1. Evaluate
\[ \int_{-3}^{0} (1 + \sqrt{9 - x^2}) \, dx \]
by interpreting it in terms of areas.

Solution. This is like unto a quiz we did the second week. Split up the integral
\[ \int_{-3}^{0} (1 + \sqrt{9 - x^2}) \, dx = \int_{-3}^{0} 1 \, dx + \int_{-3}^{0} \sqrt{9 - x^2} \, dx \]
Let’s look at the first integral: \( \int_{-3}^{0} 1 \, dx \). The graph of \( f(x) = 1 \) is a horizontal line through the point \( y = 1 \). This graph forms a rectangle on \([-3, 0]\) (draw a picture!). The area of a rectangle is \( A = lw = 3 \cdot 1 = 3 \) (notice that the area is positive because we are above the \( x \)-axis). Thus \( \int_{-3}^{0} 1 \, dx = 3 \).

Now let’s look at the second integral: \( \int_{-3}^{0} \sqrt{9 - x^2} \, dx \). The graph of \( f(x) = \sqrt{9 - x^2} \) is a semi-circle with radius \( r = 3 \). This graph forms a quarter of a circle on \([-3, 0]\) (draw a picture!). The area of a quarter circle is \( A = \pi r^2 = \frac{\pi \cdot 3^2}{4} = \frac{9\pi}{4} \) (notice that the area is positive because we are above the \( x \)-axis. Thus \( \int_{-3}^{0} \sqrt{9 - x^2} \, dx = \frac{9\pi}{4} \).

Now we want to plug into our formula above:
\[ \int_{-3}^{0} (1 + \sqrt{9 - x^2}) \, dx = \int_{-3}^{0} 1 \, dx + \int_{-3}^{0} \sqrt{9 - x^2} \, dx = 3 + \frac{9\pi}{4} \]

2. Find \( f \) if \( f'(x) = \cos x - (1 - x^2)^{-1/2} \).

Solution. We really just need to find the anti-derivative of \( f'(x) \). Let’s rewrite it so it looks a little nicer:
\[ f'(x) = \cos x - (1 - x^2)^{-1/2} = \cos x - \frac{1}{\sqrt{1 - x^2}} \]
We know how to find the anti-derivative of this:
\[ f(x) = \int f'(x) \, dx = \int \left( \cos x - \frac{1}{\sqrt{1 - x^2}} \right) \, dx = \sin x - \sin^{-1} x + C \]
3. Express the Riemann sum
\[ \lim_{n \to \infty} n \sum_{i=1}^{n} \sin x_i \Delta x \]
as a definite integral on the interval \([0, \pi]\) and then evaluate the integral.

**Solution.** Recall that the definite integral and Riemann sum are related to each other by the following formula:
\[ \lim_{n \to \infty} n \sum_{i=1}^{n} f(x_i) \Delta x = \int_{a}^{b} f(x) \, dx \]
Let’s compare our Riemann sum with a general Riemann sum:
\[ \lim_{n \to \infty} n \sum_{i=1}^{n} f(x_i) \Delta x = \lim_{n \to \infty} n \sum_{i=1}^{n} \sin x_i \Delta x \]
By comparing these two, we see that \( f(x_i) = \sin x \) \( \Rightarrow f(x) = \sin x \). We can then write out our integral on the interval \([0, \pi]\):
\[ \lim_{n \to \infty} n \sum_{i=1}^{n} \sin x_i \Delta x = \int_{0}^{\pi} \sin x \, dx = -\cos x \bigg|_{0}^{\pi} = -\cos \pi - (-\cos 0) = 1 - (-1) = 2 \]

4. Evaluate the upper and lower sums for \( f(x) = 1 + x^2, -1 \leq x \leq 1 \) with \( n = 3 \) and \( n = 4 \). Illustrate with diagrams. (Leave your answers as fractions, no need to use a calculator!)

**Solution.** (a) \( n = 3 \): First let’s find \( \Delta x \):
\[ \Delta x = \frac{b - a}{n} = \frac{1 - (-1)}{3} = \frac{2}{3} \]
We will have three intervals: \(-1 \leq x \leq -\frac{1}{3}, -\frac{1}{3} \leq x \leq \frac{1}{3}, \frac{1}{3} \leq x \leq 1\).

Upper sum: On the first interval, the highest point occurs at \( f(-1) = 2 \). On the second interval, the highest point occurs at \( f(\frac{1}{3}) = f(-\frac{1}{3}) = 1 + (\frac{1}{3})^2 = 1 + \frac{1}{9} = \frac{10}{9} \). On the third interval, the highest point occurs at \( f(1) = 2 \). So
\[ A \approx A_{upper} = \sum_{i=1}^{3} f(x_i) \Delta x = \left[ f(-1) + f\left(\frac{1}{3}\right) + f(1) \right] \Delta x = \left[ 2 + \frac{10}{9} + 2 \right] \cdot \frac{2}{3} = \frac{92}{27} \]

Lower sum: On the first interval, the lowest point occurs at \( f(-\frac{1}{3}) = \frac{10}{9} \). On the second interval, the lowest point occurs at \( f(0) = 1 \). On the third interval, the lowest point occurs at \( f(\frac{1}{3}) = \frac{10}{9} \). So
\[ A \approx A_{lower} = \sum_{i=1}^{3} f(x_i) \Delta x = \left[ f(-\frac{1}{3}) + f(0) + f\left(\frac{1}{3}\right) \right] \Delta x = \left[ \frac{10}{9} + 1 + \frac{10}{9} \right] \cdot \frac{2}{3} = \frac{58}{27} \]
(b) $n = 4$: First let’s find $\Delta x$:

$$\Delta x = \frac{b - a}{n} = \frac{1 - (-1)}{4} = \frac{2}{4} = \frac{1}{2}$$

We will have four intervals: $-1 \leq x \leq -\frac{1}{2}, -\frac{1}{2} \leq x \leq 0, 0 \leq x \leq \frac{1}{2}, \frac{1}{2} \leq x \leq 1$.

Upper sum: On the first interval, the highest point occurs at $f(-1) = 2$. On the second interval, the highest point occurs at $f(-\frac{1}{2}) = 1 + (-\frac{1}{2})^2 = 1 + \frac{1}{4} = \frac{5}{4}$. On the third interval, the highest point occurs at $f(\frac{1}{2}) = \frac{5}{4}$. On the fourth interval, the highest point occurs at $f(1) = 2$. So

$$A \approx A_{\text{upper}} = \sum_{i=1}^{4} f(x_i) \Delta x = \left[ f(-1) + f \left( -\frac{1}{2} \right) + f \left( \frac{1}{2} \right) + f(1) \right] \Delta x$$

$$= \left[ 2 + \frac{5}{4} + \frac{5}{4} + 2 \right] \cdot \frac{1}{2} = \frac{13}{4}$$

Lower sum: On the first interval, the lowest point occurs at $f(-\frac{1}{2}) = \frac{5}{4}$. On the second interval, the lowest point occurs at $f(0) = 1$. On the third interval, the lowest point occurs at $f(0) = 1$. On the fourth interval, the lowest point occurs at $f(\frac{1}{2}) = \frac{5}{4}$. So

$$A \approx A_{\text{lower}} = \sum_{i=1}^{4} f(x_i) \Delta x = \left[ f \left( -\frac{1}{2} \right) + f(0) + f(0) + f \left( \frac{1}{2} \right) \right] \Delta x$$

$$= \left[ \frac{5}{4} + 1 + 1 + \frac{5}{4} \right] \cdot \frac{1}{2} = \frac{9}{4}$$

□

5. If $\int_{0}^{6} f(x)dx = 10$ and $\int_{0}^{4} f(x)dx = 7$, find $\int_{4}^{6} f(x)dx$.

Solution. Recall that

$$\int_{0}^{4} f(x)dx + \int_{4}^{6} f(x)dx = \int_{0}^{6} f(x)dx$$

(this is a nice property of integrals). We want to solve for $\int_{4}^{6} f(x)dx$:

$$\int_{4}^{6} f(x)dx = \int_{0}^{6} f(x)dx - \int_{0}^{4} f(x)dx = 10 - 7 = 3$$

□

6. Find the derivative of

$$h(x) = \int_{e^x}^{\sqrt{x}} \frac{z^2}{\sin z + 2} dz$$
Solution. Use the Fundamental Theorem of Calculus, part 1:

\[
h'(x) = \frac{(\sqrt{x})^2}{\sin \sqrt{x} + 2} \cdot (\sqrt{x})' - \frac{(e^x)^2}{\sin e^x + 2} \cdot (e^x)' = \frac{x}{\sin \sqrt{x} + 2} \cdot \frac{1}{2\sqrt{x}} - \frac{e^{2x}}{\sin e^x + 2} \cdot e^x
\]

\[
= \frac{\sqrt{x}}{2(\sin \sqrt{x} + 2)} - \frac{e^{3x}}{\sin e^x + 2}
\]

7. A particle is moving with the given data, find the position of the particle:

\[a(t) = 10 \sin t + 3 \cos t, s(0) = 0, s(2\pi) = 12\]

Solution. Find the anti-derivatives:

\[a(t) = 10 \sin t + 3 \cos t\]
\[v(t) = -10 \cos t + 3 \sin t + C\]
\[s(t) = -10 \sin t - 3 \cos t + Ct + D\]

Need to use \(s(0) = 0\) and \(s(2\pi) = 12\) to solve for \(C\) and \(D\):

\[s(0) = 0 \Rightarrow -10 \sin 0 - 3 \cos 0 + C \cdot 0 + D = 0 \Rightarrow -3 + D = 0 \Rightarrow D = 3.\]
\[s(2\pi) = 12 \Rightarrow -10 \sin 2\pi - 3 \cos 2\pi + C \cdot 2\pi + D = 12 \Rightarrow -3 + 2\pi C + 3 = 12 \Rightarrow C = \frac{6}{\pi}.\]

Thus \(s(t) = -10 \sin t - 3 \cos t + \frac{6}{\pi} t + 3.\)

8. \(\int \tan x\,dx\)

Solution. Solve using substitution.

\[
\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx
\]

Let \(u = \cos x\), \(du = -\sin x\,dx\):

\[
\int \frac{\sin x}{\cos x}\,dx = - \int \frac{1}{u}\,du = - \ln |u| + C = - \ln |\cos x| + C
\]

We can rewrite this answer using log properties:

\[- \ln |\cos x| + C = \ln \left| \frac{1}{\cos x} \right| + C = \ln |\sec x| + C\]

Either answer is acceptable.
9. \( \int_0^1 (x^e + e^x) \, dx \)

Solution.

\[
\int_0^1 (x^e + e^x) \, dx = \left( \frac{x^{e+1}}{e+1} + e^x \right) \bigg|_0^1 = \left( \frac{1^{e+1}}{e+1} + e^1 \right) - \left( \frac{0^{e+1}}{e+1} + e^0 \right) = \frac{1}{e+1} + e - 1
\]

10. \( \int \sec t (\sec t + \tan t) \, dt \)

Solution.

\[
\int \sec t (\sec t + \tan t) \, dt = \int (\sec^2 t + \sec t \tan t) \, dt = \tan t + \sec t + C
\]

11. \( \int \frac{e^{\sqrt{x}}}{\sqrt{x}} \, dx \)

Solution. Solve using substitution. Let \( u = \sqrt{x} \), \( du = \frac{1}{2\sqrt{x}} \, dx \):

\[
\int \frac{e^{\sqrt{x}}}{\sqrt{x}} \, dx = 2 \int e^u \, du = 2e^u + C = 2e^{\sqrt{x}} + C
\]

12. \( \int_{1/\sqrt{3}}^{\sqrt{3}} \frac{8}{1 + x^2} \, dx \)

Solution.

\[
\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{8}{1 + x^2} \, dx = 8 \tan^{-1} x \bigg|_{1/\sqrt{3}}^{\sqrt{3}} = 8 \left( \tan^{-1}(\sqrt{3}) - \tan^{-1}(1/\sqrt{3}) \right) = 8 \left( \frac{\pi}{3} - \frac{\pi}{6} \right) = \frac{4\pi}{3}
\]

13. \( \int_{e}^{e^4} \frac{dx}{x \sqrt{\ln x}} \)

Solution. Solve using substitution. Let \( u = \ln x \), \( du = \frac{1}{x} \, dx \); when \( x = e, u = 1 \), when \( x = e^4, u = 4 \):

\[
\int_{e}^{e^4} \frac{dx}{x \sqrt{\ln x}} = \int_{1}^{4} \frac{1}{u} \, du = \int_{1}^{4} u^{-1/2} \, du = 2 u^{1/2} \bigg|_{1}^{4} = 2(\sqrt{4} - \sqrt{1}) = 2(2 - 1) = 2
\]