Surface Area, Surface Integral Examples

Written by Victoria Kala
vtkala@math.ucsb.edu
SH 6432u Office Hours: R 12:30 − 1:30pm
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The first example demonstrates how to find the surface area of a given surface. The second example demonstrates how to find the surface integral of a given vector field over a surface.

1. Find the surface area of the portion of the sphere of radius 4 that lies inside the cylinder $x^2 + y^2 = 12$ and above the xy-plane.

   Solution. We need to evaluate
$$A = \iint_D ||\mathbf{r}_u \times \mathbf{r}_v|| dA.$$

   We are asked to find the surface area of a portion of the sphere, this is the surface we need to parametrize. The parametrization vector is given by
$$\mathbf{r}(x, y, z) = \langle x, y, z \rangle.$$

   In spherical coordinates, $x = \rho \cos\theta \sin\phi, y = \rho \sin\theta \sin\phi, z = \rho \cos\phi$. Plug these into our parametrization vector:
$$\mathbf{r}(\rho, \theta, \phi) = \langle \rho \cos\theta \sin\phi, \rho \sin\theta \sin\phi, \rho \cos\phi \rangle.$$

   Right now our parametrization is a function of three variables, we need to narrow it down to two variables. We are given one more piece of information about the sphere: it has radius 4. This means $\rho = 4$. Plug this in:
$$\mathbf{r}(\theta, \phi) = \langle 4 \cos\theta \sin\phi, 4 \sin\theta \sin\phi, 4 \cos\phi \rangle.$$

   Now we need to find the range of θ and ϕ. Since we are going all the way around the cylinder and sphere, $0 \le \theta \le 2\pi$. To find the range of ϕ, we need to find where the cylinder and sphere intersect. The equation of the sphere is
$$x^2 + y^2 + z^2 = 16.$$

   The equation of the cylinder is $x^2 + y^2 = 12$. Plug this into the equation of the sphere:
$$12 + z^2 = 16 \quad \Rightarrow \quad z^2 = 4 \quad \Rightarrow \quad z = 2.$$

   In spherical coordinates, $z = \rho \cos\phi$, but we know that $\rho = 4$, so $z = 4 \cos\phi$. Therefore
$$z = 2 \quad \Rightarrow \quad 4\cos\phi = 2 \quad \Rightarrow \quad \cos\phi = \frac{1}{2} \quad \Rightarrow \quad \phi = \frac{\pi}{3},$$

   and we have that $0 \le \phi \le \frac{\pi}{3}$. Therefore
$$A = \int_0^{2\pi} \int_0^{\pi/3} ||\mathbf{r}_\theta \times \mathbf{r}_\phi|| d\phi d\theta.$$

   Now find $\mathbf{r}_\theta \times \mathbf{r}_\phi$:
$$\mathbf{r}_\theta = \langle -4 \sin\theta \sin\phi, 4 \cos\theta \sin\phi, 0 \rangle$$
$$\mathbf{r}_\phi = \langle 4 \cos\theta \cos\phi, 4 \sin\theta \cos\phi, -4 \sin\phi \rangle$$
$$\mathbf{r}_\theta \times \mathbf{r}_\phi = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4\sin\theta\sin\phi & 4\cos\theta\sin\phi & 0 \\ 4\cos\theta\cos\phi & 4\sin\theta\cos\phi & -4\sin\phi \end{vmatrix} = \langle -16\cos\theta\sin^2\phi, -16\sin\theta\sin^2\phi, -16\sin\phi\cos\phi \rangle$$

Find $||\mathbf{r}_\theta \times \mathbf{r}_\phi||$:

$$||\mathbf{r}_\theta \times \mathbf{r}_\phi|| = \sqrt{(-16\cos\theta\sin^2\phi)^2 + (-16\sin\theta\sin^2\phi)^2 + (-16\sin\phi\cos\phi)^2} = ... = 16\sin\phi$$

Therefore

$$A = \int_0^{2\pi}\int_0^{\pi/3} ||\mathbf{r}_\theta \times \mathbf{r}_\phi|| d\phi d\theta = \int_0^{2\pi}\int_0^{\pi/3} 16\sin\phi d\phi d\theta = 2\pi(-16\cos\phi)\Big|_0^{\pi/3} = 16\pi.$$

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2. Set up the integral $\iint_S \mathbf{F}\cdot d\mathbf{S}$ where $\mathbf{F} = y\mathbf{j} - z\mathbf{k}$ and S is the surface given by the paraboloid $y = x^2 + z^2$ between $y = 0$ and $y = 1$. Assume S has positive orientation.

Solution. We need to evaluate

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_D \mathbf{F}\cdot\mathbf{n} dA$$

where D the range of the parameters in dA.

We need to parametrize the surface. The parametrization vector is given by

$$\mathbf{r}(x, y, z) = \langle x, y, z\rangle.$$

We need to narrow this down to dependence on two variables. We are given that the surface is $y = x^2 + z^2$. Plug this into the paramterization vector:

$$\mathbf{r}(x, z) = \langle x, x^2 + z^2, z\rangle.$$

Find $\mathbf{r}_x \times \mathbf{r}_z$:

$$\begin{aligned}\mathbf{r}_x &= \langle 1, 2x, 0\rangle\\ \mathbf{r}_z &= \langle 0, 2z, 1\rangle\\ \mathbf{r}_x \times \mathbf{r}_z &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 2x & 0\\ 0 & 2z & 1\end{vmatrix} = \langle 2x, -1, 2x\rangle.\end{aligned}$$

Then

$$\iint_D \mathbf{F}\cdot\mathbf{n} dA = \iint_D \langle 0, y, -z\rangle\cdot\langle 2x, -1, 2z\rangle dA = \iint_D (-y - 2z^2) dA = \iint_D (-(x^2+z^2) - 2z^2) dA$$

Note that we have to plug in y into the integral since we will be integrating over x and z.

We can certainly find bounds for this integral in Cartesian coordinates, but it will be simpler to switch our integral into cylindrical coordinates. In cylindrical coordinates

$$x = r\cos\theta, z = r\sin\theta, \quad 0 \le r \le 1, 0 \le \theta \le 2\pi.$$

To get the bounds for r, use $y = x^2 + z^2$ and $y = 1$. Therefore

$$\iint_D \mathbf{F}\cdot\mathbf{n} dA = \iint_D (-(x^2+z^2) - 2z^2) dA = \int_0^{2\pi}\int_0^1 (-r^2 - 2r^2\sin^2\theta) dr d\theta.$$

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