Practice Problems: Integration by Parts (Solutions)

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The following are solutions to the Integration by Parts practice problems posted November 9.

1. \( \int e^x \sin x \, dx \)
   
   \textit{Solution:} Let \( u = \sin x \), \( dv = e^x \, dx \). Then \( du = \cos x \, dx \) and \( v = e^x \). Then
   
   \[ \int e^x \sin x \, dx = e^x \sin x - \int e^x \cos x \, dx \]

   Now we need to use integration by parts on the second integral. Let \( u = \cos x \), \( dv = e^x \, dx \).
   Then \( du = -\sin x \, dx \) and \( v = e^x \). Then
   
   \[ \int e^x \sin x \, dx = e^x \sin x - e^x \cos x - \int e^x \sin x \, dx \]

   The right integral is the same as the one we started with! Move it over:
   
   \[ 2 \int e^x \sin x \, dx = e^x \sin x - e^x \cos x \]

   And divide by 2:
   
   \[ \int e^x \sin x \, dx = \frac{1}{2} (e^x \sin x - e^x \cos x) \]

   This is our final solution, so make sure to add your constant \( C \):
   
   \[ \int e^x \sin x \, dx = \frac{1}{2} (e^x \sin x - e^x \cos x) + C \]

   \( \square \)

2. \( \int (\sin^{-1} x)^2 \, dx \)

   \textit{Solution:} Let \( u = (\sin^{-1} x)^2 \), \( dv = dx \). Then \( du = \frac{2 \sin^{-1} x}{\sqrt{1-x^2}} \, dx \), \( v = x \). Then
   
   \[ \int (\sin^{-1} x)^2 \, dx = x(\sin^{-1} x)^2 - \int \frac{2x \sin^{-1} x}{\sqrt{1-x^2}} \, dx \]

   We need to use a substitution on the last integral. Let \( w = \sin^{-1} x \). Then \( dw = \frac{1}{\sqrt{1-x^2}} \, dx \)
   and \( x = \sin w \). Just looking at the last integral, we have:
   
   \[ \int \frac{2x \sin^{-1} x}{\sqrt{1-x^2}} \, dx = \int 2w \sin w \, dw \]
We can use integration by parts on this last integral by letting \( u = 2w \) and \( dv = \sin wdw \). Tabular method makes it rather quick:

\[
\int 2w \sin wdw = 2w \cos w + 2 \sin w
\]

At this point you can plug back in \( w \):

\[
\int 2w \sin wdw = 2 \sin^{-1} x \cos (\sin^{-1} x) + 2 \sin (\sin^{-1} x)
\]

OR you can look at the triangle formed by our substitution for \( w \). Since \( x = \sin w \) then the hypotenuse will be 1, the opposite side will be \( x \) and the adjacent side will be \( \sqrt{1-x^2} \). Then

\[
\int 2w \sin wdw = 2 \sqrt{1-x^2} \sin^{-1} x + 2x
\]

Either of these solutions is fine. So then our integral will look like either one of the solutions below:

\[
\int (\sin^{-1} x)^2 dx = x(\sin^{-1} x)^2 - (2 \sin^{-1} x \cos (\sin^{-1} x) + 2 \sin (\sin^{-1} x)) + C
\]

\[
\int (\sin^{-1} x)^2 dx = x(\sin^{-1} x)^2 - (2 \sqrt{1-x^2} \sin^{-1} x + 2x) + C
\]

\[\square\]

3. \( \int x \tan^2 x dx \)

\textbf{Solution:} Use the identity \( \tan^2 x = \sec^2 x - 1 \):

\[
\int x \tan^2 x dx = \int x(\sec^2 x - 1) dx = \int x \sec^2 x dx - \int x dx
\]

The last integral is no problem. The first integral we need to use integration by parts. Let \( u = x, dv = \sec^2 x \). Then \( du = dx, v = \tan x \), so:

\[
\int x \sec^2 x dx = x \tan x - \int \tan x dx
\]

You can rewrite the last integral as \( \int \frac{\sin x}{\cos^2 x} dx \) and use the substitution \( w = \cos x \). \( \int \tan x dx = -\ln |\cos x| \), so:

\[
\int x \sec^2 x dx = x \tan x + \ln |\cos x|
\]

Plug that into the original integral:

\[
\int x \tan^2 x dx = x \tan x + \ln |\cos x| - \frac{1}{2} x^2 + C
\]

\[\square\]
4. \( \int_0^1 t \cosh t \, dt \)

Solution: This is quick with tabular method. Let \( u = t \), \( dv = \cosh t \):

\[
\int_0^1 t \cosh t \, dt = t \sinh t - \cosh t \bigg|_0^1 = \sinh(1) - \cosh(1) + \cosh(0)
\]

You can leave your answer like this. If you want to evaluate it further, remember that 
\[ \sinh x = \frac{e^x - e^{-x}}{2} \text{ and } \cosh x = \frac{e^x + e^{-x}}{2}. \]
Then we see that 
\[ \sinh(1) = \frac{1}{2}(e - e^{-1}), \quad \cosh(1) = \frac{1}{2}(e^1 + e^{-1}), \quad \text{and } \cosh 0 = 1. \]

Then
\[
\int_0^1 t \cosh t \, dt = \sinh(1) - \cosh(1) + \cosh(0) = 1 - \frac{1}{e}
\]

\( \square \)

5. \( \int z^3 e^z \, dx \)

Solution: Tabular is the way to go with this baby. Let \( u = z^3 \), \( dv = e^z \, dz \). Then

\[
\int z^3 e^z \, dx = z^3 e^z - 3z^2 e^z + 6ze^z - 6e^z + C = e^z(z^3 - 3z^2 + 6z - 6) + C
\]

\( \square \)

6. \( \int_1^{\sqrt{3}} \arctan(1/x) \, dx \)

Solution: Let \( u = \arctan(1/x) \), \( dv = dx \). Then \( du = -\frac{dx}{x^2 + 1} \) (using chain rule), \( v = x \):

\[
\int_1^{\sqrt{3}} \arctan(1/x) \, dx = x \arctan(1/x) \bigg|_1^{\sqrt{3}} + \int_1^{\sqrt{3}} \frac{x}{x^2 + 1} \, dx
\]

The last integral you can use the substitution \( w = x^2 + 1 \). Then:

\[
\int_1^{\sqrt{3}} \arctan(1/x) \, dx = x \arctan(1/x) + \frac{1}{2} \ln(x^2 + 1) \bigg|_1^{\sqrt{3}}
\]

\[
= \sqrt{3} \arctan(\sqrt{3}) + \frac{1}{2} \ln 4 - \arctan(1) + \frac{1}{2} \ln 2 = \frac{\sqrt{3} \pi}{3} + \frac{1}{2} \ln 2 - \frac{\pi}{4}
\]

\( \square \)

7. \( \int \cos x \ln(\sin x) \, dx \)

Solution: We first need to do a substitution. Let \( w = \sin x \), then \( dw = \cos x \, dx \):

\[
\int \cos x \ln(\sin x) \, dx = \int \ln w \, dw
\]
Next use integration by parts with $u = \ln w, dv = dw$. Then $du = \frac{1}{w}dw, v = w$:

$$\int \ln wdw = w \ln w - \int w \ln w - w$$

We need to plug back in $w$:

$$\int \cos x \ln (\sin x)dx = \sin x \ln(\sin x) - \sin x + C$$

□

8. $\int_1^2 \frac{(\ln x)^2}{x^3}dx$

You can do this problem a couple different ways. I will show you two solutions.

Solution I: First do the substitution $w = \ln x$. Then $dw = \frac{1}{x}dx$ and $x = e^w$. Then

$$\int_1^2 \frac{(\ln x)^2}{x^3}dx = \int_0^{\ln 2} \frac{w^2}{e^{2w}}dw = \int_0^{\ln 2} w^2 e^{-2w}dw$$

Tabular is easy on this guy:

$$\int_0^{\ln 2} w^2 e^{-2w}dw = -\frac{w^2}{2} e^{-2w} - \frac{w}{2} e^{-2w} - \frac{1}{4} e^{-2w}\bigg|_0^{\ln 2} = -\frac{e^{-2w}}{2} \left(w^2 + w + \frac{1}{2}\right)\bigg|_0^{\ln 2} = -\frac{1}{8} \left((\ln 2)^2 + \ln 2 + 3\right)$$

Solution II: Start of with integration by parts. Let $u = (\ln x)^2, dv = \frac{1}{x^3}dx$. Then $du = \frac{2 \ln x}{x}dx, v = -\frac{1}{2x^2}$:

$$\int_1^2 \frac{(\ln x)^2}{x^3}dx = -\frac{(\ln x)^2}{2x^2}\bigg|_1^{\ln 2} + \int_1^2 \frac{\ln x}{x^3}dx$$

Do integration by parts again. Let $u = \ln x, dv = \frac{1}{x^3}dx$. Then $du = \frac{1}{x}dx, v = -\frac{1}{2x^2}$:

$$\int \frac{\ln x}{x^3}dx = -\frac{\ln x}{2x^2}\bigg|_1^{\ln 2} + \int_1^{\ln 2} \frac{1}{2x^3}dx = \left(-\frac{\ln x}{2x^2} - \frac{1}{4x^2}\right)\bigg|_1^{\ln 2}$$

Plugging this into the original integral we get:

$$\int_1^2 \frac{(\ln x)^2}{x^3}dx = \left(-\frac{(\ln x)^2}{2x^2} - \frac{\ln x}{2x^2} - \frac{1}{4x^2}\right)\bigg|_1^{\ln 2} = -\frac{1}{2x^2} \left((\ln x)^2 + \ln x + \frac{1}{2}\right)\bigg|_1^{\ln 2} = -\frac{1}{8} \left((\ln 2)^2 + \ln 2 + 3\right)$$
9. $\int \cos \sqrt{x} \, dx$

Solution: First do the substitution $w = \sqrt{x}$. Then $dw = \frac{1}{2\sqrt{x}} \, dx \Rightarrow 2\sqrt{x} \, dw = dx \Rightarrow 2wdw = dx$:

$$\int \cos \sqrt{x} \, dx = \int 2w \cos w \, dw$$

Using tabular with $u = 2w, dv = \cos w \, dw$ we get:

$$\int 2w \cos w \, dw = 2w \sin w + 2 \cos w + C$$

Plug back in $w$ to get the final solution:

$$\int \cos \sqrt{x} \, dx = 2\sqrt{x} \sin \sqrt{x} + 2 \cos \sqrt{x} + C$$

□

10. $\int_{\frac{\pi}{2}}^{\pi} \theta^3 \cos(\theta^2) \, d\theta$

Note: There was a typo on the original, it should be $d\theta$ instead of $dx$.

Solution: Rewrite: $\int_{\frac{\pi}{2}}^{\pi} \theta^3 \cos(\theta^2) \, d\theta = \int_{\frac{\pi}{2}}^{\pi} \theta \cdot \theta^2 \cos(\theta^2) \, d\theta$. Then use the substitution $w = \theta^2$, so we have $dw = 2\theta \, d\theta$:

$$\int_{\frac{\pi}{2}}^{\pi} \theta^3 \cos(\theta^2) \, d\theta = \frac{1}{2} \int_{\frac{\pi}{2}}^{\pi} w \cos w \, dw$$

Tabular makes this easy with $u = w, dv = \cos w \, dw$:

$$\frac{1}{2} \int_{\frac{\pi}{2}}^{\pi} w \cos w \, dw = \frac{1}{2} \left( w \sin w + \cos w \right) \bigg|_{\frac{\pi}{2}}^{\pi} = -\frac{1}{2} - \frac{\pi}{4}$$

□

11. $\int x \ln(1 + x) \, dx$

Solution: Use the substitution $w = 1 + x$. Then $dw = dx$ and $x = w - 1$:

$$\int x \ln(1 + x) \, dx = \int (w - 1) \ln w \, dw$$

Next use integration by parts with $u = \ln w, dv = (w - 1) \, dw$. Then $du = \frac{1}{w} \, dw$ and $v = \left( \frac{1}{2} w^2 - w \right)$:

$$\int (w - 1) \ln w \, dw = \left( \frac{1}{2} w^2 - w \right) \ln w - \int \left( \frac{1}{2} w - 1 \right) \, dw$$
The right integral is straightforward, so
\[
\int (w - 1) \ln w \, dw = \left( \frac{1}{2} w^2 - w \right) \ln w - \frac{1}{4} w^2 + w + C
\]
Next, plug back in \( w \):
\[
\int x \ln(1 + x) \, dx = \left( \frac{1}{2} (1 + x)^2 - (1 + x) \right) \ln (1 + x) - \frac{1}{4} (1 + x)^2 + 1 + x + C
\]
This answer is fine. You can simplify it a bit more for kicks and giggles:
\[
\int x \ln(1 + x) \, dx = \frac{1}{2} (x^2 - 1) \ln (1 + x) - \frac{1}{4} x^2 + \frac{1}{2} x + C
\]
\[\square\]

12. \( \int \sin(\ln x) \, dx \)

Solution: Use the substitution \( w = \ln x \). Then \( dw = \frac{1}{x} \, dx \Rightarrow x \, dw = dx \Rightarrow e^w \, dw = dx \) since \( x = e^w \) from our substitution. Then we have:
\[
\int \sin(\ln x) \, dx = \int e^w \sin w \, dw
\]
This is the same as Problem #1, so
\[
\int e^w \sin w \, dw = \frac{1}{2} (e^w \sin w - e^w \cos w) + C
\]
Plug back in \( w \):
\[
\int \sin(\ln x) \, dx = \frac{1}{2} (x \sin (\ln x) - x \cos (\ln x)) + C
\]
\[\square\]

13. \( \int x^3 \sqrt{1 + x^2} \, dx \)

You can do this problem a couple different ways. I will show you two solutions.
Solution I: You can actually do this problem without using integration by parts. Use the substitution \( w = 1 + x^2 \). Then \( dw = 2x \, dx \) and \( x^2 = w - 1 \):
\[
\int x^3 \sqrt{1 + x^2} \, dx = \int x \cdot x^2 \sqrt{1 + x^2} \, dx = \frac{1}{2} \int (w - 1) \sqrt{w} \, dw = \frac{1}{2} \int (w^{3/2} - w^{1/2}) \, dw
\]
\[
= \frac{1}{5} w^{5/2} - \frac{1}{3} w^{3/2} + C = \frac{1}{5} (1 + x^2)^{5/2} - \frac{1}{3} (1 + x^2)^{3/2} + C
\]
Solution II: You can use integration by parts as well, but it is much more complicated. Rewrite the integral:
\[
\int x^3 \sqrt{1 + x^2} \, dx = \int \frac{1}{2} x^2 \cdot 2x \sqrt{1 + x^2} \, dx
\]
Let \( u = \frac{1}{2}x^2, dv = 2x\sqrt{1+x^2}dx \). Then \( du = xdx, v = \frac{3}{5}(1 + x^2)^{3/2} \) (using a substitution on \( dv \)):

\[
\int \frac{1}{2}x^2 \cdot 2x\sqrt{1+x^2}dx = \frac{1}{3}x^2(1 + x^2)^{3/2} - \frac{2}{3} \int x(1 + x^2)^{3/2}dx
\]

You can use a substitution on the last integral:

\[
\int \frac{1}{2}x^2 \cdot 2x\sqrt{1+x^2}dx = \frac{1}{3}x^2(1 + x^2)^{3/2} - \frac{2}{15} (1 + x^2)^{5/2} + C
\]

\[\square\]

14. Find the area between the given curves: \( y = x^2 \ln x, \ y = 4 \ln x \)

**Solution:** We need to find when the two curves intersect, so set them equal to each other:

\[
x^2 \ln x = 4 \ln x \Rightarrow (x^2 - 4) \ln x = 0 \Rightarrow (x - 2)(x + 2) \ln x = 0
\]

The solutions to this equation are \( x = -2, 2, 1 \). But, \( x = -2 \) isn’t in our domain (since \( \ln x \) has the domain \((0, \infty)\)), so we are going to toss that solution out. This means we are going to integrate from \( x = 1 \) to \( x = 2 \). You can just guess which function is on the top or bottom:

\[
A = \int_1^2 (\text{top function} - \text{bottom function}) \, dx = \int_1^2 (4 \ln x - x^2 \ln x) \, dx = \int_1^2 (4 - x^2) \ln x \, dx
\]

Using integration by parts, let \( u = \ln x, dv = (4 - x^2)dx \). Then \( du = \frac{1}{x}dx, v = 4x - \frac{1}{3}x^3 \):

\[
\int_1^2 (4 - x^2) \ln x \, dx = \left[ (4x - \frac{1}{3}x^3) \ln x \right]_1^2 - \int_1^2 \left( 4 - \frac{1}{3}x^2 \right) \, dx
\]

\[= \left[ \left( 4x - \frac{1}{3}x^3 \right) \ln x - 4x + \frac{1}{3}x^3 \right]_1^2 = \frac{16}{3} \ln 2 - \frac{29}{9}
\]

\[\square\]

15. Use the method of cylindrical shells to find the volume generated by rotating the region bounded by the given curves about the specified axis: \( y = e^{-x}, \ y = 0, \ x = -1, \ x = 0 \) about \( x = 1 \).

**Solution:** Draw a picture of what is happening. Recall that the volume for a cylinder is \( V = 2\pi RH \). In this scenario, \( R = 1 - x \) and \( H = e^{-x} \) (since \( H \) is the top function minus the bottom function). \( x \) is going from \(-1\) to 0:

\[
V = \int_{-1}^{0} 2\pi(1 - x)e^{-x} \, dx
\]

Using tabular is pretty quick with \( u = 1 - x, dv = e^{-x}dx \):

\[
\int_{-1}^{0} 2\pi(1 - x)e^{-x} \, dx = 2\pi xe^{-x}\bigg|_{-1}^{0} = 2\pi e
\]

\[\square\]