# Practice Problems: Arc Length

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*Solutions to the practice problems posted on November 30.*

Find the exact length of the curve for the problems below.

1. $x = \frac{1}{3}\sqrt{y}(y-3), 1 \leq y \leq 9$

   *Solution:* Use the formula $L = \int_1^9 \sqrt{1 + (\frac{dx}{dy})^2} dy$. Need to find $\frac{dx}{dy}$:

   $$x = \frac{1}{3}\sqrt{y}(y-3) = \frac{1}{3}(y^{3/2} - 3y^{1/2}) \Rightarrow \frac{dx}{dy} = \frac{1}{3}\left(\frac{3}{2}y^{1/2} - \frac{3}{2}y^{-1/2}\right) = \frac{1}{2}\left(\sqrt{y} - \frac{1}{\sqrt{y}}\right)$$

   Then find $\sqrt{1 + (\frac{dx}{dy})^2}$:

   $$\sqrt{1 + \frac{1}{4}\left(\sqrt{y} - \frac{1}{\sqrt{y}}\right)^2} = \sqrt{\frac{1}{4y} + \frac{1}{2} + \frac{1}{4y}} = \sqrt{\frac{1}{4}\left(\sqrt{y} + \frac{1}{\sqrt{y}}\right)^2} = \frac{1}{2}\left(\sqrt{y} + \frac{1}{\sqrt{y}}\right)$$

   Now find $L$:

   $$L = \int_1^9 \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy = \frac{1}{2}\int_1^9 \left(\sqrt{y} + \frac{1}{\sqrt{y}}\right) dy = \frac{1}{2}\left(\frac{2}{3}y^{3/2} + 2y^{1/2}\right)\Big|_1^9 = \frac{32}{3}$$

   □

2. $y = 3 + \frac{1}{2}\cosh 2x, 0 \leq x \leq 1$

   *Solution:* Use the formula $L = \int_0^1 \sqrt{1 + (\frac{dy}{dx})^2} dx$. Need to find $\frac{dy}{dx}$:

   $$\frac{dy}{dx} = \frac{1}{2}\sinh 2x \cdot 2 = \sinh 2x$$

   Then find $\sqrt{1 + (\frac{dy}{dx})^2}$:

   $$\sqrt{1 + \sinh^2 2x} = \sqrt{\cosh^2 2x} = \cosh 2x$$

   Now find $L$:

   $$L = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_0^1 \cosh 2x dx = \frac{1}{2}\sinh 2x\Big|_0^1 = \frac{1}{2}(\sinh 2 - \sinh 0) = \frac{1}{2}\sinh 2$$

   □

3. $y = \ln(1-x^2), 0 \le x \le \frac{1}{2}$

*Solution:* Use the formula $L = \int_0^{1/2} \sqrt{1+(\frac{dy}{dx})^2}dx$. Need to find $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{1}{1-x^2} \cdot -2x = \frac{-2x}{1-x^2}$$

Then find $\sqrt{1+(\frac{dy}{dx})^2}$:

$$\sqrt{1+\frac{4x^2}{(1-x^2)^2}} = \sqrt{\frac{(1-x^2)^2+4x^2}{(1-x^2)^2}} = \sqrt{\frac{1+2x^2+x^4}{(1-x^2)^2}} = \sqrt{\frac{(1+x^2)^2}{(1-x^2)^2}} = \frac{1+x^2}{1-x^2}$$

Now find $L$:

$$L = \int_0^{1/2} \sqrt{1+(\frac{dy}{dx})^2}dx = \int_0^{1/2} \frac{1+x^2}{1-x^2}dx = \int_0^{1/2} \left(-1+\frac{2}{1-x^2}\right)dx = \int_0^{1/2} \left(-1-\frac{2}{x^2-1}\right)dx$$

We need to use partial fractions to integrate $\frac{2}{x^2-1}$. $x^2-1$ factors into $(x-1)(x+1)$:

$$\frac{2}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$$

$$2 = A(x+1) + B(x-1) \Rightarrow 2 = (A+B)x + A - B$$

We get the system of equations $\begin{cases} A+B=0 \\ A-B=2 \end{cases}$. Solving this system yields the solution $A = 1, B = -1$. So then

$$L = \int_0^{1/2} \left(-1-\frac{2}{x^2-1}\right)dx = \int_0^{1/2} \left(-1-\frac{1}{x-1}+\frac{1}{x+1}\right)dx = -x - \ln|x-1| + \ln|x+1|\Big|_0^{1/2}$$

$$= -\frac{1}{2} - \ln\left|\frac{1}{2}\right| + \ln\left|\frac{3}{2}\right| = -\frac{1}{2} + \ln 3$$

□

4. $y = 1 - e^{-x}, 0 \le x \le 2$

*Solution:* Use the formula $L = \int_0^{2} \sqrt{1+(\frac{dy}{dx})^2}dx$. Need to find $\frac{dy}{dx}$:

$$\frac{dy}{dx} = e^{-x}$$

Then find $\sqrt{1+(\frac{dy}{dx})^2}$:

$$\sqrt{1+(e^{-x})^2}$$

Now find $L$:

$$L = \int_0^{2} \sqrt{1+(\frac{dy}{dx})^2}dx = \int_0^{2} \sqrt{1+(e^{-x})^2}dx$$

Let $u = e^{-x}$. Then $du = -e^{-x}dx \Rightarrow -e^{x}du = dx \Rightarrow -\frac{1}{u}du = dx$. When $x = 0, u = 1$. When $x = 2, u = e^{-2}$:

$$-\int_1^{e^{-2}} \sqrt{1+u^2}\ \frac{1}{u}du$$

Need to use trig substitution. Let $u = \tan\theta$, then $du = \sec^2\theta d\theta$ (we won't change the bounds on this one):

$$\int \sqrt{1+u^2}\ \frac{1}{u}du = \int \sqrt{1+\tan^2\theta}\ \frac{1}{\tan\theta}\sec^2\theta d\theta = \int \frac{\sec^3\theta}{\tan\theta}d\theta = \int \frac{(1+\tan^2\theta)\sec\theta}{\tan\theta}d\theta$$

$$= \int \left(\sec\theta\tan\theta + \frac{\sec\theta}{\tan\theta}\right)d\theta = \int (\sec\theta\tan\theta + \csc\theta)\,d\theta = \sec\theta + \ln|\csc\theta - \cot\theta|$$

$$= \sqrt{1+u^2} + \ln\left|\frac{\sqrt{1+u^2}-1}{u}\right|$$

So then

$$-\int_1^{e^{-2}} \sqrt{1+u^2}\ \frac{1}{u}du = -\sqrt{1+u^2} - \ln\left|\frac{\sqrt{1+u^2}-1}{u}\right|\Bigg|_1^{e^{-2}}$$

$$= -\sqrt{1+e^{-4}} - \ln\left|\frac{\sqrt{1+e^{-4}}-1}{e^{-2}}\right| - \left(-\sqrt{2} - \ln\left|\sqrt{2}-1\right|\right)$$

$$= -\sqrt{1+e^{-4}} - \ln(\sqrt{1+e^{-4}}-1) + \ln e^{-2} + \sqrt{2} + \ln(\sqrt{2}-1)$$

$$= -\sqrt{1+e^{-4}} - \ln(\sqrt{1+e^{-4}}-1) - 2 + \sqrt{2} + \ln(\sqrt{2}-1)$$

□