Practice Problems: Trig Integrals (Solutions)

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November 9, 2014

The following are solutions to the Trig Integrals practice problems posted on November 9.

1. $\int \sec x \, dx$
   
   **Note:** This is an integral you should just memorize so you don’t need to repeat this process again.

   **Solution:**
   
   $$\int \sec x \, dx = \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x} \, dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx$$

   Let $w = \sec x + \tan x$, so $dw = (\sec x \tan x + \sec^2 x) \, dx$:

   $$\int \sec^2 x + \sec x \tan x \, dx = \int \frac{1}{w} \, dw = \ln |w| + C$$

   Plug back in $w$:

   $$\int \sec x \, dx = \ln |\sec x + \tan x| + C$$

   □

2. $\int \sec^3 x \, dx$

   **Solution:** Rewrite:

   $$\int \sec^3 x \, dx = \int \sec x \cdot \sec^2 x \, dx$$

   Use integration by parts. Let $u = \sec x$, $dv = \sec^2 x \, dx$. Then $du = \sec x \tan x \, dx$ and $v = \tan x$:

   $$\int \sec^3 x \, dx = \sec x \tan x - \int \sec x \tan^2 x \, dx = \sec x \tan x - \int \sec x(\sec^2 x - 1) \, dx$$

   $$= \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx = \sec x \tan x + \ln |\sec x + \tan x| - \int \sec^3 x \, dx$$

   Notice on the right side we have the same integral as what we started with, so move it over to the left side:

   $$2 \int \sec^3 x \, dx = \sec x \tan x + \ln |\sec x + \tan x|$$

   Divide by 2 and add $C$:

   $$\int \sec^3 x \, dx = \frac{1}{2} (\sec x \tan x + \ln |\sec x + \tan x|) + C$$

   □

1
3. \( \int \cos^4 x \, dx \)

**Solution:** Since we have an even power of \( \cos \), we need to use the half angle identity:
\[
\int \cos^4 x \, dx = \int (\cos^2 x)^2 \, dx = \frac{1}{4} \int (1 + 2 \cos 2x + \cos^2 2x) \, dx
\]

Use half angle again:
\[
\frac{1}{4} \int \left( 1 + 2 \cos 2x + \frac{1}{2} (1 + \cos 4x) \right) \, dx = \frac{1}{4} \int \left( \frac{3}{2} + 2 \cos 2x + \frac{1}{2} \cos 4x \right) \, dx
\]
\[
= \frac{1}{4} \left( \frac{3}{2} x + \sin 2x + \frac{1}{8} \sin 4x \right) + C
\]

□

4. \( \int t \sin^2 t \, dt \)

**Solution:** Use half angle identity:
\[
\int t \sin^2 t \, dt = \int t \left( \frac{1}{2} (1 - \cos 2t) \right) \, dt = \frac{1}{2} \left( \int t \, dt - \int t \cos 2t \, dt \right)
\]

The first integral is straightforward, use integration by parts (tabular method) on the second with \( u = t, dv = \cos 2t \, dt \):
\[
\int t \sin^2 t \, dt = \frac{1}{2} \left( \frac{1}{2} t^2 - \frac{1}{2} t \sin 2t - \frac{1}{4} \cos 2t \right) + C
\]

□

5. \( \int \frac{\sin^3 \sqrt{x}}{\sqrt{x}} \, dx \)

**Solution:** Let \( w = \sqrt{x}, \) so \( dw = \frac{1}{2\sqrt{x}} \, dx \Rightarrow 2 \, dw = \frac{1}{\sqrt{x}} \, dx: \)
\[
\int \frac{\sin^3 \sqrt{x}}{\sqrt{x}} \, dx = 2 \int \sin^3 w \, dw = 2 \int \sin w \cdot \sin^2 w \, dw = 2 \int \sin w (1 - \cos^2 w) \, dw
\]

Let \( y = \cos w, \) so \( dy = -\sin w \, dw: \)
\[
2 \int \sin w (1 - \cos^2 w) \, dw = -2 \int (1 - y^2) \, dy = -2 \left( y - \frac{1}{3} y^3 \right)
\]

Plug back in \( w: \)
\[
-2 \left( \sqrt{\cos w} - \frac{1}{3} \cos^3 w \right) = -2 \left( \cos w - \frac{1}{3} \cos^3 w \right)
\]

Plug back in \( x \) and add \( C: \)
\[
-2 \left( \cos \sqrt{x} - \frac{1}{3} \cos^3 \sqrt{x} \right) = -2 \left( \cos \sqrt{x} - \frac{1}{3} \cos^3 \sqrt{x} \right) + C
\]

□
6. $\int_0^\pi \sin^2 t \cos^4 t dt$

   **Solution:** You can use half angle identity on this problem, but you would need to use it several times. I don’t think you would see a problem like this on your exam, but it is nice to practice anyway.

   \[
   \int_0^\pi \sin^2 t \cos^4 t dt = \int_0^\pi \sin^2 t \cos^2 t \cos^2 t dt = \int_0^\pi (\sin t \cos t)^2 \left(\frac{1}{2} (1 + \cos 2t)\right) dt \\
   = \frac{1}{2} \int_0^\pi \left(\frac{1}{2} \sin 2t\right)^2 (1 + \cos 2t) dt = \frac{1}{8} \int_0^\pi (\sin 2t)^2 (1 + \cos 2t) dt \\
   \]

   Let’s look at these integrals separately. The left integral we need to use half angle identity:

   \[
   \int_0^\pi \sin^2 2t dt = \frac{1}{2} \int_0^\pi (1 - \cos 2t) dt = \frac{1}{2} \left( t - \frac{1}{2} \sin 2t \right) \bigg|_0^\pi = \frac{\pi}{2}
   
   \]

   Now let’s look at the right integral. Use the substitution $w = \sin 2t$, then $dw = 2 \cos 2t dt$:

   \[
   \int_0^\pi (\sin 2t)^2 \cos 2t dt = \frac{1}{2} \int_0^0 w^2 dw = 0
   
   \]

   So the final answer is:

   \[
   \int_0^\pi \sin^2 t \cos^4 t dt = \frac{1}{8} \left( \frac{\pi}{2} \right) = \frac{\pi}{16}
   
   \]

   □

7. $\int_0^{\pi/2} (2 - \sin \theta)^2 d\theta$

   **Solution:** Multiply this all out and use half angle identity:

   \[
   \int_0^{\pi/2} (2 - \sin \theta)^2 d\theta = \int_0^{\pi/2} (4 - 4 \sin \theta + \sin^2 \theta) d\theta = \int_0^{\pi/2} \left( 4 - 4 \sin \theta + \frac{1}{2} - \frac{1}{2} \cos 2\theta \right) d\theta \\
   = \int_0^{\pi/2} \left( \frac{9}{2} - 4 \sin \theta - \frac{1}{2} \cos 2\theta \right) d\theta = \frac{9}{2} \theta + 4 \cos \theta - \frac{1}{4} \sin 2\theta \bigg|_0^{\pi/2} = \frac{9\pi}{4} - 4
   
   \]

   □

8. $\int \cos^2 x \sin 2x dx$

   **Solution:**

   \[
   \int \cos^2 x \sin 2x dx = \int \cos^2 x \cdot 2 \sin x \cos x dx = 2 \int \sin x \cos^3 x dx
   
   \]

   □
Let \( w = \cos x \), \( dw = -\sin x \, dx \):

\[
2 \int \sin x \cos^3 x \, dx = -2 \int w^3 \, dw = -\frac{1}{2} w^4 + c = -\frac{1}{2} \cos^4 x + C
\]

\[ \square \]

9. \( \int \tan x \sec^3 x \, dx \)

Solution:

\[
\int \tan x \sec^3 x \, dx = \int \sec^2 x \cdot \sec x \tan x \, dx
\]

Let \( w = \sec x \), \( dw = \sec x \tan x \, dx \):

\[
\int \sec^2 x \cdot \sec x \tan x \, dx = \int w^2 \, dw = \frac{1}{3} w^3 + C = \frac{1}{3} \sec^3 x + C
\]

\[ \square \]

10. \( \int x \sec x \tan x \, dx \)

Solution: Use integration by parts with \( u = x \), \( dv = \sec x \tan x \, dx \). Then \( du = dx \), \( v = \sec x \):

\[
\int x \sec x \tan x \, dx = x \sec x - \int \sec x \, dx = x \sec x - \ln |\sec x + \tan x| + C
\]

\[ \square \]

11. \( \int \csc x \, dx \)

Note: This is similar to the first problem. This is an integral you should just memorize so you don’t need to repeat this process again.

Solution:

\[
\int \csc x \, dx = \int \csc x \frac{\csc x - \cot x}{\csc x - \cot x} \, dx = \int \frac{\csc^2 x - \csc x \cot x}{\csc x - \cot x} \, dx
\]

Let \( w = \csc x - \cot x \). Then \( dw = (- \csc x \cot x + \csc^2 x) \, dx \):

\[
\int \frac{\csc^2 x - \csc x \cot x}{\csc x - \cot x} \, dx = \int \frac{1}{w} \, dw = \ln |w| + C = \ln |\csc x - \cot x| + C
\]

\[ \square \]

12. \( \int \cot^3 x \, dx \)

Solution:

\[
\int \cot^3 x \, dx = \int \cot x \cot^2 x \, dx = \int \cot x (\csc^2 x - 1) \, dx = \int \cot x \csc^2 x \, dx - \int \cot x \, dx
\]
\[
\int \csc x \cdot \csc x \cot x \, dx = \cot x dx
\]

Let’s look at the first integral. Let \( w = \csc x \), then \( dw = -\csc x \cot x \, dx \):

\[
\int \csc x \cdot \csc x \cot x \, dx = - \int wdw = - \frac{1}{2} w^2 = - \frac{1}{2} \csc^2 x
\]

Now let’s look at the second integral. Rewrite it and let \( y = \sin x \) so \( dy = \cos x \, dx \):

\[
\int \cot x \, dx = \int \frac{\cos x}{\sin x} \, dx = \int \frac{1}{y} \, dy = \ln |y| = \ln |\sin x|
\]

Now combine the two answers and add \( C \):

\[
\int \cot^3 x \, dx = - \frac{1}{2} \csc^2 x + \ln |\sin x| + C
\]

\[\square\]

13. \( \int \sin 8x \cos 5x \, dx \)

\textit{Solution}: I don’t think you would see a problem like this on your exam, but it is nice to practice anyway. There is a trig identity listed on page 476 of your text: \( \sin A \cos B = \frac{1}{2}[\sin (A - B) - \sin (A + B)] \). You can also derive this equation yourself.

\[
\int \sin 8x \cos 5x \, dx = \frac{1}{2} \left( \int (\sin 3x + \sin 13x) \, dx \right) = \frac{1}{2} \left( -\frac{1}{3} \cos 3x - \frac{1}{13} \cos 13x \right) + C
\]

\[\square\]

14. \( \int \cos \pi x \cos 4\pi x \, dx \)

\textit{Solution}: Similar to the previous problem, I don’t think you would see a problem like this on your exam. On page 476 of your text is the identity \( \cos A \cos B = \frac{1}{2}[\cos(A - B) + \cos(A + B)] \).

\[
\int \cos \pi x \cos 4\pi x \, dx = \frac{1}{2} \left( \int (\cos (-3\pi x) + \cos 5\pi x) \, dx \right) = \frac{1}{2} \left( \int (\cos 3\pi x + \cos 5\pi x) \, dx \right)
\]

\[
= \frac{1}{2} \left( \frac{1}{3\pi} \sin 3\pi x + \frac{1}{5\pi} \sin 5\pi x \right) + C
\]

\[\square\]

15. \( \int_{0}^{\pi/6} \sqrt{1 + \cos 2\pi x} \, dx \)

\textit{Solution}: This is using the half identity backwards.
\[
\int_0^{\pi/6} \sqrt{1 + \cos 2x} \, dx = \int_0^{\pi/6} \sqrt{2 \cdot \frac{1}{2} (1 + \cos 2x)} \, dx = \int_0^{\pi/6} \sqrt{2 \cos^2 x} \, dx = \sqrt{2} \int_0^{\pi/6} \cos x \, dx
\]

\[= \sqrt{2} \sin x \bigg|_0^{\pi/6} = \frac{\sqrt{2}}{2} \]

\[
\int_0^{\pi/4} \sqrt{1 - \cos 4\theta} \, d\theta
\]

**Solution:** This is using the half identity backwards.

\[
\int_0^{\pi/4} \sqrt{1 - \cos 4\theta} \, d\theta = \int_0^{\pi/4} \sqrt{2 \cdot \frac{1}{2} (1 - \cos 4\theta)} \, d\theta = \int_0^{\pi/4} \sqrt{2 \sin^2 2\theta} \, d\theta = \sqrt{2} \int_0^{\pi/4} \sin 2\theta \, d\theta
\]

\[= \sqrt{2} \left( -\frac{1}{2} \cos 2\theta \right) \bigg|_0^{\pi/4} = \frac{\sqrt{2}}{2} \]

\[
\int \frac{1 - \tan^2 x}{\sec^2 x} \, dx
\]

**Solution:**

\[
\int \frac{1 - \tan^2 x}{\sec^2 x} \, dx = \int (\cos^2 x - \sin^2 x) \, dx = \int \cos 2x \, dx = \frac{1}{2} \sin 2x + C
\]

\[
\int \frac{dx}{\cos x - 1}
\]

**Solution:** Multiply by the conjugate:

\[
\int \frac{dx}{\cos x - 1} = \int \frac{1}{\cos x - 1} \cdot \frac{\cos x + 1}{\cos x + 1} \, dx = \int \frac{\cos x + 1}{\cos^2 x - 1} \, dx = \int \frac{\cos x + 1}{-\sin^2 x} \, dx
\]

\[= \int (-\csc x \cot x - \csc^2 x) \, dx = \csc x + \cot x + C \]

\[
\int x \tan^2 x \, dx
\]

**Solution:** Use the identity \( \tan^2 x = \sec^2 x - 1 \):

\[
\int x \tan^2 x \, dx = \int x(\sec^2 x - 1) \, dx = \int x \sec^2 x \, dx - \int x \, dx
\]
The last integral is no problemo. The first integral we need to use integration by parts. Let 
\( u = x \), \( dv = \sec^2 x \). Then \( du = dx \), \( v = \tan x \), so:
\[
\int x \sec^2 x \, dx = x \tan x - \int \tan x \, dx
\]
You can rewrite the last integral as \( \int \frac{\sin x}{\cos x} \, dx \) and use the substitution \( w = \cos x \). \( \int \tan x \, dx = -\ln |\cos x| \), so:
\[
\int x \sec^2 x \, dx = x \tan x + \ln |\cos x|
\]
Plug that into the original integral:
\[
\int x \tan^2 x \, dx = x \tan x + \ln |\cos x| - \frac{1}{2} x^2 + C
\]
\( \square \)

20. \( \int x \sin^2 (x^2) \, dx \)

Solution: Let \( w = x^2 \), \( dw = 2x \, dx \):
\[
\int x \sin^2 (x^2) \, dx = \frac{1}{2} \int \sin^2 w \, dw = \frac{1}{2} \cdot \frac{1}{2} \int (1 - \cos 2w) \, dw = \frac{1}{4} \left( w - \frac{1}{2} \sin 2w \right) + C
\]
\[
= \frac{1}{4} (w - \sin w \cos w) + C = \frac{1}{4} (x^2 - \sin (x^2) \cos (x^2)) + C
\]
\( \square \)

21. Find the area of the region bounded by the given curves: \( y = \sin^3 x \), \( y = \cos^3 x \), \( \pi/4 \leq x \leq 5\pi/4 \)

Solution: To find the area we need to subtract the bottom function from the top function and then integrate over our domain:
\[
A = \int_{\pi/4}^{5\pi/4} (\sin^3 x - \cos^3 x) \, dx = \int_{\pi/4}^{5\pi/4} \sin^3 x \, dx - \int_{\pi/4}^{5\pi/4} \cos^3 x \, dx
\]
Let’s look at the first integral:
\[
\int_{\pi/4}^{5\pi/4} \sin^3 x \, dx = \int_{\pi/4}^{5\pi/4} \sin x \cdot \sin^2 x \, dx = \int_{\pi/4}^{5\pi/4} \sin x (1 - \cos^2 x) \, dx
\]
Let \( w = \cos x \), \( dw = -\sin x \, dx \):
\[
\int_{\pi/4}^{5\pi/4} \sin x (1 - \cos^2 x) \, dx = -\int_{\sqrt{2}/2}^{-\sqrt{2}/2} (1 - w^2) \, dw = \int_{-\sqrt{2}/2}^{\sqrt{2}/2} (1 - w^2) \, dw
\]
\[
= \left( w - \frac{1}{3} w^3 \right) \bigg|_{-\sqrt{2}/2}^{\sqrt{2}/2} = \sqrt{2} - \frac{\sqrt{2}}{6}
\]
Now let’s look at the second integral:

\[ \int_{\pi/4}^{5\pi/4} \cos^3 x \, dx = \int_{\pi/4}^{5\pi/4} \cos x \cdot \cos^2 x \, dx = \int_{\pi/4}^{5\pi/4} \cos x (1 - \sin^2 x) \, dx \]

Let \( y = \sin x, \, dy = \cos x \, dx \):

\[ \int_{\pi/4}^{5\pi/4} \cos x (1 - \sin^2 x) \, dx = \int_{\sqrt{2}/2}^{-\sqrt{2}/2} (1 - y^2) \, dy \]

Now plug this back in:

\[ A = \int_{\pi/4}^{5\pi/4} \sin^3 x \, dx - \int_{\pi/4}^{5\pi/4} \cos^3 x \, dx = \left( \sqrt{2} - \frac{\sqrt{2}}{6} \right) - \left( -\sqrt{2} + \frac{\sqrt{2}}{6} \right) = 2\sqrt{2} - \frac{2\sqrt{2}}{3} = \frac{5\sqrt{2}}{3} \]

\[ \square \]

22. Find the volume obtained by rotating the region bounded by the given curves about the specified axis: \( y = \sec x, \, y = \cos x, \, 0 \leq x \leq \pi/3 \) about \( y = -1 \).

**Solution:** Use the washer method. The outer area is given by \( A = \pi r^2 = \pi (\sec x + 1)^2 \), and the inside area is given by \( A = \pi r^2 = \pi (\cos x + 1)^2 \):

\[ V = \int_0^{\pi/3} \text{(outer area – inside area)} \, dx = \pi \int_0^{\pi/3} ((\sec x + 1)^2 - (\cos x + 1)^2) \, dx \]

\[ = \pi \int_0^{\pi/3} (\sec^2 x + 2 \sec x - \cos^2 x - 2 \cos x) \, dx = \pi \int_0^{\pi/3} (\sec^2 x + 2 \sec x - \frac{1}{2} - \frac{1}{2} \cos 2x - 2 \cos x) \, dx \]

\[ = \pi \left( \tan x + 2 \ln |\sec x + \tan x| - \frac{1}{2} x - \frac{1}{4} \sin 2x - 2 \sin x \right) \bigg|_0^{\pi/3} \]

\[ = \pi \left( \tan \frac{\pi}{3} + 2 \ln \left| \sec \frac{\pi}{3} + \tan \frac{\pi}{3} \right| - \frac{\pi}{6} - \frac{1}{4} \sin \frac{2\pi}{3} - 2 \sin \frac{\pi}{3} \right) = \pi \left( 2 \ln(2 + \sqrt{3}) - \frac{\pi}{6} - \frac{\sqrt{3}}{8} \right) \]

\[ \square \]