Math 6A Practice Problems I

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Answers

This page contains answers only. Detailed solutions are on the following pages.

1. (a) $y = \dfrac{1}{x}, 0 < x < 1$
   (b) $y = (x+1)^2, x > -1$
2. $y = 2x + 1$ or $x = t, y = 2t + 1$
3. $L = e^3 - e^{-8} + 11$
4. $\mathbf{r}'(t) = \frac{1}{\sqrt{1-t^2}}\ \mathbf{i} - \frac{2}{\sqrt{1-t^2}}\ \mathbf{j}$
5. $\mathbf{v}(t) = \sqrt{2}\ \mathbf{i} + e^t\ \mathbf{j} - e^{-t}\ \mathbf{k}$
   $\mathbf{a}(t) = e^t\ \mathbf{j} + e^{-t}\mathbf{k}$
   $s = e^t + e^{-t}$
6. $\mathbf{r}(t) = \left(\frac{t^2}{2} + 1\right)\ \mathbf{i} + e^t\ \mathbf{j} + (te^t - e^t + 2)\ \mathbf{k}$
7. $\mathbf{T}(0) = \frac{1}{3}\langle 1, 2, 2\rangle$
8. (a) $L = 2\pi\sqrt{2}$
   (b) $\mathbf{r}\left(t(s)\right) = \cos\left(\frac{s}{\sqrt{2}}\right)\ \mathbf{i} + \sin\left(\frac{s}{\sqrt{2}}\right)\ \mathbf{j} + \frac{s}{\sqrt{2}}\ \mathbf{k}$
   (c) $\kappa = \frac{1}{2}$
9. (a) $\mathbf{r}(t) = \langle 1, 3, 2\rangle + t\langle -5, 0, -2\rangle$
   (b) $\mathbf{r}(t) = \langle 1, 0, 6\rangle + t\langle 1, 3, 1\rangle$
10. (a) Parallel
    (b) Skew
11. $-2x + y + 5z = 1$
12. $\theta = \cos^{-1}\left(\frac{2}{\sqrt{42}}\right)$
13. (a) $f_x < 0, f_y < 0, f_{xx} = f_{xy} = f_{yy} = 0$
    (b) $f_x = f_y = 0, f_{xx} < 0, f_{yy} < 0, f_{xy} = 0$
14. (a) DNE
    (b) DNE
15. $f_{xx} = 6xy^5 + 24x^2y$
    $f_{xy} = 15x^2y^4 + 8x^3$
    $f_{yx} = 15x^2y^4 + 8x^3$
    $f_{yy} = 20x^3y^3$
    Yes, $f_{xy} = f_{yx}$
16. $L(x, y) = \frac{2}{3} + \frac{1}{9}(x-2) - \frac{2}{9}(y-1)$
17. $\Delta A \approx 5.4\text{cm}^2$
18. (a) $\nabla f = \langle e^{2yz}, 2xze^{2yz}, 2xye^{2yz}\rangle$
    (b) $\nabla f(3, 0, 2) = \langle 1, 12, 0\rangle$
    (c) $\frac{26}{3}$
19. $\nabla f(1, 1, -1) = \langle -1, -1, -2\rangle$
    $||\nabla f(1, 1, -1)|| = sqrt6$
20. (a) $f(0, 0) = 2$ saddle, $f(\pm 1, \pm 1) = 0$ local min
    (b) $f(3, \sqrt[3]{3}) = 83 - 9\sqrt[3]{3}$ absolute max
    $f(1, 1) = 0$ absolute min

1. For the following parametrizations:

   (i) Eliminate the parameter to find a Cartesian equation of the curve.

   (ii) Sketch the curve and indicate with an arrow the direction in which the curve is traced as the parameter increases.

   (a) $x = \sin t, y = \csc t$, $0 < t < \pi/2$

   Solution. y can be rewritten as

$$y = \csc t = \frac{1}{\sin t} = \frac{1}{x}.$$

   $t = 0$ corresponds with $x = \sin 0 = 0$; so since $t > 0$ we must have $x > 0$. Similarly, $t = \frac{\pi}{2}$ corresponds with $x = \sin \frac{\pi}{2} = 1$; so since $t < \frac{\pi}{2}$ we must have $x < 1$. Thus $y = \dfrac{1}{x}, 0 < x < 1$.

   The graph is not included here. The curve has vertical asymptote at $x = 0$. The curve "moves" to the right with increasing time. □

   (b) $x = e^t - 1, y = e^{2t}$

   Solution. We can write $e^t = x + 1$ and substitute into the equation for y:

$$y = e^{2t} = (e^t)^2 = (x + 1)^2.$$

   Since $e^t > 0$ then $x > -1$. Thus $y = (x + 1)^2$ for $x > -1$.

   The graph is not included here. The curve is a right half of a parabola with vertex $(-1, 0)$ (open at the vertex) and "moves" to the right with increasing time. □

2. Find an equation of the tangent line to the curve $x = 1 + \ln t, y = t^2 + 2$ at the point $(1, 3)$.

   Solution. We need to find slope of the line, i.e. the derivative $\frac{dy}{dx}$ at the point $(1, 3)$. Recall that

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}.$$

   We have the derivatives

$$\frac{dx}{dt} = \frac{1}{t} \quad \text{and} \quad \frac{dy}{dt} = 2t,$$

   hence

$$\frac{dy}{dx} = \frac{2t}{\frac{1}{t}} = 2t^2.$$

   The point $(1, 3)$ occurs when $t = 1$. Plugging this in to the equation gives us the slope $m = 2$. Thus

$$y - 3 = 2(x - 1) \quad \Rightarrow \quad y = 2x - 2 + 3 \quad \Rightarrow \quad y = 2x + 1.$$

   This can also be written in parametric form $x = t, y = 2t + 1$. □

3. Find the exact length of the curve $x = e^t - t, y = 4e^{t/2}, -8 \le t \le 3$.

   Solution. The derivatives of x and t are

$$\frac{dx}{dt} = e^t - 1 \quad \text{and} \quad \frac{dy}{dt} = 2e^{t/2}.$$

   Then

$$\begin{aligned}\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} &= \sqrt{(e^t - 1)^2 + (2e^{t/2})^2} = \sqrt{e^{2t} - 2e^t + 1 + 4e^t} \\ &= \sqrt{e^{2t} + 2e^t + 1} = \sqrt{(e^t + 1)^2} = e^t + 1\end{aligned}$$

$$L = \int_{-8}^{3} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_{-8}^{3} (e^t + 1)dt = (e^t + t)\Big|_{-8}^{3} = e^3 - e^{-8} + 11$$

   □

4. Find the derivative of the vector function $\mathbf{r}(t) = \sin^{-1} t\ \mathbf{i} + \sqrt{1 - t^2}\ \mathbf{j} + \mathbf{k}$.

   Solution. Take the derivative of each component:

$$\mathbf{r}'(t) = \frac{1}{\sqrt{1 - t^2}}\ \mathbf{i} - \frac{2}{\sqrt{1 - t^2}}\ \mathbf{j}.$$

   □

5. Find the velocity, acceleration, and speed of a particle where the position function is given by $\mathbf{r}(t) = \sqrt{2}t\ \mathbf{i} + e^t\ \mathbf{j} + e^{-t}\ \mathbf{k}$.

   Solution. Take the first two derivatives to find velocity and acceleration:

$$\begin{aligned}\mathbf{v}(t) &= \mathbf{r}'(t) = \sqrt{2}\ \mathbf{i} + e^t\ \mathbf{j} - e^{-t}\ \mathbf{k} \\ \mathbf{a}(t) &= \mathbf{r}''(t) = e^t\ \mathbf{j} + e^{-t}\mathbf{k}\end{aligned}$$

   Speed is the magnitude of velocity:

$$s = ||\mathbf{v}(t)|| = \sqrt{(\sqrt{2})^2 + (e^t)^2 + (e^{-t})^2} = \sqrt{e^{2t} + 2e^t e^{-t} + e^{-2t}} = \sqrt{(e^t + e^{-t})^2} = e^t + e^{-t}.$$

   □

6. Find $\mathbf{r}(t)$ if $\mathbf{r}'(t) = t\ \mathbf{i} + e^t\ \mathbf{j} + te^t\ \mathbf{k}$ and $\mathbf{r}(0) = \mathbf{i} + \mathbf{j} + \mathbf{k}$.

Solution. Integrate each term:

$$\mathbf{r}(t) = \left(\frac{t^2}{2} + c_1\right)\ \mathbf{i} + \left(e^t + c_2\right)\ \mathbf{j} + \left(te^t - e^t + c_3\right)\ \mathbf{k}$$

Set $\mathbf{r}(0) = \mathbf{i} + \mathbf{j} + \mathbf{k}$ to solve for c_1, c_2, c_3:

$$c_1\ \mathbf{i} + (1 + c_2)\ \mathbf{j} + (-1 + c_3)\ \mathbf{k} = \mathbf{i} + \mathbf{j} + \mathbf{k}.$$

Therefore $c_1 = 1, c_2 = 0, c_3 = 2$, and

$$\mathbf{r}(t) = \left(\frac{t^2}{2} + 1\right)\ \mathbf{i} + e^t\ \mathbf{j} + \left(te^t - e^t + 2\right)\ \mathbf{k}.$$

□

7. Find the unit tangent vector $\mathbf{T}(t)$ to the curve $\mathbf{r}(t) = \langle te^{-t}, 2\arctan t, 2e^t\rangle$ when $t = 0$.

Solution. We need to find

$$\mathbf{T}(0) = \frac{\mathbf{r}(0)}{||\mathbf{r}(0)||}.$$

The derivative of $\mathbf{r}(t)$ is

$$\mathbf{r}'(t) = \langle e^{-t} - te^{-t},\ \frac{2}{1+t^2},\ 2e^t\rangle \quad \Rightarrow \quad \mathbf{r}'(0) = \langle 1, 2, 2\rangle,$$

and

$$||\mathbf{r}'(0)|| = \sqrt{1^2 + 2^2 + 2^2)} = 3.$$

Hence

$$\mathbf{T}(0) = \frac{1}{3}\langle 1, 2, 2\rangle.$$

□

8. Consider the circular helix with vector equation $\mathbf{r}(t) = \cos t\ \mathbf{i} + \sin t\ \mathbf{j} + t\ \mathbf{k}$.

   (a) Find the arc length of the helix $\mathbf{r}(t)$ from the point $(1, 0, 0)$ to the point $(1, 0, 2\pi)$.

   Solution. The point $(1, 0, 0)$ occurs when $t = 0$, and $(1, 0, 2\pi)$ occurs when $t = 2\pi$. $\mathbf{r}'(t) = -\sin t\ \mathbf{i} + \cos t\ \mathbf{j} + \mathbf{k}$, hence

$$||\mathbf{r}'(t)|| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1} = \sqrt{2}.$$

   Therefore

$$L = \int_0^{2\pi} \sqrt{2}dt = 2\pi\sqrt{2}.$$

□

   (b) Reparametrize the helix $\mathbf{r}(t)$ with respect to arc length measured from $(1, 0, 0)$ in the direction of increasing t.

Solution. We first need to find the arc length function $s(t)$ given by

$$s(t) = \int_0^t ||\mathbf{r}'(u)||du = \int_0^t \sqrt{2}du = \sqrt{2}t.$$

Solve for t in terms of s:

$$t = \frac{s}{\sqrt{2}},$$

and substitute back into the equation for $\mathbf{r}(t)$:

$$\mathbf{r}\left(t(s)\right) = \cos\left(\frac{s}{\sqrt{2}}\right)\ \mathbf{i} + \sin\left(\frac{s}{\sqrt{2}}\right)\ \mathbf{j} + \frac{s}{\sqrt{2}}\ \mathbf{k}.$$

□

(c) Find the curvature κ of $\mathbf{r}(t)$.

Solution. From part (a) we know that

$$\mathbf{r}'(t) = \mathbf{r}'(t) = -\sin t\ \mathbf{i} + \cos t\ \mathbf{j} + \mathbf{k} \quad \text{and} \quad ||\mathbf{r}'(t)|| = \sqrt{2},$$

thus $\mathbf{T} = \dfrac{1}{\sqrt{2}}\langle -\sin t, \cos t, 1\rangle$. Then

$$\frac{d\mathbf{T}}{dt} = \frac{1}{\sqrt{2}}\langle -\cos t, -\sin t, 0\rangle$$

and

$$\left|\left|\frac{d\mathbf{T}}{dt}\right|\right| = \frac{1}{\sqrt{2}}\sqrt{\cos^2 t + \sin^2 t} = \frac{1}{\sqrt{2}}.$$

Therefore

$$\kappa = \frac{\frac{d\mathbf{T}}{dt}}{||\mathbf{r}'(t)||} = \frac{1}{2}.$$

You can also use the reparamatrization from part (b) to find $\mathbf{T}(s)$ and use the formula $\kappa = \left|\left|\dfrac{d\mathbf{T}}{ds}\right|\right|$.

□

9. Find an equation of the line that

(a) passes through the points $(1,3,2), (-4,3,0)$.

Solution. Set $\mathbf{r}_0 = \langle 1,3,2\rangle$. The direction of the line is the vector between the two points:

$$\mathbf{v} = \langle -4-1, 3-3, 0-2\rangle = \langle -5, 0, -2\rangle.$$

The line is therefore

$$\mathbf{r}(t) = \langle 1,3,2\rangle + t\langle -5,0,-2\rangle.$$

□

(b) passes through the point $(1,0,6)$ and is perpendicular to the plane $x+3y+z=5$.

Solution. To be perpendicular to the plane $x+3y+z=5$ is to be parallel to the normal of that plane; that is, the direction of our line will be equal to the normal of the given plane. The normal to the plane is $\mathbf{n}=\langle 1,3,1\rangle$. Set $\mathbf{v}=\mathbf{n}$ and $\mathbf{r}_0=\langle 1,0,6\rangle$. Then

$$\mathbf{r}(t)=\langle 1,0,6\rangle+t\langle 1,3,1\rangle.$$

□

10. Determine whether the lines L_1 and L_2 are parallel, skew, or intersecting. If they intersect, find the point of intersection.

(a) $L_1: x=-6t, \quad ,y=1+9t, \quad z=-3t$
$L_2: x=1+2s, \quad y=4-3s, \quad z=s$

Solution. Set the equations equal to each other to get the system

$$\begin{cases} -6t &= 1+2s \\ 1+9t &= 4-3s \\ -3t &= s \end{cases}$$

The third equation says $s=-3t$. Substituting this into the first equation yields

$$-6t=1+2(-3t) \quad \Rightarrow \quad -6t=1-6t \quad \Rightarrow \quad 0=1,$$

which is a contradiction. The system is inconsistent, so there is no intersection.
Now we must decide whether these lines are skew or parallel. The direction of L_1 is $\mathbf{v}_1=\langle -6,9,-3\rangle$ and the direction of L_2 is $\mathbf{v}_2=\langle 2,-3,1\rangle$. Notice that $\mathbf{v}_1=-3\mathbf{v}_2$. The lines are in the same direction, hence the lines are parallel. □

(b) $L_1: x=1+t, \quad y=-2+3t, \quad z=4-t$
$L_2: x=2s, \quad y=3+s, \quad z=-3+4s$

Solution. Set the equations equal to each other to get the system

$$\begin{cases} 1+t &= 2s \\ -2+3t &= 3+s \\ 4-t &= -3+4s \end{cases}$$

This system is inconsistent, there is no intersection (you can check by using row reduction). The direction of L_1 is $\langle 1,3,-1\rangle$ and the direction of L_2 is $\langle 2,1,4\rangle$. The two directions are not multiples of each other, hence the lines are not parallel, they must be skew. □

11. Find an equation of the plane through the point $(6,3,2)$ and perpendicular to the vector $\langle -2,1,5\rangle$.

Solution. To be perpendicular to $\mathbf{v} = \langle -2, 1, 5\rangle$ is for the normal of the plane to be parallel to $\mathbf{v}$. Thus $\mathbf{n} = \mathbf{v}$. Therefore the equation of the plane is

$$\langle -2,1,5\rangle \cdot (x-6, y-3, z-2) = 0 \quad \Rightarrow \quad -2(x-6) + y - 3 + 5(z-2) = 0$$
$$-2x + y + 5z = 1$$

□

12. Find the angle between the planes $x + y + z = 1$ and $x - 2y + 3z = 1$. (Hint: find the angle between the normal vectors of the planes.)

Solution. To find the angle between two planes, we need to find the angle between the normal vectors of the planes. The first plane has normal vector $\mathbf{n}_1 = \langle 1, 1, 1\rangle$ and the second plane has normal vector $\mathbf{n}_2 = \langle 1, -2, 3\rangle$. We then have

$$\begin{aligned} \mathbf{n}_1 \cdot \mathbf{n}_2 &= \langle 1,1,1\rangle \cdot \langle 1,-2,3\rangle = 1 - 2 + 3 = 2 \\ ||\mathbf{n}_1|| &= \sqrt{1^2+1^2+1^2} = \sqrt{3} \\ ||\mathbf{n}_2|| &= \sqrt{1^2+(-2)^2+3^2} = \sqrt{14}. \end{aligned}$$

Therefore
$$\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{||\mathbf{n}_1|| ||\mathbf{n}_2||} = \frac{2}{\sqrt{3}\sqrt{42}} \quad \Rightarrow \quad \theta = \cos^{-1}\left(\frac{2}{\sqrt{42}}\right).$$

□

13. For the following functions:

(i) Sketch the level curves $f(x,y) = k$ for the given k values.

(ii) Use your contour map from (i) to estimate $f_x, f_y, f_{xx}, f_{xy}, f_{yy}$ at the point $P(0,0)$.

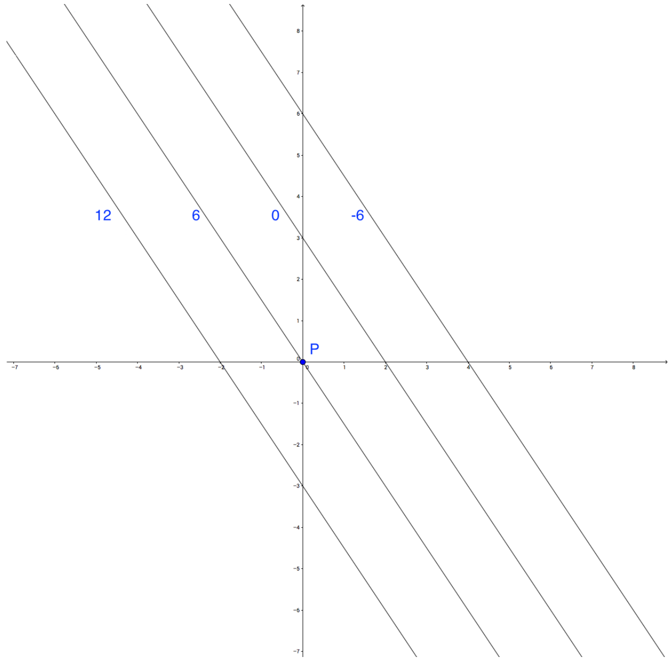
(a) $f(x,y) = 6 - 3x - 2y$ for $k = -6, 0, 6, 12$

Solution. The level curves $f(x,y) = k$ are the lines

$$\begin{aligned} y &= -\frac{3}{2}x + 6, && k = -6 \\ y &= -\frac{3}{2}x + 3, && k = -0 \\ y &= -\frac{3}{2}x, && k = 6 \\ y &= -\frac{3}{2}x - 3, && k = 12 \end{aligned}$$

As we move to the right of P the values of $f(x,y)$ are decreasing, hence $f_x < 0$. As we move up from P the values of $f(x,y)$ are decreasing, hence $f_y < 0$.

f_{xx} means to look at f_x and the bunching or spreading of the lines as we move in the x direction. As we move in the x direction, the lines remain a constant distance from each other, hence there is no bunching or spreading. Thus $f_{xx} = 0$.



f_{yy} means to look at f_y and the bunching or spreading of the lines as we move in the y direction. As we move in the y direction, the lines remain a constant distance from each other, hence there is no bunching or spreading. Thus $f_{yy} = 0$.

f_{xy} means to look at f_x and the bunching or spreading of the lines as we move in the y direction. There was no bunching or spreading in this direction, thus $f_{xy} = 0$. $\square$

(b) $f(x, y) = \sqrt{9 - x^2 - y^2}$ for $k = 0, 1, 2, 3$

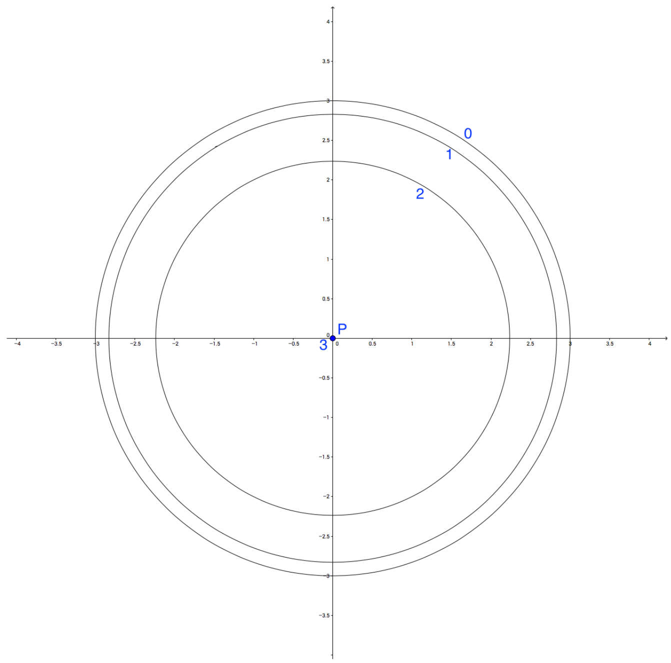
Solution. The level curves $f(x, y) = k$ are the circles

$$\begin{aligned} x^2 + y^2 &= 9, \quad k = 0 \\ x^2 + y^2 &= 8, \quad k = 1 \\ x^2 + y^2 &= 5, \quad k = 2 \\ x^2 + y^2 &= 0, \quad k = 3 \end{aligned}$$

As we move to the right of P, the values of $f(x, y)$ are decreasing, hence $f_x < 0$. But if we start from the left of P and move right, the values of $f(x, y)$ are increasing, hence $f_x > 0$. These together imply that $f_x = 0$ at P.

As we move up from P, the values of $f(x, y)$ are decreasing, hence $f_y < 0$. But if we start from the bottom of P and move up, the values of $f(x, y)$ are increasing, hence $f_y < 0$. These together imply that $f_y = 0$ at P.

f_{xx} means to look at f_x and the bunching or spreading of the contours as we move in the x direction. Let's start on the right of P: As we move to the right of P, the lines



bunch together (increasing rate); $f_x < 0$ on the right of P, hence f is decreasing at an increasing rate, therefore $f_{xx} < 0$. Now let's look at the left of P: If we start on the left of P and move to the right, the contours spread (decreasing rate); $f_x > 0$ on the left of P, hence f is increasing at a decreasing rate, therefore $f_{xx} < 0$. In either case, we have $f_{xx} < 0$.

f_{yy} means to look at f_y and the bunching or spreading of the contours as we move in the x direction. Let's start above P: As we move up from P the lines bunch together (increasing rate); $f_y < 0$ above P, hence f is decreasing at an increasing rate, therefore $f_{yy} < 0$. Now let's look below P: If we start below P and move up, the contours spread (decreasing rate); $f_y > 0$ below P, hence f is increasing at a decreasing rate, therefore $f_{yy} < 0$.

f_{xy} means to look at f_x and the bunching or spreading of the contours as we move in the y direction. Let's start above P: As we move up from P the lines bunch together (increasing rate); but $f_x = 0$ above P, hence $f_{xy} = 0$. Same is true if we start below P.

□

14. Find the limit, if it exists, or show that the limit does not exist:

(a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y^4}{x^4 + 3y^4}$

Solution. If we take the path $y = 0$:

$$\lim_{x\to 0} \frac{0}{x^4 + 0} = 0.$$

But, if we take the path $y = x$:

$$\lim x \to 0 \frac{x^4}{4x^4} = \frac{1}{4}.$$

These two limits do not equal each other, hence the limit does not exist. $\square$

(b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy\cos y}{3x^2 + y^2}$

Solution. If we take the path $y = 0$:

$$\lim_{x\to 0} \frac{0}{3x^2 + 0} = 0.$$

But, if we take the path $y = x$:

$$\lim_{x\to 0} \frac{x^2 \cos x}{4x^2} = \lim_{x\to 0} \frac{1}{4} \cos x = \frac{1}{4}.$$

These two limits do not equal each other, hence the limit does not exist. $\square$

15. Find all of the second partial derivatives of $f(x, y) = x^3y^5 + 2x^4y$. Does Clairaut's Theorem hold?

Solution. We begin by finding the first partial derivatives:

$$\begin{aligned} f_x &= 3x^2y^5 + 8x^3y \\ f_y &= 5x^3y^4 + 2x^4 \end{aligned}$$

Now we find the second partials:

$$\begin{aligned} f_{xx} &= 6xy^5 + 24x^2y \\ f_{xy} &= 15x^2y^4 + 8x^3 \\ f_{yx} &= 15x^2y^4 + 8x^3 \\ f_{yy} &= 20x^3y^3 \end{aligned}$$

Yes, Clairaut's Theorem holds since $f_{xy} = f_{yx}$. $\square$

16. Find the linearization $L(x, y)$ of the function $f(x, y) = \dfrac{x}{x + y}$ at the point $(2, 1)$.

Solution. The linearization $L(x, y)$ of f at the point $(2, 1)$ is given by the formula

$$L(x, y) = f(2, 1) + f_x(2, 1)(x - 2) + f_y(2, 1)(y - 1).$$

Find f_x and f_y:

$$\begin{aligned} f_x &= \frac{(x + y)x' - x(x + y)'}{(x + y)^2} = \frac{x + y - x}{(x + y)^2} = \frac{y}{(x + y)^2} \\ f_y &= -\frac{x}{(x + y)^2} \end{aligned}$$

Evaluate at $(2,1)$:

$$\begin{aligned} f(2,1) &= \frac{2}{2+1} = \frac{2}{3} \\ f_x(2,1) &= \frac{1}{(2+1)^3} = \frac{1}{9} \\ f_y(2,1) &= -\frac{2}{(2+1)^3} = -\frac{2}{9} \end{aligned}$$

Therefore

$$L(x,y) = \frac{2}{3} + \frac{1}{9}(x-2) - \frac{2}{9}(y-1).$$

□

17. The length and width of a rectangle are measured as 30 cm and 24 cm respectively, with an error in measurement of at most 0.1 cm each. Use differentials to estimate the maximum error in the calculated area of the rectangle.

Solution. The area of a rectangle is given by $A = LW$. The differential dA is given by

$$dA = \frac{\partial A}{\partial L}dL + \frac{\partial A}{\partial W}dW \quad \Rightarrow \quad dA = WdL + LdW$$

Since $dA \approx \Delta A$, $dL \approx \Delta L, dW \approx \Delta W$, and we are told that $|\Delta L| \leq 0.1$ cm, $|\Delta W| \leq 0.1$ cm, then

$$\Delta A \approx W\Delta L + L\Delta W = (24)(0.1) + 30(0.1)\text{cm}^2 = 5.4\text{cm}^2$$

□

18. Let $f(x,y,z) = xe^{2yz}$, $\mathbf{u} = \langle \frac{2}{3}, \frac{-2}{3}, \frac{1}{3} \rangle$.

(a) Find the gradient of f.

Solution.

$$\begin{aligned} \nabla f &= \langle \frac{\partial}{\partial x}(xe^{2yz}), \frac{\partial}{\partial y}(xe^{2yz}), \frac{\partial}{\partial z}(xe^{2yz}) \rangle \\ &= \langle e^{2yz}, 2xze^{2yz}, 2xye^{2yz} \rangle \end{aligned}$$

□

(b) Find the gradient at the point $P(3,0,2)$.

Solution. $\nabla f(3,0,2) = \langle 1, 12, 0 \rangle$ □

(c) Find the rate of change of f at P in the direction of the vector $\mathbf{u}$. (In other words, find the directional derivative.)

Solution. The directional derivative is

$$\nabla f(3,0,2) \cdot \mathbf{u} = \langle 1, 12, 0 \rangle \cdot \langle \frac{2}{3}, \frac{2}{3}, -\frac{1}{3} \rangle = \frac{26}{3}$$

□

19. Find the maximum rate of change of $f(x,y,z) = \dfrac{x+y}{z}$ at the point $(1, 1-1)$ and the direction in which it occurs.

Solution. The maximum rate of change is $||\nabla f||$ and the direction in which it occurs is ∇f.

$$\nabla f = \langle \frac{1}{z}, \frac{1}{z}, -\frac{x+y}{z^2} \rangle \quad \Rightarrow \quad \nabla f(1,1,-1) = \langle -1, -1, -2 \rangle$$
$$||\nabla f(-1,-1,-2)|| = \sqrt{(-1)^2 + (-1)^2 + (-2)^2} = \sqrt{6}.$$

□

20. (a) Find the local maximum and minimum values and saddles points of $f(x,y) = x^4 + y^4 - 4xy + 2$

Solution. Find the critical points by setting $f_x, f_y = 0$:

$$f_x = 4x^3 - 4y = 0 \quad \text{and} f_y = 4y^3 - 4x = 0$$

The first equation says $y = x^3$, the second equation says $x = y^3$. Substituting the first equation into the second equation yields

$$x = (x^3)^3 \quad \Rightarrow \quad x = x^9 \quad \Rightarrow \quad x - x^9 = 0 \quad \Rightarrow \quad x(1-x^8) = 0.$$

This gives us $x = 0, x = 1, x = -1$. Substituting this back into the first equation yields $y = 0, y = 1, y = -1$. Thus the critical points are $(0,0), (1,1), (-1,-1)$.
Calculate the second derivatives:

$$\begin{aligned} f_{xx} &= 12x^2 \\ f_{xy} &= -4 \\ f_{yy} &= 12y^2 \end{aligned}$$

We will now use the Second Derivative Test to determine whether the critical points are a min, max, or saddle. I will do this using the determinant of the Hessian matrix:

$$D = f_{xx} f_{yy} - (f_{xy})^2$$

You can also use the eigenvalues of the Hessian matrix to classify the critical points, but that is not included here.
At $(0,0)$, $D(0,0) = 0 - 16 < 0$, thus $(0,0)$ is a saddle. The function value is $f(0,0) = 2$.
At $(1,1)$, $D = 144 - 16 > 0$ and $f_{xx}(1,1) = 12 > 0$, thus $(1,1)$ is a local min. The function value is $f(1,1) = 0$.
At $(-1,-1)$, $D = 144 - 16 > 0$ and $f_{xx}(-1,-1) = 12 > 0$, thus $(-1,-1)$ is a local min. The function value is $f(-1,-1) = 0$. □

(b) Find the absolute maximum and minimum values of $f(x,y) = x^4 + y^4 - 4xy + 2$ on the set $D = \{(x,y) : 0 \leq x \leq 3, 0 \leq y \leq 2\}$.

Proof. D forms a rectangle. We need to find the critical points along the boundary of D.
Along $y = 0$ (bottom of the rectangle), we have $f(x,0) = g_1(x) = x^4$. Take the derivative and set it equal to zero to get the critical point:

$$4x^3 = 0 \quad \Rightarrow \quad x = 0.$$

The critical point here is $(x,y) = (0,0)$.
Along $x = 3$ (right side of the rectangle), we have $f(3,y) = g_2(y) = 3^4 + y^4 - 4(3)y + 2 = y^4 - 12y + 83$. Take the derivative and set it equal to zero to get the critical point:

$$4y^3 - 12 = 0 \quad \Rightarrow \quad y^3 = 3 \quad \Rightarrow \quad y = \sqrt[3]{3}.$$

The critical point here is $(x,y) = (3, \sqrt[3]{3})$.
Along $y = 2$ (top of the rectangle), we have $f(x,2) = g_3(y) = x^4 + 2^4 - 4x(2) + 2 = x^4 - 8x + 18$. Take the derivative and set it equal to zero to get the critical point:

$$4x^3 - 8 = 0 \quad \Rightarrow \quad x^3 = 2 \quad \Rightarrow \quad x = \sqrt[3]{2}.$$

The critical point here is $(x,y) = (\sqrt[3]{2}, 2)$.
Along $x = 0$, (left side of the rectangle), we have $f(0,y) = g_4(y) = y^4 + 2$. Take the derivative and set it equal to zero to get the critical point:

$$4y^3 = 0 \quad \Rightarrow \quad y = 0.$$

The critical point here is $(x,y) = (0,0)$.
The critical points from the boundary are $(0,0), (3, \sqrt[3]{3}), (\sqrt[3]{2}, 2)$. the critical points from part (a) are $(0,0), (1,1), (-1,-1)$, but $(-1,-1)$ is not in the given rectangle so we only need to test $(0,0)$ and $(1,1)$. We need to evaluate are function at each of these points. The largest function value is the max, the lowest function value is the min.

$$\begin{aligned}
f(0,0) &= 2 \\
f(3, \sqrt[3]{3}) &= 3^4 + (\sqrt[3]{3})^4 - 4(3)(\sqrt[3]{3}) + 2 = 83 - 9\sqrt[3]{3} \\
f(\sqrt[3]{2}, 2) &= (\sqrt[3]{2})^4 + 2^4 - 4(\sqrt[3]{2})(2) + 2 = 18 - 6\sqrt{2} \\
f(1,1) &= 0
\end{aligned}$$

The absolute max and min on the set D therefore occur at $f(3, \sqrt[3]{3}) = 83 - 9\sqrt[3]{3}$ and $f(1,1) = 0$, respectively. $\square$