.
\]
The algebra used to solve exercise 5 will be useful. You need to say explicitly how big \( N \) must be, in terms of \( \epsilon \) and \( z \), so that the condition of the definition holds.

**Exercise 111.** Use algebra to write
\[
\frac{1}{2} - \frac{z}{2} = \frac{1}{2} \cdot \frac{1}{1 - (z/2)} \quad \text{and} \quad \frac{1}{2} - z^3,
\]
as power series.

To answer questions (2) and (3) we will need also to consider a new series formed by taking absolute values, and say that the series \( \sum c_n \) CONVERGES ABSOLUTELY if the series of absolute values \( \sum |c_n| \) converges, i.e. if the sequence
\[
s_0 = |c_0| \\
s_1 = |c_0| + |c_1| \\
s_2 = |c_0| + |c_1| + |c_2| \\
\vdots
\]
converges.

For example, it is trivially true that if \(|z| < 1\), then \(||z|| < 1\), which implies that \( \sum |z|^n \) converges to \( 1/(1-|z|) \) by exercise 110, and so the Geometric series converges absolutely for \(|z| < 1\). Already we can have an example with positive answers to questions (1)-(4); this is a power series if we define the \( a_n = 1 \) for all \( n \), and take the base point \( a = 0 \). The function \( f(z) = 1/(1-z) \) is holomorphic in the open unit disk \(|z| < 1\), and has a power series expansion given by the Geometric series.

**Exercise 112.** Let \( t_n = n(n+1)/2 \) be the \( n \)th triangular number, and
\[
S_N = 1/t_1 + 1/t_2 + \ldots + 1/t_n + \ldots + 1/t_N.
\]
the sum of the reciprocals of the first \( N \) triangular numbers.
(1) Find a closed form expression for $S_N$. (Hint: partial fractions on $2/(n(n + 1))$.)

(2) Compute

$$\sum_{n=1}^{\infty} \frac{2}{n(n + 1)}.$$ 

**Exercise 113.** In this exercise we show that the Harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges.

(1) View each term $1/n$ being summed as the area of a rectangle with base the interval $(n, n + 1)$ of width 1 on the x axis, and height $1/n$. Draw a rough diagram of the first few rectangles. On this same graph, sketch the function $y = 1/x$. It should lie under all the rectangles.

(2) Thus the area under $y = 1/x$, from 1 up to some big integer $N$, is less than the sum of the areas of the rectangles, which is the $N$-th partial sum of the series $s_N = 1 + 1/2 + 1/3 + \cdots + 1/N$. Use calculus to compute the area under $y = 1/x$ from $x = 1$ up to $x = N$.

(3) Use this to show that as $N \to \infty$, $s_N \to \infty$.

This trick can often be made to work, its called the **INTEGRAL TEST** for convergence.

The Harmonic series shows that for infinite series $\sum_{n} c_n$, the fact that the terms $c_n$ tend to 0 is **not** sufficient to show the series converges. It is, however, a **necessary** condition, which we will now show.

A sequence $\{s_n\}$ is called a **CAUCHY SEQUENCE** if, for all $\epsilon > 0$, there is some $M$ so that for all pairs of integers $m, n > M$, we have that $|s_m - s_n| < \epsilon$.

**Exercise 114.** Suppose the sequence $\{s_n\}$ converges, say to $S$. Show it is a Cauchy sequence. Hint: by hypothesis for all $\epsilon > 0$ we have an $N = N(\epsilon)$ so the definition of convergence holds. You must specify an $M = M(\epsilon)$ so the Cauchy criterion holds. The ‘add and subtract’ trick may be useful.

**Exercise 115.** Suppose an infinite series $\sum_{n} c_n$ converges to some number $S$. So by definition, the sequence $\{s_n\}$ of partial sums converges to $S$, and thus by the above exercise the sequence $\{s_n\}$ is Cauchy. Show the sequence $\{c_n\}$ of terms in the series has limit $0$. 
An application of the idea of absolute convergence is the \textsc{Comparison Test}. It uses information about one series to get information about another. Here’s how it works. Suppose \( \sum c_n \) is some series that we know converges absolutely. Given another series \( \sum b_n \), if \( |b_n| \leq |c_n| \) for all \( n \), then the comparison test says that the series \( \sum b_n \) also converges absolutely. The reason this works is that if \( \sum |c_n| \) converges, its partial sums don’t increase to infinity. The partial sums for \( \sum |b_n| \) are all less than the partial sums for \( \sum |c_n| \).

Conversely if the series with the \textit{smaller} terms \( \sum b_n \) does \textit{not} converge absolutely, then neither does the series with the larger terms \( \sum c_n \).

The next exercise proves a lemma which shows that any power series has a \textsc{Radius of Convergence} \( R \), with \( 0 \leq R \leq \infty \), so that the series will converge absolutely for \( |z - a| < R \).

\textbf{Exercise 116.} Suppose the power series \( \sum a_n(z-a)^n \) converges for \( z = z_1 \) some point in the complex plane. Show the series converges \textit{absolutely} for all \( z \) with \( |z-a| < |z_1-a| \), i.e. in a disk of radius \( |z_1-a| \) centered at \( a \). Hints:

(1) First justify the following claim: The exists some bound \( B \) so that \( |a_n(z_1-a)^n| < B \) for all \( n \).

(2) In the series \( \sum_{n=0}^{\infty} |a_n(z-a)^n| \),

\hspace{1cm} do the ‘multiply and divide’ trick, use (1), and compare to the only series we already know converges.

The Geometric series is the tool that makes \textit{everything} work in this section. It is the answer to every question. We love the Geometric series. And now the radius of convergence \( R \) is just the largest \( R \) that works, or more precisely the supremum of the set \( \{|z-a| \mid \text{the power series converges at } z\} \).

The series converges absolutely if \( |z-a| < R \), diverges if \( |z-a| > R \). on the boundary \( |z-a| = R \), we have no information.

The \textsc{Ratio Test} for convergence of series say the following: If \( \sum_{n=0}^{\infty} c_n \) is any series, let \( L \) be the limit

\[ L = \lim_{n \to \infty} \frac{|c_{n+1}|}{|c_n|}. \]

Then the series converges absolutely if \( L < 1 \), and it diverges if \( L > 1 \). There is no information if \( L = 1 \), anything can happen.
**Exercise 117.** This exercise proves the convergence in case \( L < 1 \). Suppose \( L \) as in the theorem satisfies \( L < 1 \); i.e.

\[
L = \lim_{n \to \infty} \frac{|c_{n+1}|}{|c_n|} < 1.
\]

Since \( L \) is less than 1, it is a property of real numbers that there is another number \( r \) between \( L \) and 1. (For example, \( r = (L + 1)/2 \) works.)

1. Convince yourself that since \( r \) is bigger than the limit \( L \), its true that
   \[
   \frac{|c_{n+1}|}{|c_n|} < r
   \]
   for \( n \) big enough, say bigger than some \( N \). (\( N \) depends on \( r \), but is then fixed independent of everything else.)

2. Use this to show that for any \( k \)
   \[
   |c_{N+k}| < r^k |c_N|.
   \]

3. Now compare to a Geometric series.

In the case when \( L > 1 \), one can show similarly that the individual terms \( |c_n| \) are increasing, so the series diverges.

We typically use this with \( c_n = a_n(z-a)^n \) coming from some power series \( \sum_n a_n(z-a)^n \) centered at \( z = a \). The ratio test is extremely useful because so many power series involve \( n! \), and the ratios cancel so nicely.

In the series for \( e^z \), we have

\[
\frac{|a_{n+1} z^{n+1}|}{|a_n z^n|} = \frac{|z|^{n+1}/(n+1)!}{|z|^n/n!} = \frac{|z|^{n+1} n!}{|z|^n (n+1)!} = \frac{|z|}{n+1}.
\]

So the limit \( L = 0 \) for any \( z \) and therefore the series always converges absolutely, for any \( z \). We say the radius \( R = \infty \).

For the geometric series \( a_n = 1 \) for every \( n \), so

\[
\frac{|a_{n+1} z^{n+1}|}{|a_n z^n|} = \frac{|z|^{n+1}|n|}{|z|^n|n+1|} = |z|.
\]

Thus the the series converges absolutely if \( |z| < 1 \), as we already knew. The radius of convergence is 1.

For power series \( \sum_n a_n(z-a)^n \), if all the \( a_n \) are different from 0, then the ratio test implies that if

\[
\lim_n \left| \frac{a_{n+1}}{a_n} \right| = L, \quad \text{then} \quad R = \frac{1}{L}.
\]
If the coefficients are 0 in some regular way, one can still apply the ratio test, but it is a little more complicated. For example, consider the power series
\[ \sum_{n=0}^{\infty} \frac{z^{3n}}{2^n}. \]
(See exercise 111). The only coefficients which are not 0 are those indexed by a multiple of 3. We can instead view this as a series in powers of \( z^3 \), with all the coefficients nonzero. We have
\[ \frac{|z|^{3(n+1)}/2^{n+1}}{|z^{3n}|/2^n} = |z|^3/2. \]
Thus the series converges absolutely if \( |z|^3 < 2 \), or, \( |z| < 2^{1/3} \). This shows the radius of convergence is \( R = 2^{1/3} \).

20. ANALYTIC FUNCTIONS

We say a function \( f(z) \) is ANALYTIC if it is given by a power series expansion, convergent in some open disk of radius \( R > 0 \). (Exercise 156 below shows that \( R = 0 \) can happen, but this is boring. It defines a function \( f \) only at the base point \( a \), and \( f(a) = a_0 \), the constant term.)

To study the ‘analytic’ properties of analytic functions, we generalize from sequences of numbers to sequences of functions. We say a sequence of functions \( \{s_n(z)\} \) defined on some set \( \Omega \) CONVERGES POINTWISE to a function \( f(z) \) if, for each \( z \) in \( \Omega \), the sequence of numbers converges. More precisely, for every \( z \), and for every \( \epsilon > 0 \), there is an \( N = N(\epsilon, z) \) so that for \( n > N \),
\[ |s_n(z) - f(z)| < \epsilon. \]
If it seems to you like we’ve done nothing here, you’re right. Pointwise convergence is the obvious definition, but it needs a name.

More subtle is the idea of UNIFORM CONVERGENCE. We say the sequence \( \{s_n(z)\} \) converges to \( f(z) \) uniformly on \( \Omega \), if, for every \( \epsilon > 0 \), there is an \( N = N(\epsilon) \) so that for \( n > N \) and for all \( z \) in \( \Omega \),
\[ |s_n(z) - f(z)| < \epsilon. \]
The idea is that the same \( N \) works for all the \( z \) simultaneously. Here’s the standard example (works even for real variables.)

Exercise 118. Let \( \Omega = (0, 1) \) be the open unit interval, and \( s_n(x) = x^n \).
FIGURE 19. Pointwise, not uniform, convergence

(1) Show that $x^n$ converges pointwise to 0. That is, given an $x$ and an $\epsilon$, how big does $N = N(\epsilon, x)$ have to be so that

$$x^n < \epsilon \quad \text{for } n > N?$$

(2) Show that $x^n$ does not converge uniformly to 0. That is, for every $\epsilon > 0$ and for every $N$, find an $x$ in $(0, 1)$ so that

$$x^N > \epsilon.$$ 

Figure 19 shows a graph of the functions $x^n$, for $n = 1, \ldots, 10$.

We care about uniform convergence because it preserves properties of functions, lets us do stuff like pass limits through integrals or through derivatives. Pointwise convergence does not even preserve the property of continuity, as a modification of the example above will show: Let $\Omega$ be the closed unit interval $[0, 1]$, and again consider the sequence of function $\{x^n\}$. Each is continuous because it is a polynomial. For $x < 1$, the sequence converges pointwise to 0, as we saw above. For $x = 1$, the sequence is the constant sequence $\{1\}$ which converges to 1. Thus the sequence of continuous functions
converges pointwise to the function
\[
\delta_1(x) = \begin{cases} 
0 & \text{if } x \neq 1 \\
1 & \text{if } x = 1,
\end{cases}
\]
which is not continuous.

To prove the theorem in the case of uniform convergence, we better have a precise definition of continuity. So, a function \( f(z) \) is CONTINUOUS at \( w \), if, for every sequence \( z_n \) converging to \( w \), we have that the sequence \( f(z_n) \) converges to \( f(w) \). In other words, for all convergent sequences \( \{z_n\} \), we have that
\[
\lim_{n} f(z_n) = f(\lim_{n} z_n).
\]
So \( f \) is continuous if and only if we can pass limits through it. This definition is equivalent to the \( \delta \)-\( \epsilon \) version of continuity, but I prefer it because it is more conceptual.

We can now show that if \( f_n(z) \) is a sequence of continuous functions which converges uniformly on \( \Omega \) to \( f(z) \), then \( f(z) \) is also continuous. To show this, suppose \( \{z_n\} \) is any sequence which converges to \( w \) in \( \Omega \). We need to show the sequence \( f(z_n) \) converges to \( f(w) \). Suppose \( \epsilon > 0 \) is given.

1. By hypothesis, the sequence of functions \( f_n(z) \) converges uniformly to \( f(z) \). So there exists some \( N = N(\epsilon) \) so that
\[
|f_N(z) - f(z)| < \epsilon/3 \quad \text{for all } z \text{ in } \Omega.
\]
The parameter \( N \) is now fixed.

2. By hypothesis the function \( f_N(z) \) is continuous, so the sequence \( f_N(z_n) \) converges to \( f_N(w) \). This means there exists some \( M = M(N, \epsilon) \) so that for all \( n > M \),
\[
|f_N(z_n) - f_N(w)| < \epsilon/3.
\]

3. Thus
\[
|f(z_n) - f(w)| = |f(z_n) - f_N(z_n) + f_N(z_n) - f_N(w) + f_N(w) - f(w)| \leq |f(z_n) - f_N(z_n)| + |f_N(z_n)| - f_N(w) + |f_N(w) - f(w)| = \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.
\]
And so the sequence \( f(z_n) \) converges to \( f(w) \), as required.

**Exercise 119.** Let \( \Omega \) be the open interval \((0, 1)\), and define a sequence \( f_n(x) \) so the graph of \( y = f_n(x) \) is a straight line from the point \((0, 2n)\) to the point \((1/n, 0)\), and \( f_n(x) = 0 \) for \( 1/n \leq x < 1 \). (We could write
a formula for $f_n(x)$, but the geometric description is easier.) Figure 20 shows a graph of the functions $f_n(x)$, for $n = 1, \ldots, 10$.

1. Compute $\int_0^1 f_n(x)\,dx$ for each $n$. Geometry will be easier than calculus.
2. Compute $\lim_n \int_0^1 f_n(x)\,dx$. This should be trivial if you did (1) correctly.
3. Show that $f_n(x)$ converges pointwise to 0. Specifically, given $0 < x < 1$ and $\epsilon > 0$, provide an integer $N = N(x)$ so that if $n > N$, then $f_n(x) = 0$ and so is less than $\epsilon$.
4. Deduce that

$$\lim_n \int_0^1 f_n(x)\,dx \neq \int_0^1 \lim_n f_n(x)\,dx$$

in this case. The difficulty is that we merely have pointwise, not uniform convergence.

Because this next exercise proves the theorem, instead of merely providing a counterexample, we will do it in the complex setting.
**Exercise 120.** Suppose \( \{s_n(z)\} \) is a sequence of continuous functions, uniformly convergent to \( f(z) \) in \( \Omega \), and \( \Gamma \) is a curve in \( \Omega \). Show that
\[
\int_{\Gamma} s_n(z) \, dz \quad \text{converges to} \quad \int_{\Gamma} f(z) \, dz.
\]
Equivalently, this says that
\[
\lim_n \int_{\Gamma} s_n(z) \, dz = \int_{\Gamma} \lim_n s_n(z) \, dz.
\]
Hint: It is enough to prove that
\[
\int_{\Gamma} (s_n(z) - f(z)) \, dz \quad \text{converges to} \quad 0.
\]
The M-L inequality saves the day again. Be sure to notice exactly how uniform convergence is required.

We treated sequences of functions generally above but right now we care about the special case when the sequence of functions \( s_n(z) \) is given by partial sums of a power series:
\[
s_n(z) = a_0 + a_1(z - a) + \cdots + a_n(z - a)^n = \sum_{j=0}^{n} a_j(z - a)^j.
\]
In this case the functions \( s_n(z) \) are polynomials, thus continuous and holomorphic. We already know the convergence is absolute in the disk \( |z - a| < R \). But is it pointwise or uniform?

**Exercise 121.** Suppose a power series
\[
f(z) = \sum_j a_j(z - a)^j
\]
has radius of convergence \( R > 0 \), and \( r < R \) is some positive number. Then the partial sums \( s_n(z) \) converge to \( f(z) \) uniformly on the closed disk \( |z - a| \leq r \). Hints:

1. For \( |z - a| = r < R \) we have absolute convergence. Write down explicitly what that means in terms of the ‘tail’ of the series, i.e. the terms from \( N + 1 \) to \( \infty \).
2. Now for all \( z \) with \( |z - a| \leq r \), apply the triangle inequality to \( |s_n(z) - f(z)| \).

This result is subtle. The convergence is *not* uniform on the entire domain, the open disk \( |z - a| < R \). None the less, for *any* \( r < R \), the convergence *is* uniform on the smaller closed disk \( |z - a| \leq r \). Since each \( s_n(z) \) is a polynomial; it is continuous. Thus the uniform limit \( f(z) \) is continuous at \( w \) for every \( w \) with \( |w - a| \leq r \).
But \( r \) was arbitrary! Given a \( w \) with \( |w - a| < R \), there is some \( r \) so that \( |w - a| \leq r < R \). So \( f(z) \) is continuous at \( w \) for all \( w \) in the open disk \( |w - a| < R \).

**Exercise 122.** This exercise provides the counterexample to show that convergence of a power series need not be uniform on the entire open disk. Specifically, consider \( f(z) = \sum_{n=0}^{\infty} z^n = 1/(1 - z) \) for \( |z| < 1 \), and restrict attention to \( z \) a real number \( x, 0 \leq x < 1 \). Negating the definition of uniform convergence, it suffice to show there is some \( \epsilon > 0 \) such that, for all \( N \), there is some \( x = x(N) \) so that the \( N \)'th partial sum

\[
\sum_{n=0}^{N} x^n > \epsilon.
\]

Try \( \epsilon = 1/2 \), and don’t forget exercise 5.

**Exercise 123.** This exercise proves one half of a very important theorem: Analytic functions are holomorphic functions on \( \Omega = \) the open disk \( |z - a| < R \).

1. Let \( \Gamma \) be any closed curve in \( \Omega \), and \( r < R \) be such that \( \Gamma \) actually lies in the closed disk \( |z - a| \leq r \). What does the Cauchy Integral Theorem tell you about the integral of the polynomial\(^{10}\)

\[
\int_{\Gamma} s_n(z)dz?
\]

2. What do exercises 120 and 121 tell you about the relationship between

\[
\int_{\Gamma} f(z)dz = \int_{\Gamma} \lim_{n} s_n(z)dz \quad \text{and} \quad \lim_{n} \int_{\Gamma} s_n(z)dz?
\]

3. What does Morera’s Theorem (exercise 98) tell you about \( f(z) \)?

So the limit \( f(z) \) has a derivative, but what is it? You might expect it is the limit of the derivatives \( s'_n(z) \). This is true, but needs to be proven.

**Exercise 124.** Suppose we have a sequence \( \{f_n(z)\} \) of holomorphic functions on \( \Omega \), which converge to \( f(z) \), uniformly on every closed disk \( D \) of radius \( r \) contained in \( \Omega \). By the footnote to the previous exercise, \( f(z) \) is holomorphic. Then the sequence \( \{f'_n(z)\} \) converges

\(^{10}\)In fact this same proof works for any sequence \( \{f_n(z)\} \) of holomorphic functions, uniformly convergent. There is nothing special about polynomials here.
to $f'(z)$, uniformly on every closed disk $D$. Hints: Given $\varepsilon > 0$ and a closed disk $D$, choose a simple closed curve $\Gamma$ so that

$$D \subset \text{inside} \quad \Gamma \subset \Omega.$$ 

Let $d$ be the difference between the radius of $\Gamma$, and the radius $r$ of $D$. So for $z$ in $D$ and $w$ on $\Gamma$, $|z - w| \geq d$.

(1) Apply the Cauchy Formula for derivatives to the difference $f'(z) - f'_n(z)$.

(2) Apply the M-L inequality.

Exercise 124 tells us we can differentiate a power series ‘term by term’. So if

$$f(z) = \sum_{j=0}^{\infty} a_j(z - a)^j \quad \text{then} \quad f'(z) = \sum_{j=0}^{\infty} ja_j(z - a)^{j-1}.$$ 

The ratio test quickly shows this new power series also has radius of convergence $R$, (since $\lim_j (j+1)/j = 1$.) Thus $f'(z)$ can also be differentiated term by term, and we proceed by induction.

Exercise 125. (Taylor’s Theorem for power series) Suppose $f(z) = \sum_n a_n(z - a)^n$ is a power series with radius of convergence $R > 0$. Show that

$$\frac{f^{(n)}(a)}{n!} = a_n.$$ 

Exercise 126. (Taylor’s Theorem for holomorphic functions) Suppose $f(z)$ is any holomorphic function on a domain $\Omega$. Let $a$ be a point in $\Omega$, and $R$ the largest radius such that the open disk $D : |z - a| < R$ is a subset of $\Omega$. This exercise proves the other half of the theorem alluded to above: the function $f(z)$ is analytic at $a$. By the result of exercise 125, once we know $f(z)$ is analytic, the coefficients are necessarily given by

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (z - a)^n.$$ 

Hints: Let $z$ be in $D$, then there is some $r$ (for example $r = (R + |z - a|)/2$) so that $z$ is in the disk $D' : |z - a| < r$. Let $\Gamma$ be a circle centered at $a$ such that

$$D' \subset \text{inside} \quad \Gamma \subset D \subset \Omega.$$ 

(1) For $z$ in $D'$ and dummy variable of integration $w$ on $\Gamma$, write down the Cauchy Integral Formula.
(2) Verify that
\[ \frac{1}{w-z} = \frac{1}{w-a} \frac{1}{1 - \frac{z-a}{w-a}}. \]

(3) What can you say about the absolute value of \((z-a)/(w-a)\)?
What does that suggest to you? Think hard about the only example we explicitly know so far.

(4) Switch sum and integral. Why is this legal?

(5) Now apply the \(n\)-th order Cauchy Integral Formula, exercise 97.

This equivalence between holomorphic functions and analytic functions is deep and beautiful. Now that we’ve got it, we will use these two terms interchangeably.

The zeros of analytic functions can’t get ‘too close to each other’, in the sense that is made precise by this exercise:

**Exercise 127.** Suppose \(z_0\) is in \(\Omega\), and \(f(z)\) is analytic in \(\Omega\) with \(f(z_0) = 0\). If \(f(z)\) is not the identically zero function, then there is some \(\epsilon > 0\) such that \(f(z) \neq 0\) for all \(z \neq z_0\) in the disk of radius \(\epsilon\) around \(z_0\).

Hints:

(1) Write
\[ f(z) = \sum_{k=N}^{\infty} a_k(z-z_0)^k \]
\[ = a_N(z-z_0)^N + O(z-z_0)^{N+1} \]
with \(N \geq 1\) and \(a_N \neq 0\). Why can you assume this?

(2) Define a function
\[ g(z) = \sum_{k=0}^{\infty} a_{N+k}(z-z_0)^k \]
\[ = a_N + O(z-z_0) \]

(3) What is the simple algebraic relationship between \(f(z)\) and \(g(z)\)?

(4) \(g(z)\) is another convergent power series (ratio test), therefore analytic, holomorphic and thus continuous. Use this to produce the desired \(\epsilon\).

(5) Now what about \(f(z)\)?

**Exercise 128.** Let \(f(z)\) be an analytic function on \(\Omega\), different from the 0 function. Let
\[ S = \{ z \in \Omega \mid f(z) = 0 \}. \]
Then for every point \( z_0 \) in \( S \), there is some \( \epsilon = \epsilon(z_0) \) so that
\[
S \cap D_{\epsilon}(z_0) = \{ z_0 \}.
\]
(We say that the set \( S \) has no accumulation point.) Hint:

1. Suppose by contradiction, that this is not true. What does continuity of \( f \) tell you about \( f(z_0) \)?
2. Now use the previous exercise.

We can extend this idea from one function being zero to two functions being equal.

**Exercise 129.** Suppose \( f(z) \) and \( g(z) \) are analytic functions on \( \Omega \), and \( f(z) = g(z) \) for all \( z \) in some smaller set \( \Omega' \) which is still big enough to contain, say, a disk. Show that \( f(z) = g(z) \) for all \( z \) in \( \Omega \).

This says in particular that if there is some way to extend the definition of an analytic function from a smaller set \( \Omega' \) to a larger set \( \Omega \) (this is called **analytic continuation**), then it is unique.

**Exercise 130.** Suppose a holomorphic function \( f(z) \) has power series expansion at \( a \) given by
\[
f(z) = \sum_{n=0}^{\infty} c_n (z - a)^n
\]
for \( |z - a| < R \). We write \( z - a = r \exp(i\theta) \) in polar coordinates. The following three steps will show Parseval’s identity\(^{11}\) that
\[
\frac{1}{2\pi} \int_0^{2\pi} |f(a + r \exp(i\theta))|^2 \, d\theta = \sum_{n=0}^{\infty} |c_n|^2 r^{2n}.
\]

1. Show that
\[
f(z) \bar{f}(z) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} c_m \bar{c}_n r^{m+n} \exp(i(m - n)\theta).
\]
2. Show that for integer \( k \neq 0 \),
\[
\int_0^{2\pi} \exp(ik\theta) \, d\theta = 0.
\]

(N.B. This is not a complex line integral. Why the hypothesis \( k \neq 0 \)?)

---

\(^{11}\)This is the Fourier series analog of the Math 108B theorem for orthonormal bases that
\[
\vec{v} = c_1 \vec{e}_1 + \cdots + c_n \vec{e}_n \quad \text{implies that} \quad ||\vec{v}||^2 = |c_1|^2 + \cdots + |c_n|^2.
\]
Integrate both sides of (a) from 0 to 2π.

**Exercise 131.** Letting \( f(z) \) be as in the previous problem, show that
\[
\left( \sum_{n=0}^{\infty} |c_n|^2 r^{2n} \right)^{1/2} \leq \max_{|z-a|=r} |f(z)|.
\]

**Exercise 132.** Suppose the function \( f(z) \) in exercise 130 is entire, so the series expansion holds for all \( R \) (i.e. \( R = \infty \)). Suppose also that \( f(z) \) is bounded, i.e. there is some \( M \) so that \( |f(z)| \leq M \) for all complex \( z \). Use Parseval’s identity (22) to show that the \( c_n = 0 \) for all \( n > 0 \), and thus \( f(z) \) is constant. (This is another proof of Liouville’s Theorem.)

**Exercise 133.** Suppose the function \( f(z) \) in exercise 130 has a local maximum at \( z = a \), so that for some \( \epsilon > 0 \)
\[
|f(a + r \exp(i\theta))| \leq |f(a)| \quad \text{for all } \theta, \text{ and for all } r \leq \epsilon.
\]
Use Parseval’s identity (22) to show that the \( c_n = 0 \) for all \( n > 0 \), and thus \( f(z) \) is constant. (This is another proof of the Maximum Principle.)

**Dirichlet series.** Given a sequence \( \{a_n\} \) of complex numbers, we can form instead of a power series a different kind of function called a **Dirichlet series**
\[
L(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}.
\]
Here, for historical reasons we use \( s \) for the name of the complex variable, and write \( s = \sigma + it \) for the real and imaginary parts. By \( n^{-s} \) we mean \( \exp(-s \log(n)) \). The first example of a Dirichlet series is the **Riemann Zeta function**
\[
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.
\]
We have already observed that \( \zeta(1) \) is the Harmonic series, which diverges.

**Exercise 134.** Show that for \( \sigma = \text{Re}(s) > 1 \), the series defining \( \zeta(s) \) converges absolutely. Hint: Integral test.

**Exercise 135.** Suppose the series (23) converges for some \( s_1 = \sigma_1 + it_1 \).
Show that (23) converges absolutely for all \( s \) with \( \sigma > \sigma_1 + 1 \). This is the analog for Dirichlet series of exercise 116 for power series. It says that (except for pathological situations) a Dirichlet series converges absolutely in some half plane. Hint: Imitate exercise 116.
Exercise 136. With $\sigma_1$ exactly as in the previous exercise, let $B = \sigma_1 + 1$, so $L(s)$ converges absolutely for $\sigma > B$.

(1) For any $\beta > B$, show that the convergence is uniform on the (closed) half plane defined by $\sigma \geq \beta$.

(2) Show that $L(s)$ is a continuous function of $s$ on the (open) half plane defined by $\sigma > B$.

This is the analog for Dirichlet series of exercise 121.

Exercise 137. With $B$ as in the previous exercise, show that $L(s)$ is holomorphic on the half plane $\sigma > B$. Hint: which exercise for power series are we imitating?

In fact, one can prove a little more. Using Summation by Parts, it can be shown that if $L(s_1)$ converges, then $L(s)$ converges for $\sigma > \sigma_1$. And for any $\delta > 0$, the convergence is uniform in the angle $\arg(s - s_1) < \pi/2 - \delta$, from which $L(s)$ is holomorphic on the half plane $\sigma > \sigma_1$.

Unlike the case of power series, there can be a region of only conditional convergence between the region of absolute convergence, and the region of divergence. For example,

$$\phi(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}$$

converges for $\sigma > 0$ by the alternating series test, but converges absolutely only for $\sigma > 1$, by exercise 134. The series diverges for $\sigma \leq 0$, since the terms do not have limit 0. By the previous (unproven) remarks, $\phi(s)$ is a holomorphic function in the half plane $\sigma > 0$.

Exercise 138. Show that for $\sigma > 1$,

$$(1 - 2^{1-s})\zeta(s) = \phi(s).$$

Since $1 - 2^{1-s}$ is equal 0 only for $s = 1 + 2\pi in/ \log(2)$, for integer $n$, we know that $(1 - 2^{1-s})^{-1}$ is holomorphic except for these values. Thus we can extend the function $\zeta(s)$ to the set $\Omega$ defined by \{Re($s$) > 0, $s \neq 1 + 2\pi in/ \log(2)$\} via

$$\zeta(s) = (1 - 2^{1-s})^{-1}\phi(s).$$

Exercise 139. Dirichlet series are used as generating functions to study sequences $\{a_n\}$ with interesting multiplicative properties. For example, show that

(1)

$$\zeta(s)^2 = \sum_{n=1}^{\infty} \frac{\tau(n)}{n^s},$$
where \( \tau(n) = \sum_{d|n} 1 \), the DIVISOR FUNCTION, counts the number of divisors of an integer \( n \).

\[
\zeta(s)\zeta(s-1) = \sum_{n=1}^{\infty} \frac{\sigma(n)}{n^s},
\]

where \( \sigma(n) = \sum_{d|n} d \), the SIGMA FUNCTION, is the sum of the divisors of an integer \( n \).

21. EXAMPLES AND SHORTCUTS

Most of this section is stolen from the Math 3C reader. We won’t go over it in class; you are expected to review on your own.

Every derivative of \( e^z \) is \( e^z \), and \( e^0 = 1 \). So the Taylor series for \( e^z \) at \( z = 0 \) is

\[
1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \cdots + \frac{z^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{z^n}{n!}.
\]

The functions \( \sin(z) \) and \( \cos(z) \) have a nice pattern in their Taylor polynomials, because the derivatives repeat after the fourth one: \( \sin(z), \cos(z), -\sin(z), -\cos(z), \sin(z), \cos(z), \) etc. When we plug in \( 0 \) we get a sequence \( 0, 1, 0, -1, 0, 1, 0, -1, \ldots \). So by formula (20) the Taylor series for \( \sin(z) \) at \( z = 0 \) is

\[
z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots + (-1)^n \frac{z^{2n+1}}{(2n+1)!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}.
\]

and the Taylor series for \( \cos(z) \) at \( z = 0 \) is

\[
1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots + (-1)^n \frac{z^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}.
\]

We already know about the Geometric series

\[
(1 - z)^{-1} = 1 + z + z^2 + \cdots + z^n + \cdots = \sum_{n=0}^{\infty} z^n.
\]

The Taylor series (24), (25), (26), and (27) come up so often that you will certainly need to know them.
For an example of a new function, consider

\[
\begin{align*}
\frac{1}{(0!)^2} - \frac{(z/2)^2}{(1!)^2} + \frac{(z/2)^4}{(2!)^2} - \frac{(z/2)^6}{(3!)^2} + & \cdots \\
\cdots + (-1)^n \frac{(z/2)^{2n}}{(n!)^2} + & \cdots = \\
\sum_{n=0}^{\infty} (-1)^n \frac{(z/2)^{2n}}{(n!)^2}.
\end{align*}
\]

We will eventually use these to define new functions. The power series defined by (28) is the Bessel function, \(J_0(z)\), with applications to many PDEs that arise in physics. In fact, the level curves for the real and imaginary parts of \(J_0(z)\) are shown in Figure 14.

We can add power series together, multiply, or make substitutions. This will give the Taylor series of new functions from old ones. For example, to get the Taylor series of \(\sin(2z)\), we just take (25) and substitute \('2z'\) for \('z'\). Thus the series for \(\sin(2z)\) is

\[
2z - \frac{8z^3}{3!} + \frac{32z^5}{5!} - \cdots + (-1)^n \frac{2^{2n+1}z^{2n+1}}{(2n+1)!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n+1}z^{2n+1}}{(2n+1)!}.
\]

Similarly if we want the series for \(e^{z^3}\), we take (24) and substitute \('z^3'\) for \('z'\). Thus the series for \(e^{z^3}\) is

\[
1 + z^3 + \frac{z^6}{2} + \frac{z^9}{6} + \cdots + \frac{z^{3n}}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{z^{3n}}{n!}.
\]

We can also add or subtract series. The series for \(e^z - 1 - z\) is

\[
\frac{z^2}{2} + \frac{z^3}{6} + \cdots + \frac{z^n}{n!} + \cdots = \sum_{n=2}^{\infty} \frac{z^n}{n!}.
\]

since \(1 + z\) is its own Taylor series. To get the series for \(z^2e^z\), just multiply every term in the series for \(e^z\) by \(z^2\), to get

\[
z^2 + z^3 + \frac{z^4}{2} + \frac{z^5}{6} + \cdots + \frac{z^{n+2}}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{z^{n+2}}{n!}.
\]

MORAL: It is almost always easier to start with a known series and use these shortcuts, than it is to take lots of derivatives and use (20).

Exercise 140. Use (27) to compute the Taylor series at \(z = 0\) for each of these functions: \((1+z)^{-1}, (1-z^2)^{-1}, (1+z^2)^{-1}\)
Exercise 141. Use the methods of this section to compute the Taylor series at $z = 0$ of $z^2 \cos(3z)$. Compare this to trying to compute from the definition (20). Re-read the moral at the end of this section.

Exercise 142. Compute the Taylor series at $z = 0$ of $\sinh(z) = (e^z - e^{-z})/2$, $z/(z + 1)$, $ze^{-z}$, and $z^2 \sin(z^2)$.

Exercise 143. Compute the Taylor series at $z = 0$ of $\sin(z)/z$ and $(e^z - 1)/z$.

Exercise 144. Consider the function $f(z) = (1 + z)^t$, where $t$ is some real number. We will use this function to define the binomial series. Notice $f^{(0)}(0) = f(0) = 1$, and for $k > 1$ the $k$th derivative at $0$ is
\[
f^{(k)}(0) = \frac{t(t-1)(t-2)\cdots(t-k+1)}{k!},
\]
exactly $k$ terms

Thus if we define
\[
\binom{t}{0} = 1, \quad \binom{t}{k} = \frac{t(t-1)(t-2)\cdots(t-k+1)}{k!},
\]
then the Taylor series for $(1 + z)^t$ is exactly $\sum_{k=0}^{\infty} \binom{t}{k} z^k$.

Write out explicitly $\binom{t}{2}$, $\binom{t}{3}$, and $\binom{t}{4}$. When you multiply it out, $\binom{t}{4}$ is about 0.0453135.

It becomes clear that if $t$ is an integer, and $k > t$, then $\binom{t}{k} = 0$. In this case $f(z)$ is a polynomial, and its Taylor series is finite. These binomial coefficients $\binom{t}{k}$ may be familiar to you, from Pascal’s triangle. If $t$ is not an integer, the Taylor series has infinitely many terms.

Write out the first four terms for the binomial series for $(1 + z)^{1/2}$, i.e. $t = 1/2$. This is, of course, just the degree 4 Taylor polynomial. Now plug in $z = .037$ to get a good approximation to $\sqrt{1.037}$. Compare this to what your calculator gives.

Exercise 145. Use the binomial series with $t = -1/2$ to find the Taylor series for $1/\sqrt{1 - z^2}$.

Above we saw how to add, subtract, and make substitutions in power series. Now we will take derivatives, integrate, and multiply.

Derivatives are easy, as you already know the rules for derivatives of polynomials. Thus, since the derivative of $(1 - z)^{-1}$ is $(1 - z)^{-2}$, we get the Taylor series for $(1 - z)^{-2}$ by taking the derivative of every
term in (27):

$$\frac{d}{dz} (1 + z + z^2 + z^3 + \cdots + z^n + \cdots) = 0 + 1 + 2z + 3z^2 + \cdots + nz^{n-1} + \cdots = \sum_{n=1}^{\infty} nz^{n-1} = \sum_{n=0}^{\infty} (n+1)z^n.$$  

Notice there are two ways to write the series in Σ notation, by keeping track of the power of $z$, or by keeping track of the coefficient. Since the derivative of $e^z$ is $e^z$, we should have that the Taylor series (24) is its own derivative, and it is:

$$\frac{d}{dz} (1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \cdots + \frac{z^n}{n!} + \cdots) = 0 + 1 + z + \frac{z^2}{2} + \cdots + \frac{nz^{n-1}}{n!} + \cdots.$$  

Every term shifts down by one. Notice that $n/n!$ is just $1/(n-1)!$.

We can also integrate power series ‘term by term’. For example, since the antiderivative of $(1 - z)^{-1}$ is $-\ln(1 - z)$, we get the Taylor series for $-\ln(1 - z)$ by computing the antiderivative of each term in (27):

$$z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots + \frac{z^n}{n} \cdots = \sum_{n=1}^{\infty} \frac{z^n}{n}.$$  

If we want instead $\ln(1 - z)$, of course we have to multiply every term by $-1$. There is a subtle point here; the antiderivative of a function is determined only up to a constant (the ‘$+C$’ term of Math 3B). In this example, $-\ln(1 - z)$ is the unique choice of antiderivative that is zero at $z = 0$. That value of the function determines the constant term of the series expansion. So in this case, the constant term is 0.

This idea lets us get a handle on functions that do not have a simple antiderivative. For example, you may have learned in Math 3B that the function $e^{-z^2}$ has no simple antiderivative. The graph of this function is the ‘bell shaped’ curve, with applications in statistics. If we take a Taylor series for $e^{-z^2}$:

$$1 - z^2 + \frac{z^4}{2} - \frac{z^6}{6} + \cdots + \frac{(-1)^n z^{2n}}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{n!}.$$  

and integrate term by term we get:

\[
z - \frac{z^3}{3} + \frac{z^5}{10} - \frac{z^7}{42} + \cdots + \frac{(-1)^n z^{2n+1}}{(2n+1)n!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)n!}.
\]

This is the Taylor series for the antiderivative of \(e^{-z^2}\) which is 0 at \(z = 0\). The function which has this as its Taylor series is called the ‘Error function’ \(\text{Erf}(z)\). By the Fundamental Theorem of Calculus, another way to write this is

\[
\text{Erf}(z) = \int_0^z e^{-t^2} dt
\]

since \(\text{Erf}(z)\) is just the function that computes area under the bell curve up to a variable point \(z\). We have renamed the variable of integration \(t\) to keep it separate. Another way to think about this is that we have used power series to solve the differential equation

\[
y' = e^{-z^2}, \quad y(0) = 0.
\]

Instead of multiplying a power series by a number, or a power of \(z\), we can multiply two different series together. In general the coefficients of the product are complicated, so we will not use the \(\Sigma\) notation to write the general coefficient; we will just write the first few. For example, the series for \(\sin(z) \cos(z)\) comes from multiplying

\[
(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots)(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots)
\]

\[
= z - \left(\frac{1}{2!} + \frac{1}{3!}\right)z^3 + \left(\frac{1}{5!} + \frac{1}{2!3!} + \frac{1}{4!}\right)z^5 + \cdots = z - \frac{2}{3}z^3 + \frac{2}{15}z^5 + \cdots.
\]

Notice there are two ways of getting an \(z^3\) term in the series for \(\sin(z) \cos(z)\): one from the product of the \(z\) term in \(\sin(z)\) and the \(z^2\) term in \(\cos(z)\), and another from the product of the constant term in \(\cos(z)\) and the \(z^3\) term in \(\sin(z)\). There are three ways of getting an \(z^5\) term, etc.

**Exercise 146.** Compute the Taylor series at \(z = 0\) for \((1 - z)^{-3}\).

**Exercise 147.** Compute the Taylor series at \(z = 0\) for \(\ln(1 + z), \ln(1 - z^3), \text{ and } \ln((1 + z)/(1 - z))\). (Hint: These are easy, don’t make them into hard problems.)

**Exercise 148.** In exercise 140 you computed the Taylor series for \(1/(1 + z^2)\). Use this to compute the Taylor series for \(\arctan(z)\).
Exercise 149. Compute the Taylor series at \( z = 0 \) for \( \arcsin(z) \). (Hint: exercise 145.)

Exercise 150. In exercise 143 you computed the Taylor series for \( (e^z - 1)/z \). Use the techniques of this section to find the Taylor series for the ‘Exponential integral’ function \( \text{Ei}(z) \) defined by

\[
\text{Ei}(z) = \int_0^z \frac{e^t - 1}{t} \, dt.
\]

In other words, \( \text{Ei}(z) \) is the antiderivative of \( (e^z - 1)/z \) which is 0 at \( z = 0 \). This is another function which comes up in physics and engineering.

Exercise 151. Euler’s dilogarithm \( L(z) \) is defined by

\[
L(z) = \int_0^z \frac{-\ln(1-t)}{t} \, dt,
\]

in other words, it is the antiderivative of \( -\ln(1-z)/z \) which is 0 at \( z = 0 \). It has applications in my area of research, number theory. Compute the Taylor series expansion for \( L(z) \) at \( z = 0 \).

Exercise 152. Compute the first few terms of the Taylor series at \( z = 0 \) for \( \sin(z)e^{2z} \), and for \( e^{-z}/(1 + z) \).

In addition to multiplying series together, one can also do long division. Divide the series for \( \cos(z) \) into the series for \( \sin(z) \) to get the first few terms of the series for \( \tan(z) \). It starts \( z + \frac{z^3}{3} + \cdots \), what is the next nonzero term? The series for \( \sec(z) \) comes from dividing the series for \( \cos(z) \) into 1 = 1 + 0z + 0z^2 + \cdots. The constant term is 1, what are the next two nonzero terms? You can check your answer by also using formula (20).

Exercise 153. Find the first three terms in the series expansion for \( z/(\exp(z) - 1) \).

Exercise 154. The point of these sections was to develop shortcuts for computing Taylor series, because it is hard to compute a lot of derivatives. We can turn this around, and use a known Taylor series at \( z = a \) to tell what the derivatives at \( z = a \) are: the \( n \)-th coefficient is \( n \)-th derivative at \( a \), divided by \( n! \). Use this idea and your answer to exercise 141 to compute the 10th derivative of \( z^2 \cos(3z) \) at \( z = 0 \).

Exercise 155. Try to show that the power series \( J_0(z) \) given by formula (28) satisfies Bessel’s equation:

\[
J_0''(z) + \frac{1}{z} J_0'(z) + J_0(z) = 0
\]
by collecting all like powers of $z$ on the left side, and showing everything cancels.

**Exercise 156.**

1. Use the ratio test to find the radius of convergence for $\sinh(z)$ and for $\sin(z)$.
2. Find the radius of convergence of the Taylor series for $z^2 \cos(3z)$ you computed in exercise 141.
3. Find the radius of convergence of the Taylor series for $\sin(z)/z$ and $(e^z - 1)/z$ you computed in exercise 143.
4. Show that the radius of convergence of $\sum_{n=1}^{\infty} n! z^n$ is 0. This shows the worst case can happen.
5. Compute the radius of convergence for the Bessel function $J_0(z)$ given by formula (28). This shows that $J_0(z)$ really does define a function. Use the first 4 terms and your calculator to estimate $J_0(1)$.
6. Use the ratio test to show that $R = 1$ for the series $L(z)$ you found in exercise 151. Notice that the integral test (comparing to $\int_1^{\infty} 1/t^2 dt$) shows that the series $L(1)$ does converge, even though this is on the boundary, where the ratio test gives no information. The geometric series also has radius $R = 1$, but certainly does not converge when $z = 1$. (What is the $N$th partial sum for the geometric series with $z = 1$?) This shows anything can happen on the boundary.
7. Determine the radius of convergence of the binomial series (exercise 144). There are different answers depending on whether or not $t$ is an integer.

**Bernoulli numbers.** Sometimes it happens that we are more interested in the coefficients of the Taylor series for some function than in the function itself. Identities relating functions can supply information about the coefficients. In this case the function is called the generating function for the coefficients. For example, the **Bernoulli numbers** are the coefficients in the series expansion for $z/\exp(z) - 1$:

\begin{equation}
\frac{z}{\exp(z) - 1} = \sum_{k=0}^{\infty} \frac{B_k z^k}{k!}.
\end{equation}

What are the $B_k$? Well, multiply both sides by $\exp(z) - 1$ and write

\[ 1 \cdot z + 0 \cdot z^2 + 0 \cdot z^3 + \cdots = z = (\exp(z) - 1) \sum_{k=0}^{\infty} \frac{B_k z^k}{k!} =
\]

\[ (z + \frac{z^2}{2} + \frac{z^3}{6} + \cdots)(B_0 + B_1 z + \frac{B_2 z^2}{2} + \frac{B_3 z^3}{6} + \cdots). \]
In exercise 153 you already computed $B_0 = 1$, $B_1 = -1/2$, and $B_2 = 1/6$. The Bernoulli numbers show up in number theory; we will see them again in §24 below.

Above you computed that $B_1 = -1/2$, and here is a weird fact. If you subtract the $B_1 z$ term from the function, you get

$$\frac{B_0 z^0}{0!} + \sum_{k=2}^{\infty} \frac{B_k z^k}{k!} = \frac{z}{\exp(z) - 1} + \frac{z}{2} = \frac{z \exp(z) + 1}{2 \exp(z) - 1}$$

when you combine everything over a common denominator and simplify. If you now multiply numerator and denominator by $\exp(-z/2)$ you get

$$\frac{z \exp(z/2) + \exp(-z/2)}{2 \exp(z/2) - \exp(-z/2)}$$

But this last expression is an even function of $z$, invariant when you change $z$ to $-z$. (Check this.) So it can have only even powers of $z$ in its series expansion. This says that

$$B_k = 0 \quad \text{if } k \text{ is odd, } k > 1 \quad (30)$$

$$\frac{z}{\exp(z) - 1} = 1 - \frac{z}{2} + \sum_{k=1}^{\infty} \frac{B_{2k} z^{2k}}{2k!} \quad (31)$$

**Exercise 157.** Since the Bernoulli numbers are so useful, use the identity

$$z = (\exp(z) - 1) \sum_{k=0}^{\infty} \frac{B_k z^k}{k!}$$

to try to prove the following:

$$\sum_{k=0}^{m} \binom{m+1}{k} B_k = \begin{cases} 1, & \text{in case } m = 0 \\ 0, & \text{in case } m > 0 \end{cases} \quad (32)$$

**Exercise 158.** The previous exercise implicitly defines a recurrence relation. For example, since you already know $B_0$, $B_1$, $B_2$ and $B_3$, the equation

$$\binom{5}{0} B_0 + \binom{5}{1} B_1 + \binom{5}{2} B_2 + \binom{5}{3} B_3 + \binom{5}{4} B_4 = 0$$

lets you solve for $B_4$. Now use (32) with $m = 7$ to compute $B_6$. Don’t forget about (30).
22. SINGULARITIES

So far we’ve looked at analytic functions. Now we will study the failure to be analytic. The simplest case is when this failure to be analytic is restricted to a single point. We say $z_0$ in $\Omega$ is an ISOLATED SINGULARITY of a function $f(z)$ if $f(z)$ is analytic on $\Omega \setminus \{z_0\}$. There are three possibilities:

I: It might be true that there is some complex number $w_0$ in $\mathbb{C}$ such that
$$\lim_{z \to z_0} f(z)$$
exists and is equal to $w_0$.

In this case we call $z_0$ a REMOVABLE SINGULARITY of $f(z)$. We can simply define $f(z_0) = w_0$ and we have a continuous function. In fact, exercise 162 below shows that defining $f(z_0)$ this way actually makes $f$ analytic at $z_0$, and thus, analytic on all of $\Omega$. For example, $f(z) = \sin(z)/z$ does not at first appear to be defined at $z = 0$. But from the series expansion for $\sin(z)$, we have that for $z \neq 0$,
$$f(z) = \sum_{k=0}^{\infty} \frac{(-1)^{2k+1} z^{2k}}{(2k+1)!}.$$ 

But the series on the right side above defines an entire function ($R = \infty$), by the ratio test.

II: It might be true that
$$\lim_{z \to z_0} |f(z)| = +\infty.$$

In this case we say $f(z)$ has a POLE at $z_0$. For example, $1/z$ has a pole at $z = 0$.

III: If neither I nor II holds, then we say that $z_0$ is an ESSENTIAL SINGULARITY. For example, consider the function $f(z) = \exp(1/z)$ for $z \neq 0$. As $z \to 0$ along the positive real axis, $f(z)$ is real and goes to $+\infty$. But as $z \to 0$ along the negative real axis, $f(z)$ is real and goes to 0. So we are not in either case I or II. In fact, as $z \to 0$ along the imaginary axis, $f(z)$ wraps around the unit circle infinitely often.

We next investigate the possibility of a series expansion of $f(z)$ in the presence of singularities.

**Exercise 159.** Suppose $f(z)$ is analytic in a punctured disk at $z_0$:
$$\Omega_r = \{z \in \mathbb{C} \mid 0 < |z - z_0| < r\}.$$
Then \( f(z) \) has a \textsc{Laurent series} expansion

\[
f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k
\]

where

\[
a_k = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(w)}{(w - z_0)^{k+1}} \, dw,
\]

\( \Gamma \) any simply closed curve contained in the punctured disk \( \Omega_r \) which goes around \( z_0 \) once. Hints:

1. Let \( z \) be in \( \Omega_r \), and let \( C_1 \) and \( C_2 \) be two circles (oriented counterclockwise, as usual) around \( z_0 \) so that \( z \) lies between them. Say \( C_1 \) is the smaller.
2. Draw a picture of \( \Omega \), \( C_1 \), \( C_2 \), and \( z \).
3. Which \textsc{Cauchy} theorems justify

\[
f(z) = \frac{1}{2\pi i} \int_{C_2} \frac{f(w)}{w - z} \, dw - \frac{1}{2\pi i} \int_{C_1} \frac{f(w)}{w - z} \, dw?
\]

Pay attention to the minus sign between the two integrals.
4. Convince yourself the integral over \( C_2 \) gives the contribution of the \( k \geq 0 \), exactly as in the proof of \textsc{Taylor}’s theorem.
5. For the integral over \( C_1 \), observe we have the reverse inequality

\[
|w - z_0| < |z - z_0|.
\]

Modify the proof of \textsc{Taylor}’s theorem accordingly to produce the series over \( k \leq -1 \).
6. To get the result of the theorem, with a single arbitrary \( \Gamma \) vs. \( C_1 \) and \( C_2 \), apply some \textsc{Cauchy} theorems.

Observe the similarity to \textsc{Taylor}’s theorem. The difference is that we now allow negative exponents in the series expansion. Convergence of the doubly infinite series is simple defined as

\[
\lim_{N \to \infty} \sum_{k=0}^{N} a_k (z - z_0)^k + \lim_{-N \to -\infty} \sum_{k=-N}^{-1} a_k (z - z_0)^k.
\]

The theorems we proved in §20 about absolute and uniform convergence of power series carry over to \textsc{Laurent} series unchanged. In particular, we can differentiate term by term.

**Exercise 160.** If \( f(z) \) is actually analytic in the disk \( |z - z_0| < R \), show that the \textsc{Laurent} expansion reduces to the \textsc{Taylor} series expansion. Hint: consider the integral defining \( a_k \) for \( k \leq -1 \).
The techniques for computing Taylor series carry over unchanged for Laurent series. For example, from the series expansion for \( \exp(z) \) we deduce that the Laurent expansion for \( \exp(1/z) \) is

\[
\exp(1/z) = \sum_{k=0}^{\infty} \frac{1}{k! z^k}.
\]

In fact, we could have defined Laurent expansions a little more generally, for functions which are holomorphic in an annulus around \( z_0 \) (rather than merely a punctured disk.) For example, a careful consideration of branch cuts leads one to deduce that

\[
f(z) = \sqrt{1 - 1/z^2} = \sqrt{1 - 1/z} \sqrt{1 + 1/z}
\]
is a single valued holomorphic function on

\[\Omega = \mathbb{C} \setminus [-1, 1],\]
that is, the complex plane with the real interval \(-1 \leq x \leq 1\) deleted. The binomial series expansion of exercise 144 with \( t = 1/2 \) then gives the Laurent expansion for \( z \) in \( \Omega \)

\[
\sqrt{1 - 1/z^2} = \sum_{k=0}^{\infty} \binom{1/2}{k} (-1)^k z^{-2k}.
\]

There are three possibilities for the coefficients of a Laurent expansion:

(A) It may be the case that \( a_k = 0 \) for all \( k < 0 \).

(B) It make be the case that there is some negative integer \(-N\), so that \( a_{-N} \neq 0 \) but \( a_k = 0 \) for all \( k < -N \).

(C) If neither (A) nor (B) is true, then there must be infinitely many negative integers \( k \) such that \( a_k \neq 0 \).

It turns out that the three types of singularities correspond exactly to the three cases for the Laurent expansion:

\[I \leftrightarrow (A) \quad II \leftrightarrow (B) \quad III \leftrightarrow (C)\]

**Exercise 161.** Show that (A) \(\Rightarrow\) I. Hint: exercise 116.

**Exercise 162.** Show that I \(\Rightarrow\) (A). Hints:

(1) Use the hypothesis

\[
\lim_{z \to z_0} f(z) = w_0
\]

to show that there is some \( B > 0 \) and \( r > 0 \) so that \( |f(z)| < B \) for \( |z - z_0| \leq r \).

(2) Use the M-L inequality to show that \( a_k = 0 \) for \( k < 0 \).
Since \( f(z) \) has a Taylor series expansion, it is analytic. And
\[
a_0 = f(z_0) = \lim_{z \to z_0} f(z) = w_0.
\]

For the equivalence II \( \iff \) (B), the following lemma will be helpful: Suppose \( f(z) \) is analytic in \( \Omega \setminus \{z_0\} \). Then \( f(z) \) has a pole at \( z_0 \) \( \iff \) \( 1/f(z) \) is analytic in some disk around \( z_0 \), and has an isolated zero at \( z_0 \). Equivalently, \( g(z) \) is analytic at \( z_0 \) and \( g(z_0) = 0 \) \( \iff \) \( 1/g(z) \) has a pole at \( z_0 \). In short, a function has a pole if and only if the reciprocal has a zero.

Proof. We have
\[
\lim_{z \to z_0} |f(z)| = \infty \iff \lim_{z \to z_0} \frac{1}{f(z)} = 0.
\]
And the infinite limit on the left implies that \( f(z) \neq 0 \) in some small disk of radius \( \epsilon \) around \( z_0 \). So \( 1/f(z) \) is analytic in the punctured disk, and \( z_0 \) is actually a removable singularity, with \( w_0 = 0 \). Conversely, if \( g(z) \) is analytic at \( z_0 \) and \( g(z_0) = 0 \), then by exercise 127 there is some small disk of radius \( \epsilon > 0 \) around \( z_0 \) so that \( g(z) \neq 0 \) in the punctured disk. This means that \( 1/g(z) \) is analytic on that same punctured disk, and
\[
\lim_{z \to z_0} \frac{1}{g(z)} = \infty,
\]
so \( 1/g(z) \) has a pole at \( z_0 \).

Exercise 163. Show that II \( \Rightarrow \) B. Hint: If \( f(z) \) has a pole at \( z_0 \), then \( 1/f(z) \) has a zero. Imitate what you did in exercise 127, this time for the function \( 1/f(z) \). Now use this to understand the Laurent expansion for \( f(z) \). The converse B \( \Rightarrow \) II is similar.

We have shown that
\[
I \iff (A) \quad II \iff (B)
\]
is true, so necessarily III is true if and only if (C) is true.

23. Residue Theorem and Corollaries

We say that \( f(z) \) is meromorphic in \( \Omega \) if it is holomorphic except for isolated singularities (i.e., the set \( S \) of singularities satisfies the property of (21).) Roughly speaking, meromorphic functions are quotients of holomorphic functions. We now extend our investigation of integrals to functions with singularities.
**Exercise 164.** Suppose $f(z)$ has an isolated singularity at $z_0$ in $\Omega$. Let $\Gamma$ a simple closed curve with $z_0$ inside $\Gamma$, but no other singularity of $f(z)$ inside. Suppose the Laurent expansion is

$$f(z) = \sum_k a_k(z - z_0)^k.$$ 

Then

$$\int_{\Gamma} f(z) dz = 2\pi i a_{-1}.$$ 

The coefficient $a_{-1}$ of $(z - z_0)^{-1}$ in the Laurent expansion is called the **residue** of $f(z)$ at $z_0$, denoted $\text{Res}_{z_0}(f(z))$. The residue is what is left over when you integrate. As a special case of II (or (B)), we say that $f(z)$ has a **simple pole** at $z_0$ if the residue $a_{-1} \neq 0$, but $a_k = 0$ for $k' \leq -2$. Similarly we say that $f(z)$ has a **simple zero** at $z_0$ if $a_k = 0$ for $k < 1$ but $a_1 \neq 0$.

More generally, we have

**Exercise 165.** (Residue Theorem). Suppose $f(z)$ is analytic in $\Omega$ except for isolated singularities at the points $z_1, z_2, \ldots, z_m$. Suppose $\Gamma$ is a simple closed curve enclosing the singularities. Then

$$\int_{\Gamma} f(z) dz = 2\pi i \sum_{j=1}^{m} \text{Res}_{z_j}(f(z)).$$ 

Hint: Use a Cauchy Theorem.

**Exercise 166.** What if the closed curve $\Gamma$ is not simple? Conjecture and prove a homology version of the Residue Theorem involving winding numbers. Hints:

1. Use the homology version of the Cauchy Integral Theorem (exercise 109) to convert

$$\int_{\Gamma} f(z) dz$$

to a sum of integrals of simple curves around each isolated singularity.

2. Apply the Residue Theorem above to each.

The Residue Theorem is extremely powerful because the left side, the integral, is analysis. The right side, coming from the Laurent expansion, is algebra. Because the Residue theorem is so useful, techniques for computing residues are important. The ideas are the same as in §21.
Exercise 167.  
(1) Above we computed that
\[ \exp\left(\frac{1}{z}\right) = 1 + \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{6z^3} + \ldots \]
What is the residue at 0?
(2) Use the Taylor expansion of \(\sin(z)\) at 0 to find the Laurent expansion of \(\sin\left(\frac{2}{z^2}\right)\) at 0. What is the residue at 0?

Exercise 168. Suppose \(f(z)\) is analytic at \(z_0\), and \(g(z)\) has a simple pole at \(z_0\). Then
\[ \text{Res}_{z_0}(f \cdot g(z)) = f(z_0)\text{Res}_{z_0}(g(z)). \]
Hint: Write
\[ f(z) = a_0 + O(z - z_0), \quad g(z) = \frac{b_{-1}}{z - z_0} + b_0 + O(z - z_0). \]
Multiply.

Exercise 169. Use the previous result to compute the residue at 0 of
\[ \frac{(z - i)^3(z + 1)}{z(z - 2)^4}. \]
Hint: Take \(g(z) = 1/z\).

Here’s another trick. Suppose \(h(z)\) has an \(N\)th order pole at \(z_0\):
\[ h(z) = \frac{a_{-N}}{(z - z_0)^N} + \ldots + \frac{a_{-1}}{z - z_0} + a_0 + O(z - z_0). \]
So \((z - z_0)^Nh(z)\) is analytic at \(z_0\):
\[ (z - z_0)^Nh(z) = a_{-N} + \ldots + a_{-1}(z - z_0)^{N-1} + O(z - z_0)^N. \]
By Taylor’s Theorem
\[ a_{-1} = \frac{\left((z - z_0)^Nh(z)\right)^{(N-1)}(z_0)}{(N-1)!}. \]
This is not as bad as it looks, the point of multiplying by \((z - z_0)^N\) is to cancel the bad terms which might appear in a denominator.

Exercise 170. Use this to compute
\[ \text{Res}_i \left( \frac{z + 1}{z(z - i)^2} \right). \]

Exercise 171. If you’re a graduate student studying for a qualifying exam in complex analysis, do exercises 1-7 on p. 268 of the text.
**Exercise 172.** In section §20 of the notes I defined a function $J_0(z)$, the zero order $J$-Bessel function, by a power series. A tedious computation (exercise 155) showed that $J_0(z)$ satisfies a second order ODE with non-constant coefficients, namely Bessel’s equation. In this exercise you will show that

$$J_0(z) = \frac{1}{2\pi i} \int_{\Gamma} \exp(z(w^{-1})/2) \frac{dw}{w},$$

where $\Gamma$ is any simple closed curve around the origin counterclockwise.

Hints:

1. Expand $\exp(zw/2)$ and $\exp(-z/(2w))$ as power series. (Call the summation index in the first series ‘$j$’, and the summation index in the second series ‘$k$’.)
2. Multiply them (formally) to get a Laurent expansion, in the variable $w$, for $\exp(z(w - 1/w)/2)$. This is a double series in $j$ and $k$.
3. Switch the sums and integral.
4. Apply the Residue Theorem. Don’t forget the extra $1/w$ in the integral. You should have a power series in the parameter $z$.

Suppose $f(z)$ is a meromorphic function on $\Omega$, with no zeros or poles actually lying on the curve $\Gamma$. We will consider, for a simple closed curve $\Gamma$, the integral

$$\frac{1}{2\pi i} \int_{\Gamma} \frac{f'(z)}{f(z)} \,dz = \frac{1}{2\pi i} \int_{f(\Gamma)} \frac{dw}{w}. \tag{33}$$

Here we have simply done a change of variables

$$w = f(z), \quad dw = f'(z)dz.$$

The curve $\Gamma$ being integrated over is transformed, as a set of points, into the set of points $f(\Gamma)$. (In terms of parametrizations, if $\Gamma$ is parametrized by some $z(t)$, $a \leq t \leq b$, then $f(\Gamma)$ is parametrized by $f(z(t))$, $a \leq t \leq b$.) From our definition of winding number in §18,

$$\frac{1}{2\pi i} \int_{f(\Gamma)} \frac{dw}{w} = n(f(\Gamma), 0).$$

If we can interpret the left side of (33) in a different way, we will get a theorem.

**Exercise 173.** If $f(z)$ has a zero of order $n$ at $z_0$, then

$$\text{Res}_{z_0} f'(z)/f(z) = n.$$
Hint: by hypothesis we can write

\[ f(z) = (z - z_0)^n g(z) \]

with \( g(z) \) analytic and nonzero at \( z_0 \). Use the product rule.

In fact we can extend this to poles as well if we make the following definition. We say that meromorphic function \( f(z) \) has a zero of order \( n \) at \( z_0 \), for \( n \) in \( \mathbb{Z} \) (i.e. positive or negative), if the Laurent expansion looks like

\[ f(z) = a_n (z - z_0)^n + O((z - z_0)^{n+1}). \]

Thus a simple zero is a zero of order 1, and a simple pole is a zero of order \(-1\). If the function is analytic and nonzero at \( z_0 \), then we say there is a zero of order zero at \( z_0 \). The notation is \( n = \text{ord}_{z_0}(f) \).

**Exercise 174.** Check that the previous exercise goes through as before with this new terminology.

With \( \Gamma \) and \( f(z) \) as above, let

\[ \mathcal{N}_0(\Gamma, f) = \# \{ \text{zeros of } f \text{ inside } \Gamma \}, \]

where \( \# \) means cardinality, and we count zeros with multiplicity using our new terminology: a double zero at a point counts as 2, a simple pole counts as \(-1\), etc.

**Exercise 175.** Show that

\[ \frac{1}{2\pi i} \int_\Gamma \frac{f'(z)}{f(z)} \, dz = \mathcal{N}_0(\Gamma, f). \]

**Exercise 176.** (Argument Principle) Show that

\[ n(f(\Gamma), 0) = \mathcal{N}_0(\Gamma, f). \]

In words, the winding number of the curve \( f(\Gamma) \) around the point 0 is equal to the number of zeros (counted with appropriate multiplicity) of \( f(z) \) inside \( \Gamma \).

**Exercise 177.** Suppose \( f(z) \) is meromorphic in \( \Omega \), and \( \Gamma \) is a simple closed curve. Show that

\[ \int_\Gamma \frac{zf'(z)}{f(z)} \, dz = 2\pi i \sum_j \text{ord}_{z_j}(f) \cdot z_j, \]

where, as above, \( \text{ord}_{z_j}(f) \) is the ‘order’ of the zero of \( f(z) \) at the point \( z_j \): positive if \( f(z_j) = 0 \), negative if \( f(z) \) has a pole at \( z_j \).
**Exercise 178.** (Rouché’s Theorem) Let \( \Omega \) a domain and \( \Gamma \) a simple closed curve with the inside of \( \Gamma \) a subset of \( \Omega \). Suppose \( f(z) \) is holomorphic on \( \Omega \) and nonzero on \( \Gamma \). Suppose that \( h(z) \) is another holomorphic function on \( \Omega \) such that
\[
|h(z)| < |f(z)| \quad \text{for all } z \text{ on } \Gamma.
\]
Then the functions \( f(z) \) and \( f(z) + h(z) \) have the same number of zeros inside \( \Gamma \). Hints:

1. Write \( f(z) + h(z) \) as \( f(z)(1 + h(z)/f(z)) \) and apply the argument principle.
2. Reduce the question to showing some integral is zero.
3. Now use the hypothesis \( |h(z)/f(z)| < 1 \).

The text (p.302) describes this as the dog walking theorem:

‘A man \( f(z) \) walks a dog \( h(z) \) around a fire hydrant \( w = 0 \). If the leash is sufficiently short, the man \( f(z) \) and the dog on the end of the leash \( f(z) + h(z) \) circle the hydrant the same number of times.’

As an application of this we will prove

**Exercise 179.** (Fundamental Theorem of Algebra\( ^{12} \)) A polynomial of degree \( n \)
\[
p(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0
\]
has \( n \) roots in the complex plane. Hints:

1. Let \( f(z) = a_n z^n \), and
\[
h(z) = a_{n-1} z^{n-1} + \ldots + a_1 z + a_0,
\]
so \( p(z) = f(z) + h(z) \).
2. Now convince yourself that you can choose \( \Gamma \) to be a circle of radius \( R \), (for \( R \) sufficiently large, depending on \( p(z) \)) so that \( |h(z)| < |f(z)| \) on \( \Gamma \).
3. Apply Rouché. Where are the zeros of \( f(z) \), and what are the multiplicities?

Rouché’s Theorem is a standard tool to solve many problems on complex analysis qualifying exams related to finding the roots of a complex polynomial. See any complex analysis textbook, examples following Rouché’s Theorem.

\( ^{12} \)This was Gauss’ Ph.D. thesis, although he gave a different proof.
24. **Euler’s Formula**

It was discovered by Pietro Mengoli in 1650 in his book *Novae Quadraturae Arithmetica*, that the sum of the reciprocals of the triangular numbers was equal to 2,

\[ \sum_{n=1}^{\infty} \frac{2}{n(n+1)} = 1 + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \cdots = 2. \]

The proof was exercise 112. But no such simple formula was known to Mengoli for the sum of the reciprocals of the squares \( \sum_n \frac{1}{n^2} \). It seems like a similar problem; the squares and the triangular numbers are the simplest polygonal numbers. Other mathematicians including John Wallis and Gottfried Leibnitz tried to find a formula and failed. The Swiss mathematician Jacob Bernoulli discussed the problem in his *Theory of Series* in 1704. It became known as the Basel problem, named after the city.

Everyone was surprised when Euler, then still relatively young, solved the problem. Even more surprising was the nature of the answer Euler found

\[ \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{\pi^2}{6}. \]

Euler eventually proved even more: For \( k \geq 2 \) an even integer

\[ \sum_{n=1}^{\infty} \frac{1}{n^k} = -\frac{(2\pi i)^k}{k!} \frac{B_k}{2}, \]

where the \( B_k \) are the Bernoulli numbers defined by (29). For convenience we will let

\[ u(z) = \frac{z}{\exp(z) - 1} \]

the function on the left side of (29). (The proof below is more modern than Euler’s.)

**Exercise 180.** Show that for each \( n \) in \( \mathbb{Z} \), \( 1/(\exp(z) - 1) \) has a simple pole at \( z = 2\pi i n \) with residue 1. Hint: first expand the reciprocal \( \exp(z) - 1 = \exp(z - 2\pi i n) - 1 \) as a series in powers of \( z - 2\pi i n \).

**Exercise 181.** Use the above to show that, for \( n \neq 0 \) in \( \mathbb{Z} \),

\[ \text{Res}_{2\pi i n} u(z) = \frac{z}{z^{k+1}} = (2\pi i)^{-k}. \]

Next we do the \( n = 0 \) case:
Exercise 182. Show that
\[ \text{Res}_0 \frac{u(z)}{z^{k+1}} = \frac{B_k}{k!} \]

We next construct, for each \( N \) in \( \mathbb{N} \), a square closed path \( C_N \) in the plane which encloses exactly \( 2N + 1 \) poles, those at \( 2\pi in \) with \( -N \leq n \leq N \). The \( C_N \) should cross the \( y \)-axis at \( \pm(2\pi iN + \pi i) \), halfway between the poles. See Figure 21 for examples with \( C_0 \) and \( C_2 \).

\[ \text{Figure 21. Contours } C_N \]

Exercise 183. Use the Residue Theorem to deduce that
\[ \frac{B_k}{k!} = \frac{1}{2\pi i} \int_{C_N} \frac{u(z)}{z^{k+1}} \, dz = \sum_{n=-N, n \neq 0}^{N} \frac{1}{(2\pi i n)^k}. \]

This holds for fixed \( k \) and fixed \( N \) in \( \mathbb{N} \). (Observe also that for odd \( k \) the sum on the left side is identically zero; the positive and negative terms cancel.)
Exercise 184. Show that for $k \geq 2$

\[ \lim_{N \to \infty} \int_{C_N} \frac{u(z)}{z^{k+1}} dz = 0. \]  

Hints:

1. Show that the length $L$ of the path $C_N$ is $2\pi(8N + 4)$.
2. I claim it suffice to show that there exists a constant $\beta$ so that for $N$ sufficiently big,

\[ |\exp(z) - 1| > \beta \quad \text{for } z \text{ on } C_N. \]

Why is that? How does the hypothesis $k \geq 2$ fit in?
3. To show the existence of such a constant, consider separately the sides of the square: right, left, top, and bottom.
Right: Use the fancy triangle inequality $|a - b| \geq ||a| - |b||$.
Left: Similar, but you also need to remember that in general $|a - b| = |b - a|$.
Top: Observe $y$ is fixed at $2\pi iN + \pi i$, and $-N - 1/2 \leq x \leq N + 1/2$ so

\[ \exp(z) = \exp(x) \exp(2\pi iN + \pi i) = -\exp(x) \quad \text{is real}. \]

Bottom: is similar.

From (38) we deduce that for even $k$,

\[ \frac{B_k}{k!} = -2 \sum_{n=1}^{\infty} \frac{1}{(2\pi in)^k}, \]

which is equivalent to Euler’s theorem. (Observe also that for odd $k > 1$ this gives another proof that $B_k = 0$.) Euler was also able to make sense of the sum $\sum_{n=1}^{\infty} 1/n^k$ for $k$ a negative odd integer. Furthermore, he found a remarkable symmetry between the values at $k$ and at $1 - k$. We will look at this next, from Riemann’s more modern point of view.

25. The Gamma Function $\Gamma(s)$

Euler was interested in the factorial function $n!$, and the question of how to extend this function beyond just positive integers. For a real number $s > 0$, Euler defined a complicated looking function $\Gamma(s)$, the Gamma Function, as follows

\[ \Gamma(s) = \int_{0}^{\infty} \exp(-x)x^{s-1}dx. \]
This is an improper integral, so we need to be a little careful. Write
\[ \int_0^\infty \exp(-x)x^{s-1}dx = \int_0^1 \exp(-x)x^{s-1}dx + \int_1^\infty \exp(-x)x^{s-1}dx \]
and consider the two pieces separately. First we need to observe that for \( x > 0 \), every term in the series for \( \exp(x) \) is positive, so since \( \exp(x) \) is the sum of all of the terms it is bigger than each of them: For every positive \( n \)
\[ \frac{x^n}{n!} < \exp(x) \quad \text{or} \quad \exp(-x) < \frac{n!}{x^n} \]
Now for fixed \( s \), pick any \( n > s \) then
\[ \int_1^\infty \exp(-x)x^{s-1}dx < n! \int_1^\infty x^{s-n-1}dx. \]
This is just the comparison test, property (v) of definite integrals, again. We can compute this last integral, it is
\[ n! \left( \frac{x^{s-n}}{s-n} \right)_{1}^{\infty} = n! \frac{1}{n-s}. \]
The key fact here is that \( x^{s-n} \) tends to 0 as \( x \) tends to \( \infty \), since we made \( s-n < 0 \). Notice this same argument works for \( s \leq 0 \) as well.
We only need \( s > 0 \) for the other piece.
Since we are interested in \( x > 0 \), \( \exp(-x) < 1 \), so \( \exp(-x)x^{s-1} < x^{s-1} \), and
\[ \int_0^1 \exp(-x)x^{s-1}dx < \int_0^1 x^{s-1}dx. \]
So it suffices to show this simpler integral is finite, again using the comparison test. But
\[ \int x^{s-1}dx = \frac{x^s}{s}, \quad \text{so} \quad \int_0^1 x^{s-1}dx = \frac{x^s}{s} \bigg|_0^1 = \frac{1}{s} - 0 \]
is finite if \( s > 0 \). To see this, write \( x^s = \exp(s \log(x)) \), and notice that as \( x \to 0 \), \( s \log(x) \to -\infty \), so \( \exp(s \log(x)) \to 0 \). But if \( s < 0 \), then \( s \log(x) \to +\infty \) and \( x^s = \exp(s \log(x)) \to +\infty \) as \( x \to 0 \).
Now that we’ve done this, we can make the same definition of \( \Gamma(s) \) for complex values of \( s \), as long as \( \Re(s) > 0 \). Remember that \( x^{s-1} = \exp((s-1)\log(x)) \). Checking that the improper integral is finite works as before, since \( |x^s| = x^{\Re(s)} \).

**Exercise 185.** Compute the improper integral to see that
\[ \Gamma(1) = 1. \]
Exercise 186.

\[ \Gamma(s + 1) = s\Gamma(s) \]

Hint: Integrate by parts.

These two exercises imply that \( \Gamma(n + 1) = n\Gamma(n) \) for any integer \( n \). By induction,

\[ \Gamma(n + 1) = n! \]

Roughly speaking, the Gamma function ‘interpolates’ the factorial; it extends the function from integers to complex numbers in a natural way. The integral above defining the Gamma function is often called Euler’s second integral.

Exercise 187. The function \( \Gamma(s) \) is a holomorphic function in the region \( \Omega : \text{Re}(s) > 0 \). Hint: Morera’s theorem, and switch the order of integration (i.e., Fubini’s theorem. This is justified by the absolute convergence we proved above.)

Our study of the Gamma functions requires another way of looking at it.

Exercise 188.

\[ \Gamma(s) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{1}{s+k} + \int_{1}^{\infty} \exp(-x)x^{s-1}dx \]

Hints: This is actually pretty easy:

1. Break the integral up in two pieces again at \( x = 1 \).
2. In the first integral, expand \( \exp(-x) \) as a series.
3. Change the order of the sum and integral.
4. Now compute each integral.

Again one must be sure it is legal to interchange the infinite sum and the integral. But in fact the integral of the sum of absolute values

\[ \int_{0}^{\infty} \sum_{k=0}^{\infty} \left| \frac{(-1)^k}{k!} x^k \right| x^{s-1}dx = \int_{0}^{1} \exp(x)x^{\text{Re}(s)-1}dx \]

is finite, so Fubini’s theorem applies again.

Notice that the series on the right side of (40) is not a Laurent expansion in the variable \( s \); it is something new. What we have obtained here is a PARTIAL FRACTIONS expansion of the function \( \Gamma(s) \), analogous to writing \( 2/(x^2 - 1) = 1/(x-1) - 1/(x+1) \). The difference is that now there can be infinitely many poles. The partial fractions expansion is the sum of infinitely many singular parts.
Exercise 189. Let \( \delta > 0 \), and define \( \Omega_\delta \) to be the complex plane with disks of radius \( \delta \) around the negative integers removed. So for \( s \) in \( \Omega_\delta, |s + k| > \delta \).

(1) Use the Weierstrass \( M \)-test to show the series in (40) converges absolutely and uniformly on \( \Omega_\delta \), thus \( \Gamma(s) \) is holomorphic on \( \Omega_\delta \). Since this is true for any \( \delta > 0 \), \( \Gamma(s) \) has isolated singularities at the negative integers.

(2) Now fix an integer \( j \); by the absolute convergence we can rearrange the terms in the series so the partial fractions expansion is written

\[
\frac{(-1)^j}{j!} \frac{1}{s+j} + \sum_{k \neq j} \frac{(-1)^k}{k!} \frac{1}{s+k}.
\]

On the disk \( |s+j| < 1/2 \), we have that \( |s+k| > 1/2 \) for all \( k \neq j \). Again, show the sum over all \( k \) different from \( j \) defines a holomorphic function on the disk \( |s+j| < 1/2 \).

(3) Show that

\[
\Gamma(s) = \frac{(-1)^j}{j!} \frac{1}{s+j} + O(1) \quad \text{near } s = -j.
\]

i.e., \( \Gamma(s) \) has a simple pole at \( s = -j \), with residue \( (-1)^j/j! \).

This is another example of an analytic continuation.

Exercise 190. Show that the property \( \Gamma(s+1) = s\Gamma(s) \) still holds for (40) in two stages. First, show that

\[
\int_1^\infty \exp(-x)x^s dx = e^{-1} + s \int_1^\infty \exp(-x)x^{s-1} dx.
\]

Next, show that

\[
\sum_{k=0}^\infty \frac{(-1)^k}{k!} \frac{1}{s+1+k} = s \sum_{j=0}^\infty \frac{(-1)^j}{j!} \frac{1}{s+j} - e^{-1}.
\]

(Hint: change variables in the sum \( j = k + 1 \), and write \( 1/(j-1)! \) as \((s+j-s)/j!\).)

26. The Riemann zeta function \( \zeta(s) \)

Riemann was the first to think of the parameter \( s \) in the infinite series

\[
\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s}
\]
as a complex variable, extending Euler’s work. For this reason the
function is called the Riemann zeta function. From what we proved
about Dirichlet series on page 88, the Riemann zeta function is holo-
monic in the half plane \( \text{Re}(s) > 1 \). The value at \( s = 1 \) corresponds
to the Harmonic series, which diverges.

In the previous section we introduced the Gamma function for
\( \text{Re}(s) > 0 \), and showed how to extend it to a larger domain. In
this section we will show how \( \Gamma(s) \) is connected to the Riemann
zeta function \( \zeta(s) \), and lets us extend the definition of \( \zeta(s) \) beyond
\( \text{Re}(s) > 1 \) as well. Riemann was the first to do this, but this variation
of the proof is due to Hermite.

**Exercise 191.** For \( \text{Re}(s) > 1 \)

\[
\Gamma(s)\zeta(s) = \frac{1}{s-1} - \frac{1}{2s} + \sum_{k=1}^{\infty} \frac{B_{2k}}{2k!} \frac{1}{s+2k-1} + \int_{1}^{\infty} \frac{1}{\exp(x)-1} x^{s-1} dx
\]

This is much easier than it looks:

1. First, for an integer \( n \) show by changing the variable that
   \[
   \int_{0}^{\infty} \exp(-nx)x^{s-1} dx = \frac{1}{n^s} \Gamma(s).
   \]
2. Sum both sides over all \( n \geq 1 \) to get \( \Gamma(s)\zeta(s) \) on the right side.
3. Now interchange the sum and integral
4. Observe you have a Geometric series in the variable \( \exp(-x) \),
   but starting with \( n = 1 \). Write it in closed form.
5. Simplify by multiplying numerator and denominator by \( \exp(x) \).
6. Now use the same trick we used on the Gamma function.
   Break the integral into two pieces at \( x = 1 \).
7. The second piece is as the theorem predicts. You need to ex-
   aminate the first piece. Write \( 1/(\exp(x) - 1) \) as a series with the
   Bernoulli numbers; we just need to factor out an \( x \) out of (31).
8. Now multiply in the \( x^{s-1} \) term.
9. Integrate each term separately (i.e. change the sum and inte-
   gral.) They are all of the same form
   \[
   \int_{0}^{1} x^{s+n} dx = \left( \frac{x^{s+n+1}}{s+n+1} \right)_{0}^{1} = \frac{1}{s+n+1} \quad \text{where } n = -2, -1, \text{ or } 2k - 2.
   \]
This theorem gives the analytic continuation of the function \( \Gamma(s)\zeta(s) \), analogous to that of \( \Gamma(s) \) by itself in the previous section. Here’s how. Let

\[
F(s) = \frac{1}{s-1} - \frac{1}{2s} + \sum_{k=1}^{\infty} \frac{B_{2k}}{2k!} \frac{1}{s+2k-1}
\]

be the sum of all the singular parts, and

\[
G(s) = \int_{1}^{\infty} \frac{1}{\exp(x) - 1} x^{s-1} dx.
\]

So the theorem says \( \Gamma(s)\zeta(s) = F(s) + G(s) \).

**Exercise 192.** My proof was a little sketchy that the integral which defines the function \( G(s) \) converges absolutely. Give a more careful argument, analogous to the one given for the Gamma function. Hint: Show that, for any integer \( n \), \( 1 + x^n/n! < \exp(x) \). Next show that \( G(s) \) defines an entire function by imitating a previous exercise.

**Exercise 193.** Let \( \epsilon > 0 \), and define \( \Omega_\epsilon \) to be the complex plane with disks of radius \( \epsilon \) around the negative odd integers removed. So for \( s \) in \( \Omega_\delta \), \( |s + 2k - 1| > \epsilon \) for all \( k \) in \( \mathbb{N} \). Use the Weierstrass \( M \)-test to show the series

\[
\sum_{k=1}^{\infty} \frac{B_{2k}}{2k!} \frac{1}{s+2k-1}
\]

converges absolutely and uniformly on \( \Omega_\delta \), thus is holomorphic on \( \Omega_\delta \). Hint: By the comparison test for series, \( \zeta(2k) < \zeta(2) \). Now use the value of \( \zeta(2k) \) and the comparison test.

Since this is true for any \( \epsilon > 0 \), there are isolated singularities at the negative odd integers.

By exactly the same argument used for the Gamma function above, we deduce that \( \Gamma(s)\zeta(s) \) extends to a meromorphic function with simple poles at \( s = 1 \), \( s = 0 \), and \( s = 1 - 2k \) for \( k \) in \( \mathbb{N} \). And

\[
\Gamma(s)\zeta(s) = \frac{1}{s-1} + O(1) \quad \text{near } s = 1,
\]

\[
\Gamma(s)\zeta(s) = \frac{-1}{2s} + O(1) \quad \text{near } s = 0.
\]

For \( k \geq 1 \)

\[
\Gamma(s)\zeta(s) = \frac{B_{2k}}{2k!} \frac{1}{(s+2k-1)} + O(1) \quad \text{near } s = 1 - 2k.
\]
Exercise 194. Show that the series
\[
\frac{z}{\exp(z) - 1} = 1 - \frac{z}{2} + \sum_{k=1}^{\infty} \frac{B_{2k}}{2k!} z^{2k}
\]
converges absolutely for \(|z| < 2\pi\). (Hint: Either use the bound on \(B_{2k}\) from the previous problem, or, quote a big theorem.) This exercise justifies our use of the series for \(1/(\exp(x) - 1)\) in the integral between 0 and 1 defining the function \(F(s)\).

What about \(\zeta(s)\) by itself? We can divide out the \(\Gamma(s)\) term,
\[
\zeta(s) = \frac{F(s) + G(s)}{\Gamma(s)}.
\]

Exercise 195. The function \(\zeta(s)\) extends to all values of \(s\) except 1. \(\zeta(s)\) has a simple pole at \(s = 1\), with residue 1. \(\zeta(0) = -1/2\). Furthermore, for \(n = 1, 2, 3 \ldots\)
\[
\zeta(-2n) = 0,
\]
\[
\zeta(-2n + 1) = \frac{-B_{2n}}{2n}
\]
where the \(B_{2n}\) are again the Bernoulli numbers.

You need one fact about \(\Gamma(s)\) that we can’t prove without more sophisticated mathematics: that \(\Gamma(s)\) is never equal to 0. With this fact it is OK to divide. Furthermore, since \(\Gamma(1) = 1\), near \(s = 1\) we have
\[
\frac{1}{\Gamma(s)} = 1 + O(s - 1) \quad \text{near } s = 1
\]
\[
F(s) + G(s) = \frac{1}{s - 1} + O(1) \quad \text{near } s = 1, \text{ so}
\]
(43)
\[
\zeta(s) = \frac{1}{s - 1} + O(1) \quad \text{near } s = 1.
\]
Near \(s = 0\) we have, by the partial fractions expansions of \(\Gamma(s)\) and \(F(s) + G(s)\)
\[
\Gamma(s) = \frac{1}{s} + O(1) \quad \text{near } s = 0, \text{ so}
\]
\[
\Gamma(s)^{-1} = s + O(s^2) \quad \text{near } s = 0.
\]
\[
F(s) + G(s) = -\frac{1}{2s} + O(1) \quad \text{near } s = 0, \text{ so}
\]
\[
\zeta(s) = (s + O(s^2)) \cdot (-\frac{1}{2s} + O(1))
\]
\[
= -\frac{1}{2} + O(s) \quad \text{near } s = 0.
\]
For $s$ near $-2n + 1$, we calculate just as with $s = 0$.

**Exercise 196.** (1) What are the singular parts for the Laurent expansions of $\Gamma(s)$ and $F(s) + G(s)$ at $s = -2n + 1$?

(2) What is the lead term of the Taylor expansion of $\Gamma(s)^{-1}$ at $s = -2n + 1$?

(3) What is $\zeta(-2n + 1)$?

For the last case, notice $s = -2n$ is not a pole of $F(s) + G(s)$, so all we can say is

$$F(s) + G(s) = F(-2n) + G(-2n) + O(s + 2n),$$

while

$$\Gamma(s) = \frac{1}{2n!} \frac{1}{s + 2n} + O(1) \quad \text{by (41)},$$

$$\Gamma(s)^{-1} = (2n)!(s + 2n) + O(s + 2n)^2.$$  

$$\zeta(s) = (F(-2n) + G(-2n))(2n)!(s + 2n) + O(s + 2n)^2.$$ 

In particular, $\zeta(-2n) = 0$. The constant $F(-2n) + G(-2n)$ is mysterious.

This theorem leads to an interesting observation. We see (as Euler did) that there is a remarkable symmetry between $\zeta(2n)$ and $\zeta(1 - 2n)$

$$\zeta(2n) = (-1)^{n+1} \frac{(2\pi)^{2n}}{2} \frac{B_{2n}}{2n!}, \quad \text{while} \quad \zeta(1 - 2n) = -\frac{B_{2n}}{2n}.$$ 

They both involve the mysterious Bernoulli number $B_{2n}$, as well as some boring powers of $2\pi$ and factorials.

Figure 22 shows the level curves for the real and imaginary parts of $\zeta(s)$. For comparison, recall the analogous pictures in Figures 11-16.

Near the point $s = 1$, we know that $\zeta(s)$ should look like $1/(s - 1)$; that is what (43) says. You can actually see this; compare Figure 22 near the point $s = 1$ to Figure 13 near $z = 0$. (There are small breaks in the level curves near $s = 1$, this is an artifact of the program that computed them, not a real phenomenon.)

Since $\zeta(\sigma)$ is real for real $\sigma > 1$ from the series definition, it turns out it must be real for all real $\sigma$. So $\text{Im}(\zeta)$ is zero on the real axis. And we see this level curve, the horizontal dotted line in Figure 22. The solid curves crossing this one are level curves for $\text{Re}(\zeta) = 0$. This means we can also see the trivial zeros of $\zeta(s)$ at the negative even integers; any point where $\text{Re}(\zeta) = 0$ and $\text{Im}(\zeta) = 0$ means that $\zeta(s) = 0$.

Finally, in the upper right corner we can see the first nontrivial zero of $\zeta(s)$ at $s = 1/2 + i14.13473\ldots$, or the point $(1/2, 14.13473\ldots)$.
in the plane. The other places where the level curves cross represent values other than 0 for either $\text{Re}(\zeta)$ or $\text{Im}(\zeta)$.

Figure 24 shows the argument $\text{arg}(\zeta(s))$ interpreted as a color, for the range $-11 \leq \sigma \leq 12$, and $-4.5 \leq t \leq 30$. This puts the critical line $\text{Re}(s) = 1/2$ exactly down the middle. For comparison purposes, see Figures 23(a)-23(f). We can see the first few nontrivial zeros, at
\( t = 14.13473 \ldots, t = 21.02204 \ldots, \text{ and } t = 25.01086 \ldots. \) Observe they are simple zeros, and you can also see the simple pole at \( s = 1 \) and the trivial zeros at the negative even integers.
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(a) $f(z) = z$

(b) $f(z) = z^2$

(c) $f(z) = z^3$

(d) $f(z) = 1/z$

(e) $f(z) = (z - 2)^2(z + 2)$

(f) $f(z) = (z + 1)/(z - 1)$

Figure 23. $\arg(f(z))$ interpreted as a color.
FIGURE 24. $\arg(\zeta(s))$ interpreted as a color.
Figure 25. \( \arg(j(i(1 + z)/(1 - z))) \) interpreted as a color.
Figure 26. Elliptic functions for the lattice $[1, \exp(2\pi i/3)]$. 

(a) $\text{arg}(\wp(z))$

(b) $\text{arg}(\wp'(z))$
Figure 27. Elliptic functions for the lattice $[1, i]$. 