The basic idea of this course is that curvature bounds give information about manifolds, which in turn gives topological results. A typical example is the Bonnet-Myers Theorem. Intuitively,

Bigger curvature $\Rightarrow$ Smaller manifold\(^1\).

1 Volume Comparison Theorem

1.1 Volume of Riemannian Manifold

Recall: For $U \subset \mathbb{R}^n$,

$$\text{vol}(U) = \int_U 1 \, dv = \int_U 1 \, dx_1 \cdots dx_n.$$  

Note - $dx_1 \cdots dx_n$ is called the volume density element. Change of variable formula: Suppose $\psi : V \rightarrow U$ is a diffeomorphism, with $U, V \subset \mathbb{R}^n$. Suppose $\psi(x) = y$. Then

$$\int_U dv = \int_U 1 \, dy_1 \cdot \cdots \cdot dy_n = \int_V |\text{Jac}(\psi)| \, dx_1 \cdots dx_n.$$  

On a Riemannian manifold $M^n$, let $\psi_\alpha : U_\alpha \rightarrow \mathbb{R}^n$ be a chart. Set $E_{ip} = (\psi^{-1}_\alpha)_* \left( \frac{\partial}{\partial x_i} \right)$. In general, the $E_{ip}$'s are not orthonormal. Let $\{e_k\}$ be an orthonormal basis of $T_pM$. Then $E_{ip} = \sum_{k=1}^n a_{ik} e_k$. The volume of

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\(^1\)This quarter we use that bigger curvature $\Rightarrow$ smaller volume
the parallelepiped spanned by \( \{ E_i \} \) is \( | \det(a_{ik}) | \). Now \( g_{ij} = \sum_{k=1}^{n} a_{ik}a_{kj} \), so
\[
\det(g_{ij}) = \det(a_{ij})^2.
\]
Thus
\[
\text{vol}(U_\alpha) = \int_{\psi(U_\alpha)} \sqrt{\det(g_{ij})} \circ (\psi_\alpha^{-1}) \; dx_1 \cdots dx_n
\]

Note - \( dv = \sqrt{\det(g_{ij})} \circ (\psi_\alpha^{-1}) \; dx_1 \cdots dx_n \) is called a volume density element, or volume form, on \( M \).

We have our first result, whose proof is left as an exercise.

**Lemma 1.1.1** Volume is well defined.

**Definition 1.1.1** Let \( M \) be a Riemannian manifold, and let \( \{ U_\alpha \} \) be a covering of \( M \) by domains of coordinate charts. Let \( \{ f_\alpha \} \) be a partition of unity subordinate to \( \{ U_\alpha \} \). The volume of \( M \) is
\[
\text{vol}(M) = \int_M 1 \; dv = \sum_\alpha \int_{\psi(U_\alpha)} f_\alpha \; dv.
\]

**Lemma 1.1.2** The volume of a Riemannian manifold is well defined.

### 1.2 Computing the volume of a Riemannian manifold

Partitions of unity are not practically effective. Instead we look for charts that cover all but a measure zero set.

**Example 1.2.1** For \( S^2 \), use stereographic projection.

In general, we use the exponential map. We may choose normal coordinates or geodesic polar coordinates. Let \( p \in M^n \). Then \( \exp_p : T_p M \to M \) is a local diffeomorphism. Let \( D_p \subset T_p M \) be the segment disk. Then if \( C_p \) is the cut locus of \( p \), \( \exp_p : D_p \to M - C_p \) is a diffeomorphism.

**Lemma 1.2.1** \( C_p \) has measure zero.

Hence we may use \( \exp_p \) to compute the volume element \( dv = \sqrt{\det(g_{ij})} \; dx_1 \cdots dx_n \).

Now polar coordinates are not defined at \( p \), but \( \{ p \} \) has measure zero. We have
\[ \exp_p : D_p - \{0\} \to M - C_p \cup \{p\}. \]

Set \( E_i = (\exp_p)_\ast \left( \frac{\partial}{\partial \theta_i} \right) \) and \( E_n = (\exp_p)_\ast \left( \frac{\partial}{\partial r} \right) \). To compute \( g_{ij} \)'s, we want \( E_i \) and \( E_n \) explicitly. Since \( \exp_p \) is a radial isometry, \( g_{nn} = 1 \) and \( g_{ni} = 0 \) for \( 1 \leq i < n \). Let \( J_i(r, \theta) \) be the Jacobi field with \( J_i(0) = 0 \) and \( J_i'(0) = \frac{\partial}{\partial r} \). Then \( E_i(\exp_p(r, \theta)) = J_i(r, \theta) \).

If we write \( J_i \) and \( \frac{\partial}{\partial r} \) in terms of an orthonormal basis \( \{e_k\} \), we have \( J_i = \sum_{k=1}^n a_{ik} e_k \). Thus

\[ \sqrt{\det(g_{ij}(r, \theta))} = |\det(a_{ik})| \prod_{j=1}^n |J_1 \wedge \cdots \wedge J_{n-1} \wedge \frac{\partial}{\partial r}| \]

The volume density, or volume element, of \( M \) is

\[ dv = ||J_1 \wedge \cdots \wedge J_{n-1} \wedge \frac{\partial}{\partial r}|| dr d\theta_{n-1} = A(r, \theta) dr d\theta_{n-1} \]

**Example 1.2.2** \( \mathbb{R}^n \) has Jacobi equation \( J'' = R(T, J)T \).

If \( J(0) = 0 \) and \( J'(0) = \frac{\partial}{\partial \theta} \), then \( J(r) = r \frac{\partial}{\partial \theta} \). Thus the volume element is \( dv = r^{n-1} dr d\theta_{n-1} \).

**Example 1.2.3** \( S^n \) has \( J_i(r) = \sin(r) \frac{\partial}{\partial \theta_i} \). Hence \( dv = \sin^{n-1}(r) dr d\theta_{n-1} \).

**Example 1.2.4** \( \mathbb{H}^n \) has \( J_i(r) = \sinh(r) \frac{\partial}{\partial \theta_i} \). Hence \( dv = \sinh^{n-1}(r) dr d\theta_{n-1} \).

**Example 1.2.5** Volume of unit disk in \( \mathbb{R}^n \)

\[ \omega_n = \int_{S^{n-1}} \int_0^1 r^{n-1} dr d\theta_{n-1} = \frac{1}{n} \int_{S^{n-1}} d\theta_{n-1} \]

Note -

\[ \int_{S^{n-1}} d\theta_{n-1} = \frac{2(\pi)^{n/2}}{\Gamma(n/2)}. \]
1.3 Comparison of Volume Elements

**Theorem 1.3.1** Suppose $M^n$ has $\text{Ric}_M \geq (n-1)H$. Let $dv = A(r, \theta) \, dr \, d\theta_{n-1}$ be the volume element of $M$ and let $dv_H = A_H(r, \theta) \, dr \, d\theta_{n-1}$ be the volume element of the model space (simply connected $n$-manifold with $K \equiv H$). Then

$$\frac{A(r, \theta)}{A_H(r, \theta)}$$

is a nonincreasing function in $r$.

**Proof.** We show that

$$\nabla \frac{\partial}{\partial r} \left( \frac{A(r, \theta)}{A_H(r, \theta)} \right)^2 \leq 0.$$

Since

$$A(r, \theta)^2 = \langle J_1 \wedge \cdots \wedge J_{n-1} \wedge \frac{\partial}{\partial r}, J_1 \wedge \cdots \wedge J_{n-1} \wedge \frac{\partial}{\partial r} \rangle,$$

we wish to show that

$$\langle J_1 \wedge \cdots \wedge J_{n-1} \wedge \frac{\partial}{\partial r}, J_1 \wedge \cdots \wedge J_{n-1} \wedge \frac{\partial}{\partial r} \rangle A_H(r, \theta)^2 -$$

$$A(r, \theta)^2 \langle J_H^1 \wedge \cdots \wedge J_H^{n-1} \wedge \frac{\partial}{\partial r}, J_H^1 \wedge \cdots \wedge J_H^{n-1} \wedge \frac{\partial}{\partial r} \rangle' \leq 0.$$  

Thus we wish to show that

$$2 \sum_{i=1}^{n-1} \frac{\langle J_1 \wedge \cdots \wedge J_i \wedge \cdots \wedge J_{n-1} \wedge \frac{\partial}{\partial r}, J_1 \wedge \cdots \wedge J_{n-1} \wedge \frac{\partial}{\partial r} \rangle}{(J_1 \wedge \cdots \wedge J_{n-1} \wedge \frac{\partial}{\partial r}, J_1 \wedge \cdots \wedge J_{n-1} \wedge \frac{\partial}{\partial r})}$$

$$\leq 2 \sum_{i=1}^{n-1} \frac{\langle J_H^1 \wedge \cdots \wedge (J_H^i)' \wedge \cdots \wedge J_H^{n-1} \wedge \frac{\partial}{\partial r}, J_H^1 \wedge \cdots \wedge J_H^{n-1} \wedge \frac{\partial}{\partial r} \rangle}{(J_H^1 \wedge \cdots \wedge J_H^{n-1} \wedge \frac{\partial}{\partial r}, J_H^1 \wedge \cdots \wedge J_H^{n-1} \wedge \frac{\partial}{\partial r})} \leq 0.$$  

(1)

At $r = r_0$, let $\bar{J}_i(r_0)$ be orthonormal such that $\bar{J}_n(r_0) = \frac{\partial}{\partial r} |_{r=r_0}$. Then for $1 \leq i < n$,

$$\bar{J}_i(r_0) = \sum_{k=1}^{n-1} b_{ik} J_k(r_0).$$

\footnote{Compare to the proof of the Rauch Comparison Theorem.}
Define \( \bar{J}_i(r) = \sum_{k=1}^{n-1} b_{ik} J_k(r) \), where the \( b_{ik} \)'s are fixed. Then each \( \bar{J}_i \) is a linear combination of Jacobi fields, and hence is a Jacobi field.

The left hand side of (1), evaluated at \( r = r_0 \) is

\[
2 \sum_{i=1}^{n-1} \frac{\langle \bar{J}_1 \wedge \cdots \wedge \bar{J}_i' \wedge \cdots \wedge \bar{J}_{n-1} \wedge \frac{\partial}{\partial r}, \bar{J}_1 \wedge \cdots \wedge \bar{J}_{n-1} \wedge \frac{\partial}{\partial r} \rangle}{\langle \bar{J}_1 \wedge \cdots \wedge \bar{J}_{n-1} \wedge \frac{\partial}{\partial r}, \bar{J}_1 \wedge \cdots \wedge \bar{J}_{n-1} \wedge \frac{\partial}{\partial r} \rangle}_{r=r_0}
\]

\[
= 2 \sum_{i=1}^{n-1} \langle \bar{J}_i'(r_0), \bar{J}_i(r_0) \rangle = 2 \sum_{i=1}^{n-1} I(\bar{J}_i, \bar{J}_i),
\]

where \( I \) is the index form \( I(v, v) = \int_0^{r_0} \langle v', v' \rangle + \langle R(T, v) T, v \rangle dt \). Note that for a Jacobi field \( J \),

\[
I(J, J) = \int_0^{r_0} \langle J', J' \rangle + \langle R(T, J) T, J \rangle dt
\]

\[
= \int_0^{r_0} \langle J', J' \rangle - \langle J'', J \rangle + \langle R(T, J) T, J \rangle dt
\]

\[
= \langle v', v \rangle \big|_{r=r_0}.
\]

Let \( E_i \) be a parallel field such that \( E_i(r_0) = \bar{J}_i(r_0) \), and let \( w_i = \frac{\sin \sqrt{n r}}{\sin \sqrt{n r_0}} E_i \). By the Index Lemma, Jacobi fields minimize the index form provided there are no conjugate points. Thus we have \( 2 \sum_{i=1}^{n-1} I(\bar{J}_i, \bar{J}_i) \leq 2 \sum_{i=1}^{n-1} I(w_i, w_i) \). By the curvature condition, \( 2 \sum_{i=1}^{n-1} I(w_i, w_i) \leq 2 \sum_{i=1}^{n-1} I(\bar{J}_i^H, \bar{J}_i^H) \), which is the right hand side of (1), evaluated at \( r = r_0 \).

Thus \( \bar{A}(r, \theta) \) is nonincreasing in \( r \).

**Remarks:**

1. \( \lim_{r \to 0} \bar{A}(r, \theta) = 1 \), so \( A(r, \theta) \leq \bar{A}(r, \theta) \).

2. (Rigidity) If \( A(r_0, \theta) = A_H(r_0, \theta) \) for some \( r_0 \), then \( A(r, \theta) = A_H(r, \theta) \) for all \( 0 \leq r \leq r_0 \). But then \( B(p, r_0) \) is isometric to \( B(r_0) \subset S^n_H \), where \( S^n_H \) is the model space. But then the Jacobi fields in \( M \) correspond to the Jacobi fields in the model space, so that \( M \) is isometric to the model space.
3. We cannot use the Index Lemma to prove an analogous result for \( \text{Ric}_M \leq (n-1)H \). In fact, there is no such result. For example, consider Einstein manifolds with \( \text{Ric} \equiv (n-1)H \).

4. If \( K_M \leq H \), we may use the Rauch Comparison Theorem to prove a similar result inside the injectivity radius.

5. (Lohkamp) \( \text{Ric}_M \leq (n-1)H \) has no topological implications. Any smooth manifold \( M^n \), with \( n \geq 3 \), has a complete Riemannian metric with \( \text{Ric}_M \leq 0 \).

6. \( \text{Ric}_M \leq (n-1)H \) may still have geometric implications. For example, if \( M \) is compact with \( \text{Ric}_M < 0 \) then \( M \) has a finite isometry group.

### 1.4 Volume Comparison Theorem

**Theorem 1.4.1 (Bishop-Gromov)** If \( M^n \) has \( \text{Ric}_M \geq (n-1)H \) then

\[
\frac{\text{vol}(B(p, R))}{\text{vol}(B^H(R))}
\]

is nonincreasing in \( R \).

**Proof.** We have

\[
\text{vol}B(p, R) = \int_{B(p, R)} 1d\nu = \int_0^R \int_{S_p(r)} A(r, \theta)d\theta_{n-1}dr,
\]

where \( S_p(r) = \{ \theta \in S_p : r\theta \in D_p \} \). Note that \( S_p(r_1) \subset S_p(r_2) \) if \( r_1 \geq r_2 \). The theorem now follows from two lemmas:

**Lemma 1.4.1** If \( f(r)/g(r) \geq 0 \) is nonincreasing in \( r \), with \( g(r) > 0 \), then

\[
\frac{\int_0^R f(r)dr}{\int_0^R g(r)dr}
\]

is nonincreasing in \( R \).
Proof of Lemma.- The numerator of the derivative is
\[
\left( \int_{0}^{R} g(r) dr \right) \left( \int_{0}^{R} f(r) dr \right)' - \left( \int_{0}^{R} f(r) dr \right) \left( \int_{0}^{R} g(r) dr \right)'
\]
\[
= f(R) \left( \int_{0}^{R} g(r) dr \right) - g(R) \left( \int_{0}^{R} f(r) dr \right)
\]
\[
= g(R) \left( \int_{0}^{R} f(r) dr \right) \left[ \frac{f(R)}{g(R)} - \frac{\int_{0}^{R} f(r) dr}{\int_{0}^{R} g(r) dr} \right]
\]
Now
\[
\frac{f(r)}{g(r)} \geq \frac{f(R)}{g(R)} \Rightarrow g(R) f(r) \geq f(R) g(r),
\]
so
\[
\int_{0}^{R} g(R) f(r) dr \geq \int_{0}^{R} f(R) g(r) dr.
\]
Thus
\[
\frac{f(R)}{g(R)} \leq \frac{\int_{0}^{R} f(r) dr}{\int_{0}^{R} g(r) dr},
\]
so the derivative is nonpositive.

Lemma 1.4.2 (Comparison of Lower Area of Geodesic Sphere) Suppose \( r \) lies inside the injectivity radius of the model space \( S_{n}^{H} \), so that if \( H > 0, r < \pi / \sqrt{H} \). Then
\[
\frac{\int_{S_{p}(r)} A(r, \theta) d\theta_{n-1}}{\int_{S_{n-1}} A^{H}(r) d\theta_{n-1}}
\]
is nonincreasing in \( r \).

Proof of Lemma. In the model space, \( A^{H}(r, \theta) \) does not depend on \( \theta \), so we write \( A^{H}(r) \). Note that if \( r \leq R, \)
\[
\frac{\int_{S_{p}(R)} A(R, \theta) d\theta_{n-1}}{\int_{S_{n-1}} A^{H}(R) d\theta_{n-1}} = \frac{1}{\int_{S_{n-1}} d\theta_{n-1}} \int_{S_{p}(R)} A(R, \theta) d\theta_{n-1}
\]
\[
\leq \frac{1}{\int_{S_{n-1}} d\theta_{n-1}} \int_{S_{p}(r)} A(r, \theta) d\theta_{n-1}
\]
\[
= \frac{\int_{S_{p}(r)} A(r, \theta) d\theta_{n-1}}{\int_{S_{n-1}} A^{H}(r) d\theta_{n-1}},
\]
since $S_p(r) \supset S_p(R)$ and $\frac{A(r, \theta)}{A^H(r)}$ is nonincreasing in $r$. The theorem now follows.

Note that if $R$ is greater than the injectivity radius then $\text{vol}B(p, R)$ decreases. Thus the volume comparison theorem holds for all $R$.

**Corollaries:**

1. (Bishop Absolute Volume Comparison) Under the same assumptions, $\text{vol}B(p, r) \leq \text{vol}B^H(r)$.

2. (Relative Volume Comparison) If $r \leq R$ then

$$\frac{\text{vol}B(p, r)}{\text{vol}B(p, R)} \geq \frac{\text{vol}B^H(r)}{\text{vol}B^H(R)}.$$  

If equality holds for some $r_0$ then equality holds for all $0 \leq r \leq r_0$, and $B(p, r_0)$ is isometric to $B^H(r_0)$.

**Proofs:**

(1) holds because $\lim_{r \to 0} \frac{\text{vol}B(p, r)}{\text{vol}B^H(r)} = 1$.

(2) is a restatement of the the volume comparison theorem.

Sometimes we let $R = 2r$ in (2). Then (2) gives a lower bound on the ratio $\frac{\text{vol}B(p, r)}{\text{vol}B(p, R)}$, called the doubling constant. If $\text{vol}(M) \geq V$ then we obtain a lower bound on the volume of small balls.

**Generalizations:**

1. The same proof shows that the result holds for $\text{vol}^\Gamma B(p, R)$, where $\Gamma \subset S_p = S^{n-1} \subset T_pM$. In particular, the result holds for annuli $(f_{r_0}^{R_0} \cdot \cdot \cdot)$ and for cones.

2. Integral Curvature

3. Stronger curvature conditions give submanifold results.
2 Applications of Volume Comparison

2.1 Cheng’s Maximal Diameter Rigidity Theorem

Theorem 2.1.1 (Cheng) Suppose $M^n$ has $\text{Ric}_M \geq (n-1)H > 0$. By the Bonnet-Myers Theorem, $\text{diam}_M \leq \pi/\sqrt{H}$. If $\text{diam}_M = \pi/\sqrt{H}$, Cheng’s result states that $M$ is isometric to the sphere $S^n_H$ with radius $1/\sqrt{H}$.

Proof. (Shiohama) Let $p, q \in M$ have $d(p, q) = \pi/\sqrt{H}$. Then

$$\frac{\text{vol} B(p, \pi/(2\sqrt{H}))}{\text{vol} M} = \frac{\text{vol} B(p, \pi/(2\sqrt{H}))}{\text{vol} B(p, \pi/\sqrt{H})} \geq \frac{\text{vol} B_H(\pi/(2\sqrt{H}))}{\text{vol} B_H(\pi/\sqrt{H})} = 1/2$$

Thus $\text{vol} B(p, \pi/2\sqrt{H}) \geq (\text{vol} M)/2$. Similarly for $q$. Hence $\text{vol} B(p, \pi/(2\sqrt{H})) = (\text{vol} M)/2$, so we have equality in the volume comparison. By rigidity, $B(p, \pi/(2\sqrt{H}))$ is isometric to the upper hemisphere of $S^n_H$. Similarly for $B(q, \pi/2\sqrt{H})$, so $\text{vol} M = \text{vol} S^n_H$.

Question: What about perturbation? Suppose $\text{Ric}_M \geq (n-1)H$ and $\text{diam}_M \geq \pi/\sqrt{H} - \varepsilon$. In general there is no result for $\varepsilon > 0$. There are spaces not homeomorphic to $S^n$, provided $n \geq 4$, with $\text{Ric} \geq (n-1)H$ and $\text{diam} \geq \pi/\sqrt{H} - \varepsilon$. Still, if $\text{Ric} \geq (n-1)H$ and $\text{vol} M \geq \text{vol} S^n_H - \varepsilon(n, H)$ then $M^n \text{ diffeo } S^n_H$.

2.2 Growth of Fundamental Group

Suppose $\Gamma$ is a finitely generated group, say $\Gamma = \langle g_1, \ldots, g_k \rangle$. Any $g \in \Gamma$ can be written as a word $g = \prod g_{k_i}^{n_i}$, where $k_i \in \{1, \ldots, k\}$. Define the length of this word to be $\sum_i |n_i|$, and let $|g|$ be the minimum of the lengths of all word representations of $g$. Note that $|\cdot|$ depends on the choice of generators.

Fix a set of generators for $\Gamma$. The growth function of $\Gamma$ is

$$\Gamma(s) = \# \{g \in \Gamma : |g| \leq s\}.$$

Example 2.2.1 If $\Gamma$ is a finite group then $\Gamma(s) \leq |\Gamma|$. 
Example 2.2.2 \( \Gamma = \mathbb{Z} \oplus \mathbb{Z} \). Then \( \Gamma = \langle g_1, g_2 \rangle \), where \( g_1 = (1, 0) \) and \( g_2 = (0, 1) \). Any \( g \in \Gamma \) can be written as \( g = s_1 g_1 + s_2 g_2 \). To find \( \Gamma(s) \), we want \( |s_1| + |s_2| \leq s \).

\[
\Gamma(s) = 2s + 1 + \sum_{t=1}^{s} 2(2(s-t) + 1) \\
= 2s + 1 + \sum_{t=1}^{s} (4s - 4t + 2) \\
= 2s + 1 + 4s^2 + 2s - 4 \sum_{t=1}^{s} t \\
= 4s^2 + 4s + 1 - 4(s(s+1)/2) \\
= 4s^2 + 4s + 1 - 2s^2 + s \\
= 2s^2 + 2s + 1
\]

In this case we say \( \Gamma \) has polynomial growth.

Example 2.2.3 \( \Gamma \) free abelian on \( k \) generators. Then \( \Gamma(s) = \sum_{i=0}^{k} \binom{k}{i} \binom{s}{i} \).

\( \Gamma \) has polynomial growth of degree \( k \).

Definition 2.2.1 \( \Gamma \) is said to have polynomial growth of degree \( \leq n \) if for each set of generators the growth function \( \Gamma(s) \leq as^n \) for some \( a > 0 \).

\( \Gamma \) is said to have exponential growth if for each set of generators the growth function \( \Gamma(s) \geq e^{as} \) for some \( a > 0 \).

Lemma 2.2.1 If for some set of generators, \( \Gamma(s) \leq as^n \) for some \( a > 0 \), then \( \Gamma \) has polynomial growth of degree \( \leq n \). If for some set of generators, \( \Gamma(s) \geq e^{as} \) for some \( a > 0 \), then \( \Gamma \) has exponential growth.

Example 2.2.4 \( \mathbb{Z}^k \) has polynomial growth of degree \( k \).

Example 2.2.5 \( \mathbb{Z} \ast \mathbb{Z} \) has exponential growth.

Note that for each group \( \Gamma \) there always exists \( a > 0 \) so that \( \Gamma(s) \leq e^{as} \).

Definition 2.2.2 A group is called almost nilpotent if it has a nilpotent subgroup of finite index.
**Theorem 2.2.1 (Gromov)** A finitely generated group $\Gamma$ has polynomial growth iff $\Gamma$ is almost nilpotent.

**Theorem 2.2.2 (Milnor)** If $M^n$ is complete with $\text{Ric}_M \geq 0$, then any finitely generated subgroup of $\pi_1(M)$ has polynomial growth of degree $\leq n$.

**Proof.** Let $\tilde{M}$ have the induced metric. Then $\text{Ric}_{\tilde{M}} \geq 0$, and $\pi_1(M)$ acts isometrically on $\tilde{M}$. Suppose $\Gamma = \langle g_1, \ldots, g_k \rangle$ be a finitely generated subgroup of $\pi_1(M)$. Pick $p \in M$.

Let $\ell = \max_i d(\tilde{g}_i\tilde{p}, \tilde{p})$. Then if $g \in \pi_1(M)$ has $|g| \leq s$, $d(g\tilde{p}, \tilde{p}) \leq s\ell$.

On the other hand, for any cover there exists $\varepsilon > 0$ such that $B(g\tilde{p}, \varepsilon)$ are pairwise disjoint for all $g \in \pi_1(M)$. Note that $gB(\tilde{p}, \varepsilon) = B(g\tilde{p}, \varepsilon)$.

Now

$$\bigcup_{|g| \leq s} B(g\tilde{p}, \varepsilon) \subset B(\tilde{p}, s\ell + \varepsilon);$$

since the $B(g\tilde{p}, \varepsilon)$’s are disjoint and have the same volume,

$$\Gamma(s) \text{vol}B(\tilde{p}, \varepsilon) \leq \text{vol}B(\tilde{p}, s\ell + \varepsilon).$$

Thus

$$\Gamma(s) \leq \frac{\text{vol}B(\tilde{p}, s\ell + \varepsilon)}{\text{vol}B(\tilde{p}, \varepsilon)},$$

$$\leq \frac{\text{vol}B_{\mathbb{R}^n}(0, s\ell + \varepsilon)}{\text{vol}B_{\mathbb{R}^n}(0, \varepsilon)}.$$

Now

$$\text{vol}(B_{\mathbb{R}^n}(0, s)) = \int_{S^{n-1}} \int_0^s r^{n-1}drd\theta_{n-1}$$

$$= \frac{1}{n} \int_{S^{n-1}} d\theta_{n-1}$$

$$= s^n \omega_n,$$

so $\Gamma(s) \leq \frac{s\ell + \varepsilon^n}{\varepsilon^n}$.

Since $\ell$ and $\varepsilon$ are fixed, we may choose $a$ so that $\Gamma(s) \leq as^n$. 

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Example 2.2.6 Let $H$ be the Heisenberg group
\[
\left\{ \left( \begin{array}{ccc} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{array} \right) : x, y, z \in \mathbb{R} \right\},
\]
and let
\[
H_{\mathbb{Z}} = \left\{ \left( \begin{array}{ccc} 1 & n_1 & n_2 \\ 0 & 1 & n_3 \\ 0 & 0 & 1 \end{array} \right) : n_i \in \mathbb{Z} \right\}.
\]

Then $H/H_{\mathbb{Z}}$ is a compact 3-manifold with $\pi_1(H/H_{\mathbb{Z}}) = H_{\mathbb{Z}}$. The growth of $H_{\mathbb{Z}}$ is polynomial of degree 4, so $H/H_{\mathbb{Z}}$ has no metric with $\text{Ric} \geq 0$.

Remarks:

1. If $\text{Ric}_{M} \geq 1/k^2 > 0$, then $M$ is compact. Thus $\pi_1(M)$ is finitely generated. It is unknown whether $\pi_1(M)$ is finitely generated if $M$ is noncompact.

2. Ricci curvature gives control on $\pi(M)$, while sectional curvature gives control on the higher homology groups. For example, if $K \geq 0$ then the Betti numbers of $M$ are bounded by dimension.

3. If $M$ is compact then growth of $\pi_1(M) \leftrightarrow$ volume growth of $\tilde{M}$.

Related Results:

1. (Gromov) If $\text{Ric}_M \geq 0$ then any finitely generated subgroup of $\pi_1(M)$ is almost nilpotent.

2. (Cheeger-Gromoll, 1972) If $M$ is compact with $\text{Ric} \geq 0$ then $\pi_1(M)$ is abelian up to finite index.

3. (Wei, 1988; Wilking 1999) Any finitely generated almost nilpotent group can be realized as $\pi_1(M)$ for some $M$ with $\text{Ric} \geq 0$.

4. Milnor’s Conjecture (Open) If $M^n$ has $\text{Ric}_M \geq 0$ then $\pi_1(M)$ is finitely generated.
In 1999, Wilking used algebraic methods to show that $\pi_1(M)$ is finitely generated iff any abelian subgroup of $\pi_1(M)$ is finitely generated (provided $\text{Ric}_M \geq 0$).

Sormani showed in 1998 that if $M^n$ has small linear diameter growth, i.e. if
\[
\limsup_{r \to \infty} \frac{\text{diam}\partial B(p, r)}{r} < s_n = \frac{n}{(n-1)^3} \left( \frac{n-1}{n-2} \right)^{n-1},
\]
then $\pi_1(M)$ is finitely generated.

### 2.2.1 Basic Properties of Covering Space

Suppose $\hat{M} \to M$ has the covering metric.

1. $M$ compact $\Rightarrow$ there is a compact set $K \subset \hat{M}$ such that $\{\gamma K \}_{\gamma \in \pi_1(M)}$ covers $\hat{M}$. $K$ is the closure of a fundamental domain.

2. $\{\gamma K \}_{\gamma \in \pi_1(M)}$ is locally finite.

**Definition 2.2.3** Suppose $\delta > 0$. Set $S = \{ \gamma : d(K, \gamma K) \leq \delta \}$. Note that $S$ is finite.

**Lemma 2.2.2** If $\delta > D = \text{diam}_M$ then $S$ generates $\pi_1(M)$. In fact, for any $a \in K$, if $d(a, \gamma K) \leq (\delta - D)s + D$, then $|\gamma| \leq s$.

**Proof.** There exists $y \in \gamma K$ such that $d(a, y) = d(a, \gamma K)$. Connect $a$ and $y$ a minimal geodesic $\sigma$. Divide $\sigma$ by $y_1, \ldots, y_{s+1} = y$, where $d(y_i, y_{i+1}) \leq \delta - D$ and $d(a, y_i) < D$.

Now $\{\gamma K \}_{\gamma \in \pi_1(M)}$ covers $\hat{M}$, so there exist $\gamma_i \in \pi_1(M)$ and $x_i \in K$ such that $\gamma_i(x_i) = y_i$. Choose $\gamma_{s+1} = \gamma$ and $\gamma_1 = \text{Id}$. Then $\gamma = \gamma_1^{-1} \gamma_2 \cdots \gamma_{s+1}^{-1}$. But $\gamma_i^{-1} \gamma_{i+1} \in S$, since
\[
\begin{align*}
d(x_i, \gamma_i^{-1} \gamma_{i+1} x_i) &= d(\gamma_i x_i, \gamma_{i+1} x_i) \\
&= d(y_i, \gamma_{i+1} x_i) \\
&\leq d(y_i, y_{i+1}) + d(y_{i+1}, \gamma_{i+1} x_i) \\
&= d(y_i, y_{i+1}) + d(x_{i+1}, x_i) \\
&\leq \delta.
\end{align*}
\]

Thus $|\gamma| \leq s$.  

Theorem 2.2.3 (Milnor 1968) Suppose $M$ is compact with $K_M < 0$. Then $\pi_1(M)$ has exponential growth.

Note that $K_M \leq -H < 0$ since $M$ is compact. The volume comparison holds for $K \leq -H$, but only for balls inside the injectivity radius. Since $K_M < 0$, though, the injectivity radius is infinite.

Proof of Theorem. By the lemma,

$$\bigcup_{|\gamma| \leq s} \gamma K \supset B(a, (\delta - D)s + D),$$

so $\Gamma(s) \text{vol}(K) \geq \text{vol}B(a, (\delta - d)s + D)$. Note that

$$\text{vol}B(a, (\delta - D)s + D) \geq \text{vol}B^{-H}(a, (\delta - D)s + D),$$

since $K \tilde{M} \leq -H < 0$.

Now

$$\text{vol}B^{-H}(r) = \int_{S^{n-1}} \int_0^r \left( \frac{\sinh \sqrt{H}r}{\sqrt{H}} \right)^{n-1} dr d\theta_{n-1}$$

$$= n\omega_n \int_0^r \left( \frac{\sinh \sqrt{H}r}{\sqrt{H}} \right)^{n-1} dr d\theta_{n-1}$$

$$\geq \frac{n\omega_n}{2(2\sqrt{H})^{n-1}(n - 1)\sqrt{H}} e^{\sqrt{H}r},$$

for $r$ large.

Thus

$$\Gamma(s) \geq \frac{\text{vol}B(a, (\delta - D)s + D)}{\text{vol}(K)} \geq C(n, H)e^{(\delta - D)\sqrt{H}s},$$

where $C(n, H)$ is constant.

Corollary The torus does not admit a metric with negative sectional curvature.

2.3 First Betti Number Estimate

Suppose $M$ is a manifold. The first Betti number of $M$ is

$$b_1(M) = \dim H_1(M, \mathbb{R}).$$
Now $H_1(M, \mathbb{Z}) = \pi_1(M)/[\pi_1(M), \pi_1(M)]$, which is the fundamental group of $M$ made abelian. Let $T$ be the group of torsion elements in $H_1(M, \mathbb{Z})$. Then $T \lhd H_1(M, \mathbb{Z})$ and $\Gamma = H_1(M, \mathbb{Z})/T$ is a free abelian group. Moreover,

$$b_1(M) = \text{rank}(\Gamma) = \text{rank}(\Gamma'),$$

where $\Gamma'$ is any subgroup of $\Gamma$ with finite index.

**Theorem 2.3.1 (Gromov, Gallot)** Suppose $M^n$ is a compact manifold with $\text{Ric}_M \geq (n-1)H$ and $\text{diam}_M \leq D$. There is a function $C(n, HD^2)$ such that $b_1(M) \leq C(n, HD^2)$ and \( \lim_{x \to 0^-} C(n, x) = n \) and $C(n, x) = 0$ for $x > 0$. In particular, if $HD^2$ is small, $b_1(M) \leq n$.

**Proof.** First note that if $M$ is compact and $\text{Ric}_M > 0$ then $\pi_1(M)$ is finite. In this case $b_1(M) = 0$. Also, by Milnor’s result, if $M$ is compact with $\text{Ric}_M \geq 0$ then $b_1(M) \leq n$.

As above, $b_1(M) = \text{rank}(\Gamma)$, where $\Gamma = \pi_1(M)/[\pi_1(M), \pi_1(M)]/T$. Set $\bar{M} = \bar{M}/[\pi_1(M), \pi_1(M)]/T$ be the covering space of $M$ corresponding to $\Gamma$. Then $\Gamma$ acts isometrically as deck transformations on $\bar{M}$.

**Lemma 2.3.1** For fixed $\tilde{x} \in \bar{M}$ there is a subgroup $\Gamma' \leq \Gamma$, $[\Gamma : \Gamma']$ finite, such that $\Gamma' = \langle \gamma_1, \ldots, \gamma_2 \rangle$, where:

1. $d(x, \gamma_i(x)) \leq 2\text{diam}_M$ and
2. For any $\gamma \in \Gamma' - \{e\}$, $d(x, \gamma(x)) > \text{diam}_M$.

**Proof of Lemma.** For each $\varepsilon \geq 0$ let $\Gamma_\varepsilon \leq \Gamma$ be generated by

$$\{ \gamma \in \Gamma : d(x, \gamma(x)) \leq 2\text{diam}_M + \varepsilon \}.$$

Then $\Gamma_\varepsilon$ has finite index. For if $\bar{M}/\Gamma_\varepsilon$ is a covering space corresponding to $\Gamma/\Gamma_\varepsilon$. Then $[\Gamma : \Gamma_\varepsilon]$ is the number of copies of $M$ in $\bar{M}/\Gamma_\varepsilon$. We show that $\text{diam}(\bar{M}/\Gamma_\varepsilon) \leq 2\text{diam}_M + 2\varepsilon$ so that $\bar{M}/\Gamma_\varepsilon$.

Suppose not, so there is $z \in \bar{M}$ such that $d(x, z) = \text{diam}_M + \varepsilon$. Then there is $\gamma \in \Gamma$ that $d(\gamma(x), z) \leq \text{diam}_M$. Then if $\pi_\varepsilon$ is the covering $\bar{M} \to M/\Gamma_\varepsilon$,

\[
\begin{align*}
    d(\pi_\varepsilon x, \pi_\varepsilon \gamma(x)) & \geq d(\pi_\varepsilon x, \pi_\varepsilon z) - d(\pi_\varepsilon z, \pi_\varepsilon \gamma(x)) \\
    & \geq \text{diam}_M + \varepsilon - \text{diam}_M \\
    & = \varepsilon.
\end{align*}
\]
Thus $\gamma \not\in \Gamma_\varepsilon$. But
\[
d(x, \gamma(x)) \leq d(x, z) + d(z, \gamma(x)) \leq 2\text{diam}_M + \varepsilon
\]
Thus $\tilde{M}/\Gamma_\varepsilon$ is compact, so $\Gamma_\varepsilon$ has finite index. Moreover,
\[
\{\gamma \in \Gamma : d(x, \gamma(x)) \leq 3\text{diam}_M\}
\]
is finite, $\Gamma_\varepsilon$ is finitely generated. Also, note that for $\varepsilon$ small,
\[
\{\gamma \in \Gamma : d(x, \gamma(x)) \leq 2\text{diam}_M\} = \{\gamma \in \Gamma : d(x, \gamma(x)) \leq 2\text{diam}_M + \varepsilon\}.
\]
Pick such an $\varepsilon > 0$.
Since $\Gamma_\varepsilon \leq \Gamma$ has finite index, $b_1(M) = \text{rank}(\Gamma_\varepsilon)$. Now $\Gamma_\varepsilon$ is finitely generated, say $\Gamma_\varepsilon = \langle \gamma_1, \ldots, \gamma_{b_1} \rangle$; pick linearly independent generators $\gamma_1, \ldots, \gamma_{b_1}$ so that $\Gamma'' = \langle \gamma_1, \ldots, \gamma_{b_1} \rangle$ has finite index in $\Gamma_\varepsilon$.
Let $\Gamma' = \langle \tilde{\gamma}_1, \ldots, \tilde{\gamma}_{b_1} \rangle$, where $\tilde{\gamma}_k = \ell_{k1}\gamma_1 + \cdots + \ell_{kk}\gamma_k$ and the coefficients $\ell_{ki}$ are chosen so that $\ell_{kk}$ is maximal with respect to the constraints:
1. $\tilde{\gamma}_k \in \Gamma'' \cap \{\gamma \in \Gamma : d(x, \gamma(x)) \leq 2\text{diam}_M\}$ and
2. $\text{span}\{\tilde{\gamma}_1, \ldots, \tilde{\gamma}_k\} \leq \text{span}\{\gamma_1, \ldots, \gamma_k\}$ with finite index.
Then $\Gamma' \leq \Gamma''$ has finite index, and $d(x, \tilde{\gamma}_i(x)) \leq 2\text{diam}_M$ for each $i$.
Finally, suppose there exists $\gamma \in \Gamma' - \{e\}$ with $d(x, \gamma(x)) \leq \text{diam}_M$, write
\[
\gamma = m_1\tilde{\gamma}_1 + \cdots + \tilde{\gamma}_k,
\]
with $m_k \neq 0$. Then $d(x, \gamma^2(x)) \leq 2d(x, \gamma(x)) \leq 2\text{diam}_M$, but
\[
\gamma^2 = 2m_1\tilde{\gamma}_1 + \cdots + 2m_k\tilde{\gamma}_k = (\text{terms involving } \gamma_i, i < k) + 2m_k\ell_{kk}\gamma_k,
\]
which contradicts the choice of the coefficients $\ell_{ki}$.

**Proof of Theorem.** Let $\Gamma' = \langle \gamma_1, \ldots, \gamma_{b_1} \rangle$ be as in the lemma. Then $d(\gamma_i(x), \gamma_j(x)) = d(x, \gamma_i^{-1}\gamma_j(x)) > D = \text{diam}_M$, where $i \neq j$. Thus
\[
B(\gamma_i(x), D/2) \cap B(\gamma_j(x), D/2) = \emptyset
\]
for $i \neq j$. Also
\[
B(\gamma_i(x), D/2) \subset B(x, 2D + D/2)
\]
for all $i$, so that
\[
\bigcup_{i=1}^{b_1} B(\gamma_i(x), D/2) \subset B(x, 2D + D/2).
\]

Hence
\[
b_1 \leq \frac{\text{vol}B(x, 2D + D/2)}{\text{vol}B(x, D/2)} \leq \frac{\text{vol}B^H(2D + D/2)}{\text{vol}B^H(D/2)}.
\]

Since the result holds for $H \geq 0$, assume $H < 0$. Then
\[
\frac{\text{vol}B^H(2D + D/2)}{\text{vol}B^H(D/2)} = \frac{\int_{S^{n-1}} \int_0^{5D/2} (\sinh \sqrt{-H} t)^{n-1} dt d\theta}{\int_{S^{n-1}} \int_0^{D/2} (\sinh \sqrt{-H} t)^{n-1} dt d\theta} = \frac{\int_0^{5D/2} (\sinh \sqrt{-H} t)^{n-1} dt}{\int_0^{D/2} (\sinh \sqrt{-H} t)^{n-1} dt} = \frac{\int_0^{5D\sqrt{-H}/2} (\sinh r)^{n-1} dr}{\int_0^{D\sqrt{-H}/2} (\sinh r)^{n-1} dr}.
\]

Let $U(s) = \{ \gamma \in \Gamma' : |\gamma| \leq s \}$. Then
\[
\bigcup_{\gamma \in U(s)} B(\gamma x, D/2) \subset B(x, 2Ds + D/2),
\]
whence
\[
\#U(s) \leq \frac{\text{vol}B(x, 2Ds + D/2)}{\text{vol}B(x, D/2)} \leq \frac{\text{vol}B^H(2Ds + D/2)}{\text{vol}B^H(D/2)} = \frac{\int_0^{(2s+\frac{1}{2})D\sqrt{-H}} (\sinh r)^{n-1} dr}{\int_0^{D\sqrt{-H}/2} (\sinh r)^{n-1} dr} \leq \frac{2(2s + \frac{1}{2})^n(D\sqrt{-H})^n}{(\frac{1}{2})^n(D\sqrt{-H})^n} = 2^{n+1}(2s + \frac{1}{2})^n.
\]

Thus $b_1(M) = \text{rank}(\Gamma') \leq n$, so that for $HD^2$ small, $b_1(M) \leq n$.

**Conjecture:** For $M^n$ with $\text{Ric}_M \geq (n-1)H$ and $\text{diam}_M \leq D$, the number of generators of $\pi_1(M)$ is uniformly bounded by $C(n, H, D)$. 

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2.4 Finiteness of Fundamental Groups

Lemma 2.4.1 (Gromov, 1980) For any compact $M^n$ and each $\tilde{x} \in \tilde{M}$ there are generators $\gamma_1, \ldots, \gamma_k$ of $\pi_1(M)$ such that $d(\tilde{x}, \gamma_i \tilde{x}) \leq 2\text{diam}_M$ and all relations of $\pi_1(M)$ are of the form $\gamma_i \gamma_j = \gamma_\ell$.

Proof. Let $0 < \varepsilon < \text{injectivity radius}$. Triangulate $M$ so that the length of each adjacent edge is less than $\varepsilon$. Let $x_1, \ldots, x_k$ be the vertices of the triangulation, and let $e_{ij}$ be minimal geodesics connecting $x_i$ and $x_j$.

Connect $x$ to each $x_i$ by a minimal geodesic $\sigma_i$, and set $\sigma_{ij} = \sigma_j^{-1} e_{ij} \sigma_i$.

Then $\ell(\sigma_{ij}) < 2\text{diam}_M + \varepsilon$, so $d(\tilde{x}, \sigma_{ij} \tilde{x}) < 2\text{diam}_M + \varepsilon$.

We claim that $\{\sigma_{ij}\}$ generates $\pi_1(M)$. For any loop at $x$ is homotopic to a 1-skeleton, while $\sigma_{jk} \sigma_{ij} = \sigma_{ik}$ as adjacent vertices span a 2-simplex. In addition, if $1 = \sigma \in \pi_1(M)$, $\sigma$ is trivial in some 2-simplex. Thus $\sigma = 1$ can be expressed as a product of the above relations.

Theorem 2.4.1 (Anderson, 1990) In the class of manifolds $M$ with $\text{Ric}_M \geq (n - 1)H$, $\text{vol}_M \geq V$ and $\text{diam}_M \leq D$ there are only finitely many isomorphism types of $\pi_1(M)$.

Remark: The volume condition is necessary. For example, $S^3/\mathbb{Z}_n$ has $K \equiv 1$ and $\text{diam} = \pi/2$, but $\pi_1(S^3/\mathbb{Z}_n) = \mathbb{Z}_n$. In this case, $\text{vol}(S^3/\mathbb{Z}_n) \to 0$ as $n \to \infty$.

Proof of Theorem. Choose generators for $\pi_1(M)$ as in the lemma; it is sufficient to bound the number of generators.

Let $F$ be a fundamental domain in $\tilde{M}$ that contains $\tilde{x}$. Then

$$\bigcup_{i=1}^{k} \gamma_i(F) \subset B(\tilde{x}, 3D).$$

Also, $\text{vol}(F) = \text{vol}(M)$, so

$$k \leq \frac{\text{vol} B(\tilde{x}, 3D)}{\text{vol} M} \leq \frac{\text{vol} B^H(3D)}{V}.$$

This is a uniform bound depending on $H$, $D$ and $V$.

Theorem 2.4.2 (Anderson, 1990) For the class of manifolds $M$ with $\text{Ric}_M \geq (n - 1)H$, $\text{vol}_M \geq V$ and $\text{diam}_M \leq D$ there are $L = L(n, H, V, D)$ and $N = N(n, H, V, D)$ such that if $\Gamma \subset \pi_1(M)$ is generated by $\{\gamma_i\}$ with each $\ell(\gamma_i) \leq L$ then the order of $\Gamma$ is at most $N$.  

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Proof. Let $\Gamma = \langle \gamma_1, \ldots, \gamma_k \rangle \subset \pi_1(M)$, where each $\ell(\gamma_i) \leq L$. Set

$$U(s) = \{ \gamma \in \Gamma : |\gamma| \leq s \},$$

and let $F \subset \tilde{M}$ be a fundamental domain of $M$. Then $\gamma_i(F) \cap \gamma_j(F)$ has measure zero for $i \neq j$. Now

$$\bigcup_{\gamma \in U(s)} \gamma(F) \subset B(\tilde{x}, sL + D),$$

so

$$\#U(s) \leq \frac{\text{vol}B^H(sL + D)}{V}.$$ 

Note that if $U(s) = U(s + 1)$, then $U(s) = \Gamma$. Also, $U(1) \geq 1$. Thus, if $\Gamma$ has order greater than $N$, then $U(N) \geq N$.

Set $L = D/N$ and $s = N$. Then

$$N \leq U(N) \leq \frac{\text{vol}B^H(2D)}{V}.$$ 

Hence $|\Gamma| \leq N = \frac{\text{vol}B^H(2D)}{V} + 1$, so $\Gamma$ is finite.

3 Laplacian Comparison

3.1 What is the Laplacian?

We restrict our attention to functions, so the Laplacian is a function

$$\Delta : C^\infty(M) \to C^\infty(M).$$

3.1.1 Invariant definition of the Laplacian

Suppose $f \in C^\infty(M)$. The gradient of $f$ is defined by $\langle \nabla f, X \rangle = Xf$. Note that the gradient depends on the metric. We may also define the Hessian of $f$ to be the symmetric bilinear form $\text{Hess} f : \chi(M) \times \chi(M) \to C^\infty(M)$ by

$$\text{Hess} f(X, Y) = \nabla^2_{X,Y} f = X(Yf) - (\nabla_X Y)f = \langle \nabla_X \nabla f, Y \rangle.$$ 

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The Laplacian of $f$ is the trace of Hess $f$, $\Delta f = \text{tr}(\text{Hess } f)$. Note that if $\{e_i\}$ is an orthonormal basis, we have

$$\Delta f = \text{tr} \langle \nabla_X \nabla f, Y \rangle = \sum_{i=1}^n \langle \nabla_{e_i} \nabla f, e_i \rangle = \text{div} \nabla f.$$ 

### 3.1.2 Laplacian in terms of geodesic polar coordinates

Fix $p \in M$ and use geodesic polar coordinates about $p$. For any $x \in M - C_p$, $x \neq p$, connect $p$ to $x$ by a normalized minimal geodesic $\gamma$ so $\gamma(0) = p$ and $\gamma(r) = x$. Set $N = \gamma'(r)$, the outward pointing unit normal of the geodesic sphere. Let $e_2, \ldots, e_n$ be an orthonormal basis tangent to the geodesic sphere, and extend $N, e_2, \ldots, e_n$ to an orthonormal frame in a neighborhood of $x$. Then if $e_1 = N$,

$$\Delta f = \sum_{i=1}^n \langle \nabla_{e_i} \nabla f, e_i \rangle = \sum_{i=1}^n (e_i(e_i f) - (\nabla_{e_i} e_i) f).$$

Note that

$$\nabla_{e_i} e_i = \langle \nabla_{e_i} e_i, N \rangle N + (\nabla_{e_i} e_i)^T = \langle \nabla_{e_i} e_i, N \rangle N + (\bar{\nabla}_{e_i} e_i),$$

where $\bar{\nabla}$ is the induced connection on $\partial B(p, r)$. Thus

$$\Delta f = N(N f) - (\nabla_N N) f + \sum_{i=2}^n (e_i(e_i f) - (\nabla_{e_i} e_i) f)$$

$$= \frac{\partial^2 f}{\partial r^2} + \sum_{i=2}^n (e_i(e_i f) - (\nabla_{e_i} e_i) f) - \left( \sum_{i=2}^n \langle \nabla_{e_i} e_i, N \rangle N \right) f$$

$$= \bar{\Delta} f + m(r, \theta) \frac{\partial}{\partial r} f + \frac{\partial^2 f}{\partial r^2},$$

where $\bar{\Delta}$ is the induced Laplacian on the sphere and $m(r, \theta) = -\sum_{i=2}^n \langle \nabla_{e_i} e_i, N \rangle$ is the mean curvature of the geodesic sphere in the inner normal direction.
3.1.3 Laplacian in local coordinates

Let $\varphi : U \subset M^n \to \mathbb{R}^n$ be a chart, and let $e_i = (\varphi^{-1})_*(\frac{\partial}{\partial x_i})$ be the corresponding coordinate frame on $U$. Then

$$\Delta f = \sum_{k,\ell} \frac{1}{\sqrt{\det g_{ij}}} \partial_k(\sqrt{\det g_{ij}} g^{k\ell} \partial_{\ell} f),$$

where $g_{ij} = \langle e_i, e_j \rangle$ and $(g^{ij}) = (g_{ij})^{-1}$.

Notes:

1. $\Delta f = \frac{\partial^2}{\partial r^2} f + m(r, \theta) \frac{\partial}{\partial r} + \bar{\Delta} f$. Let $m_H(r)$ be the mean curvature in the inner normal direction of $\partial B_H(x, r)$. Then

$$m_H(r) = (n - 1) \begin{cases} \frac{1}{r} & \text{if } H = 0 \\ \sqrt{H} \cot \sqrt{H}r & \text{if } H > 0 \\ \sqrt{-H} \coth \sqrt{-H}r & \text{if } H < 0 \end{cases}.$$

2. We have

$$m(r, \theta) = \frac{A'(r, \theta)}{A(r, \theta)},$$

where $A(r, \theta)drd\theta$ is the volume element.

3. We also have

$$m(r, \theta) = -\sum_{k=0}^n \langle \nabla e_i e_i, N \rangle.$$

In

- $\mathbb{R}^n$, $g = dr^2 + r^2 d\theta^2_{n-1}$
- $S_H^n$, $g = dr^2 + \left(\frac{\sin \sqrt{H}r}{\sqrt{H}}\right)^2 d\theta^2_{n-1}$
- $H^n_H$, $g = dr^2 + \left(\frac{\sinh \sqrt{-H}r}{\sqrt{-H}}\right)^2 d\theta^2_{n-1}$.

By Koszul’s formula,

$$\langle \nabla e_i e_i, N \rangle = -\langle e_i, [e_i, N] \rangle.$$

In Euclidean space, $N = \frac{\partial}{\partial r}, \frac{1}{r} e_i$ are orthonormal. In $S_H^n$,

$$N = \frac{\partial}{\partial r}, \frac{\sqrt{H}}{\sin \sqrt{H}r} e_i$$

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are orthonormal, while

\[ N = \frac{\partial}{\partial r}, \frac{\sqrt{-H}}{\sinh \sqrt{-H} r} e_i \]

are orthonormal in \( \mathbb{H}_H^n \).

### 3.2 Laplacian Comparison

On a Riemannian manifold \( M^n \), the most natural function to consider is the distance function \( r(x) = d(x, p) \) with \( p \in M \) fixed. Then \( r(x) \) is continuous, and is smooth on \( M - \{p\} \cup C_p \). We consider \( \Delta r \) where \( r \) is smooth.

If \( x \in M - \{p\} \cup C_p \), connect \( p \) and \( x \) with a normalized, minimal geodesic \( \gamma \). Then \( \gamma(0) = p, \gamma(r(x)) = x \) and \( \nabla r = \gamma'(r) \). In polar coordinates,

\[ \Delta = \frac{\partial^2}{\partial r^2} + m(r, \theta) \frac{\partial}{\partial r} + \bar{\Delta}. \]

Thus \( \Delta r = m(r, \theta) \).

**Theorem 3.2.1 (Laplacian Comparison, Mean Curvature Comparison)**

Suppose \( M^n \) has \( \text{Ric}_M \geq (n-1)H \). Let \( \Delta_H \) be the Laplacian of \( S^n_H \) and \( m_H(r) \) be the mean curvature of \( \partial B_H(r) \subset M^n_H \). Then:

1. \( \Delta r \leq \Delta_H r \) (Laplacian Comparison)
2. \( m(r, \theta) \leq m_H(r) \) (Mean Curvature Comparison)

**Proof.** We first derive an equation. Let \( N, e_2, \ldots, e_n \) be an orthonormal basis at \( p \), and extend to an orthonormal frame \( N, e_2, \ldots, e_n \) by parallel translation along \( N \). Then \( \nabla_N e_i = 0 \), so \( \langle \nabla_N \nabla_i N, e_i \rangle = N \langle \nabla_i N, e_i \rangle \). Also, \( \nabla_{e_i} \nabla_N N = 0 \). Thus

\[
\text{Ric}(N,N) = \sum_{i=2}^{n} \langle R(e_i, N) N, e_i \rangle
\]

\[
= \sum_{i=2}^{n} \langle \nabla_{N} \nabla_N N - \nabla_N \nabla_{e_i} N - \nabla_{[e_i, N]} N, e_i \rangle
\]

\[
= - \sum_{i=2}^{n} N \langle \nabla_{e_i} N, e_i \rangle - \sum_{i=2}^{n} \langle \nabla_{[e_i, N]} N, e_i \rangle.
\]
Now
\[ \sum_{i=2}^{n} \langle \nabla_{e_i} N, e_i \rangle = \sum_{i=2}^{n} e_i \langle N, e_i \rangle - \langle N, \nabla_{e_i} e_i \rangle \]
\[ = - \sum_{i=2}^{n} \langle N, \nabla_{e_i} e_i \rangle \]
\[ = m(r, \theta), \]
so
\[ \text{Ric}(N, N) = -m'(r, \theta) - \sum_{i=2}^{n} \langle \nabla_{e_i} N, e_i \rangle. \]

In addition,
\[ \nabla_{e_i} N = \sum_{j} \langle \nabla_{e_i} N, e_j \rangle e_j + \langle \nabla_{e_i} N, N \rangle N. \]

But
\[ 2\langle \nabla_{e_i} N, N \rangle = e_i \langle N, N \rangle = 0, \]
so
\[ \nabla_{e_i} N = \sum_{j} \langle \nabla_{e_i} N, e_j \rangle e_j. \]

Thus
\[ \sum_{i=2}^{n} \langle \nabla_{[e_i, N]} N, e_i \rangle = \sum_{i=2}^{n} \sum_{j=2}^{n} \langle \nabla_{e_i} N, E_j \rangle \langle \nabla_{e_j} N, e_i \rangle \]
\[ = \| \text{Hess} (r) \|^2, \]
where \( \| A \|^2 = \text{tr}(AA^t) \). Hence \( \text{Ric}(N, N) = -m'(r, \theta) - \| \text{Hess} (r) \|^2. \)

Now \( \| A \|^2 = \lambda_1^2 + \cdots + \lambda_n^2 \), where the \( \lambda_i \)'s are the eigenvalues of \( A \).

Let \( \lambda_1, \ldots, \lambda_n \) be the eigenvalues of \( \text{Hess} r \); since \( \nabla_{N} N = 0 \) we may assume \( \lambda_1 = 0. \) Then
\[ \| \text{Hess} (r) \|^2 = \lambda_2^2 + \cdots + \lambda_n^2 \]
\[ \geq (\lambda_2 + \cdots + \lambda_n)^2/(n-1), \]
since \( \langle A, I \rangle^2 \leq \| A \|^2 \| I \|^2 \) with \( A \) diagonal and \( \langle A, B \rangle = \text{tr}(AB^t) \). But \( \lambda_2 + \cdots + \lambda_n = m(r, \theta) \), so
\[ \| \text{Hess} (r) \|^2 \geq \frac{m(r, \theta)^2}{(n-1)}. \]
Since $\text{Ric}(N, N) \geq (n - 1)H$, we have

$$(n - 1)H + m(r, \theta)^2/(n - 1) \leq -m'(r, \theta).$$

Set $u = (n - 1)/m(r, \theta)$, so $m(r, \theta) = (n - 1)/u$. Then

$$H + 1/u^2 \leq (1/u^2)u',$$

so

$$Hu^2 + 1 \leq u',$$

which is

$$\frac{u'}{Hu^2 + 1} \geq 1.$$

Thus

$$\int_0^r \frac{u'}{Hu^2 + 1} \geq \int_0^r 1 = r.$$ If $H = 0$, we have $u \geq r$. In this case $(n - 1)/r \geq m(r, \theta)$, so

$$m(r, \theta) \leq m_H(r).$$

If $H > 0$, $(\tan^{-1}(\sqrt{Hu}))/\sqrt{h} \geq r$. Now as $r \to 0$, $m(r, \theta) \to (n - 1)/r$. Thus $u \to 0$ as $r \to 0$. Hence $\sqrt{H}u \geq \tan(\sqrt{H}r)$. Thus $\sqrt{H}(n - 1)/m(r, \theta) \geq \tan(\sqrt{H}r)$, so

$$m(r, \theta) \leq \frac{\sqrt{H}(n - 1)}{\tan(\sqrt{H}r)} = m_H(r).$$

Note that inside the cut locus of $M$, the mean curvature is positive, so the inequality is unchanged when we may multiply by $m(r, \theta)$.

If $H < 0$, similar arguments show that

$$m(r, \theta) \leq \frac{(n - 1)\sqrt{-H}}{\coth(\sqrt{-H}r)} = m_H(r).$$

### 3.3 Maximal Principle

We first define the Laplacian for continuous functions, and then relate the Laplacian to local extrema.

**Lemma 3.3.1** Suppose $f, h \in C^2(M)$ and $p \in M$. Then if
1. \( f(p) = h(p) \)

2. \( f(x) \geq h(x) \) for all \( x \) in some neighborhood of \( p \)

then

1. \( \nabla f(p) = \nabla h(p) \)

2. \( \text{Hess}(f)(p) \geq \text{Hess}(h)(p) \)

3. \( \Delta f(p) \geq \Delta h(p) \).

**Proof.** Suppose \( v \in T_pM \). Pick \( \gamma: (-\varepsilon, \varepsilon) \to M \) so that \( \gamma(0) = p \) and \( \gamma'(0) = v \). Then \( (f - h) \circ \gamma: (-\varepsilon, \varepsilon) \to \mathbb{R} \), so the result follows from the real case.

**Definition 3.3.1** Suppose \( f \in \mathcal{C}^0(M) \). We say that \( \Delta f(p) \geq a \) in the barrier sense if for any \( \varepsilon > 0 \) there exists a function \( f_\varepsilon \), called a support function, such that

1. \( f_\varepsilon \in \mathcal{C}^2(U) \) for some neighborhood \( U \) of \( p \)

2. \( f_\varepsilon(p) = f(p) \) and \( f(x) \geq f_\varepsilon(x) \) for all \( x \in U \)

3. \( \Delta f_\varepsilon(p) \geq a - \varepsilon \).

Note that \( f_\varepsilon \) is also called support from below, or a lower barrier, for \( f \) at \( p \). A similar definition holds for upper barrier.

**Theorem 3.3.1 (Maximal Principle)** If \( f \in \mathcal{C}^0(M) \) and \( \Delta f \geq 0 \) then \( f \) is constant in a neighborhood of each local maximum. In particular, if \( f \) has a global maximum, then \( f \) is constant.

**Proof.** If \( \Delta f > 0 \) then \( f \) cannot have a local maximum. Suppose \( \Delta f \geq 0 \), \( f \) has a local maximum at \( p \), but \( f \) is not constant at \( p \). We perturb \( f \) so that \( \Delta F > 0 \).

Consider the geodesic sphere \( \partial B(p, r) \). For \( r \) sufficiently small, there is \( z \in \partial B(p, r) \) with \( f(z) < f(p) \). We define \( h \) in a neighborhood of \( p \) such that

1. \( \Delta h > 0 \)

2. \( h < 0 \) on \( V = \{ x : f(x) = p \} \cap \partial B(p, r) \)
3. \( h(p) = 0 \)

To this end, set \( h = e^{\alpha \psi} - 1 \). Then

\[
\begin{align*}
\nabla h &= \alpha e^{\alpha \psi} \nabla \psi \\
\Delta h &= \alpha^2 e^{\alpha \psi} (\nabla \psi, \nabla \psi) + \alpha e^{\alpha \psi} \Delta \psi \\
&= \alpha e^{\alpha \phi} (\alpha |\nabla \psi|^2 + \Delta \psi)
\end{align*}
\]

We want \( \psi \) such that

1. \( \psi(p) = 0 \)
2. \( \psi(x) < 0 \) on some neighborhood containing \( V \)
3. \( \nabla \psi \neq 0 \)

Choose coordinates so \( p \mapsto 0 \) and \( z \mapsto (r, 0, \ldots, 0) \). Set

\[
\psi = x_1 - \beta (x_2^2 + \cdots + x_n^2),
\]

where \( \beta \) is chosen large enough that \( \psi < 0 \) on some open set in \( S^{n-1}_r - z \). Then \( \psi \) satisfies the above conditions.

Since \( |\nabla \psi| \geq 1 \) and \( \Delta \psi \) is continuous, we may choose \( \alpha \) large enough that \( \Delta h > 0 \). Now consider \( f_\delta = f + \delta h \) on \( B(p, r) \). For \( \delta \) small,

\[
f_\delta(p) = f(p) > \max_{\partial B(p, r)} f_\delta(x).
\]

Thus, for \( \delta \) small, \( f_\delta \) has a local maximum in the interior of \( B(p, r) \). Call this point \( q \), and set \( N = \Delta h(q) > 0 \). Since \( \Delta f(q) \geq 0 \), there is a lower barrier function for \( f \) at \( q \), say \( g \), with \( \Delta g > -\delta N/2 \). Then

\[
\Delta (g + \delta h)(q) = \Delta g + \delta \Delta h > \delta N/2
\]

and \( g + \delta h \) is a lower barrier function for \( f_\delta \) at \( q \). Thus \( \Delta f_\delta(q) > 0 \), which is a contradiction.

**Theorem 3.3.2 (Regularity)** If \( f \in C^0(M) \) and \( \Delta f \equiv 0 \) in the barrier sense, then \( f \) is \( C^\infty \).

If \( \Delta f \equiv 0 \), \( f \) is called harmonic.
3.4 Splitting Theorem

Definition 3.4.1 A normalized geodesic \( \gamma : [0, \infty) \to M \) is called a ray if \( d(\gamma(0), \gamma(t)) = t \) for all \( t \). A normalized geodesic \( \gamma : (-\infty, \infty) \) is called a line if \( d(\gamma(t), \gamma(s)) = s - t \) for all \( s \geq t \).

Definition 3.4.2 \( M \) is called connected at infinity if for all \( K \subset M \), \( K \) compact, there is a compact \( \tilde{K} \supset K \) such that every two points in \( M - \tilde{K} \) can be connected in \( M - K \).

Lemma 3.4.1 If \( M \) is noncompact then for each \( p \in M \) there is a ray \( \gamma \) with \( \gamma(0) = p \).

If \( M \) is disconnected at infinity then \( M \) has a line.

Example 3.4.1 A Paraboloid has rays but no lines.

Example 3.4.2 \( \mathbb{R}^2 \) has lines.

Example 3.4.3 A cylinder has lines.

Example 3.4.4 A surface of revolution has lines.

The theorem that we seek to prove is:

Theorem 3.4.1 (Splitting Theorem: Cheeger Gromoll 1971) Suppose that \( M^n \) is noncompact, \( \text{Ric}_M \geq 0 \), and \( M \) contains a line. Then \( M \) is isometric to \( N \times \mathbb{R} \) with the product metric, where \( N \) is a smooth \((n - 1)\)-manifold with \( \text{Ric}_N \geq 0 \). Thus, if \( N \) contains a line we may apply the result to \( N \).

To prove this theorem, we introduce Busemann functions.

Definition 3.4.3 If \( \gamma : [0, \infty) \to M \) is a ray, set \( b^\gamma_t(x) = t - d(x, \gamma(t)) \).

Lemma 3.4.2 We have

1. \( |b^\gamma_t(x)| \leq d(x, \gamma(0)) \).
2. For \( x \) fixed, \( b^\gamma_t(x) \) is nondecreasing in \( t \).
3. \( b^\gamma_t(x) - b^\gamma_t(y) \leq d(x, y) \).
Proof. (1) and (3) are the triangle inequality. For (2), suppose \( s < t \).

Then
\[
b^\gamma_s(x) - b^\gamma_t(x) = (s-t) - d(x, \gamma(s)) + d(x, \gamma(t))
\]
\[
= d(x, \gamma(t)) - d(x, \gamma(s)) - d(\gamma(s), \gamma(t))
\]
\[
\leq 0
\]

Definition 3.4.4 If \( \gamma : [0, \infty) \to M \) is a ray, the Busemann function associated to \( \gamma \) is
\[
b^\gamma(x) = \lim_{t \to \infty} b^\gamma_t(x) = \lim_{t \to \infty} t - d(x, \gamma(t)).
\]

By the above, Busemann functions are well defined and Lipschitz continuous. Intuitively, \( b^\gamma(x) \) is the distance from \( \gamma(\infty) \). Also, since
\[
b^\gamma(\gamma(s)) = \lim_{t \to \infty} t - d(\gamma(s), \gamma(t)) = \lim_{t \to \infty} t - (t-s) = s,
\]
\( b^\gamma(x) \) is linear along \( \gamma(t) \).

Example 3.4.5 In \( \mathbb{R}^n \), the rays are \( \gamma(t) = \gamma(0) + \gamma'(0)t \). In this case, \( b^\gamma(x) = \langle x - \gamma(0), \gamma'(0) \rangle \). The level sets of \( b^\gamma \) are hyperplanes.

Lemma 3.4.3 If \( M \) has \( \text{Ric}_M \geq 0 \) and \( \gamma \) is a ray on \( M \) then \( \Delta(b^\gamma) \geq 0 \) in the barrier sense.

Proof. For each \( p \in M \), we construct a support function of \( b^\gamma \) at \( p \). We first construct asymptotic rays of \( \gamma \) at \( p \).

Pick \( t_i \to \infty \). For each \( i \), connect \( p \) and \( \gamma(t_i) \) by a minimal geodesic \( \sigma_i \). Then \( \{\sigma'_i(0)\} \subset S^{n-1} \), so there is a subsequential limit \( \tilde{\gamma}'(0) \). The geodesic \( \tilde{\gamma} \) is called an asymptotic ray of \( \gamma \) at \( p \); note that \( \tilde{\gamma} \) need not be unique.

We claim that \( b^\tilde{\gamma}(x) + b^\gamma(p) \) is a support function of \( b^\gamma \) at \( p \). For \( b^\tilde{\gamma}(p) = 0 \), so the functions agree at \( p \). In addition, \( \tilde{\gamma} \) is a ray, so \( \tilde{\gamma}(t) \) is not a cut point of \( \tilde{\gamma} \) along \( p \). Hence \( d(\tilde{\gamma}(t), *) \) is smooth at \( p \), so \( b^\tilde{\gamma}(x) \) is smooth in a neighborhood of \( p \).
Now
\[
b^\tilde{\gamma}(x) = \lim_{t \to \infty} t - d(x, \tilde{\gamma}(t)) \\
\leq \lim_{t \to \infty} t - d(x, \gamma(s)) + d(\tilde{\gamma}(t), \gamma(s)) \\
= \lim_{t \to \infty} t + s - d(x, \gamma(s)) - s + d(\tilde{\gamma}(t), \gamma(s)) \\
= \lim_{t \to \infty} t + b^\gamma_s(x) - b^\gamma_s(\tilde{\gamma}(t));
\]
letting \( s \to \infty \), we obtain
\[
b^\tilde{\gamma}(x) \leq \lim_{t \to \infty} t + b^\gamma(x) - b^\gamma(\tilde{\gamma}(t)).
\]
We also have
\[
b^\gamma(p) = \lim_{t_i \to \infty} t_i - d(p, \gamma(t_i)) \\
= \lim_{t_i \to \infty} t_i - d(p, \sigma_i(t)) - d(\gamma(t_i)) \\
= -d(p, \tilde{\gamma}(t)) + \lim_{t_i \to \infty} t_i - d(\sigma_i(t), \gamma(t_i)) \\
= -d(p, \tilde{\gamma}(t)) + b^\gamma(\tilde{\gamma}(t)) \\
= -t + b^\gamma(\tilde{\gamma}(t)).
\]
Thus
\[
b^\tilde{\gamma}(x) + b^\gamma(p) \leq \lim_{t \to \infty} t + b^\gamma(x) - b^\gamma(\tilde{\gamma}(t)) - t + b^\gamma(\tilde{\gamma}(t)) \\
= b^\gamma(x),
\]
so \( b^\tilde{\gamma}(x) + b^\gamma(p) \) is a support function for \( b^\gamma \) at \( p \). By a similar argument, each \( b^\gamma_t(x) + b^\gamma(p) \) is a support function for \( b^\gamma \) at \( p \).

Finally, since \( \text{Ric}_M \geq 0 \),
\[
\Delta(b^\gamma_t(x) + b^\gamma(p)) = \Delta(t - d(x, \tilde{\gamma}(t))) \\
= -\Delta(x, \tilde{\gamma}(t)) \\
\geq -\frac{n-1}{d(x, \tilde{\gamma}(t))},
\]
which tends to 0 as \( t \to \infty \). Thus \( \Delta(b^\gamma) \geq 0 \) in the barrier sense.

The level sets of \( b^\gamma_t \) are geodesic spheres at \( \gamma(t) \). The level sets of \( b^\gamma_t \) are geodesic spheres at \( \gamma(\infty) \).
Lemma 3.4.4 Suppose $\gamma$ is a line in $M$, $\text{Ric}_M \geq 0$. Then $\gamma$ defines two rays, $\gamma^+$ and $\gamma^-$. Let $b^+$ and $b^-$ be the associated Busemann functions. Then:

1. $b^+ + b^- \equiv 0$ on $M$.
2. $b^+$ and $b^-$ are smooth.
3. Given any point $p \in M$ there is a unique line passing through $p$ that is perpendicular to $v_0 = \{x : b^+(x) = 0\}$ and consists of asymptotic rays.

Proof. For

1. Observe that 
\[
    b^+(x) + b^-(x) = \lim_{t \to \infty} (t - d(x, \gamma^+(t))) + \lim_{t \to \infty} (t - d(x, \gamma^-(t))) \\
    = \lim_{t \to \infty} 2t - (d(x, \gamma^+(t)) - d(x, \gamma^-(t))) \\
    \leq 2t - d(\gamma^+(t), \gamma^-(t)) = 0.
\]

Since $b^+(\gamma(0)) + b^- (\gamma(0)) = 0$, $0$ is a global maximum. But 
\[
    \Delta (b^+ + b^-) = \Delta b^+ + \Delta b^- \geq 0,
\]
so $b^+ + b^- \equiv 0$.

2. We have $b^+ = -b^-$. Thus 
\[
    0 \leq \Delta b^+ = -\Delta b^- \leq 0,
\]
so both $b^+$ and $b^-$ are smooth by regularity.

3. At $p$ there are asymptotic rays $\tilde{\gamma}^+$ and $\tilde{\gamma}^-$. We first show that $\tilde{\gamma}^+ + \tilde{\gamma}^-$ is a line. Since 
\[
    d(\tilde{\gamma}^+(s_1), \tilde{\gamma}^-(s_2)) \geq d(\tilde{\gamma}^-(s_2), \gamma^+(t)) - d(\tilde{\gamma}^+(s_1), \gamma^+(t)) \\
    = (t - d(\tilde{\gamma}^+(s_1), \gamma^+(t))) - (t - d(\tilde{\gamma}^-(s_2), \gamma^+(t)))
\]
holds for all $t$, we have 
\[
    d(\tilde{\gamma}^+(s_1), \tilde{\gamma}^-(s_2)) \geq b^+(\tilde{\gamma}^+(s_1)) - b^+(\tilde{\gamma}^-(s_2)) \\
    = b^+(\gamma^+(s_1)) + b^-(\gamma^-(s_2)) \\
    \geq b^+(\tilde{\gamma}^+(s_1)) + b^+(p) + b^-(\tilde{\gamma}^-(s_2)) + b^-(p) \\
    = s_1 + s_2.
\]
Thus $\tilde{\gamma}^+ + \tilde{\gamma}^-$ is a line. But our argument shows that any two asymptotic rays form a line, so the line is unique.

Set $\tilde{v}_{t_0} = (\tilde{b}^+)^{-1}(t_0)$. Then if $y \in \tilde{v}_{t_0}$ we have
\[
d(y, \tilde{\gamma}^+(t)) \geq |\tilde{b}^+(y) - \tilde{b}^+(\tilde{\gamma}^+(t))| = |t_0 - t| = d(\tilde{\gamma}^+(t_0), \tilde{\gamma}^+(t)),
\]
which shows $\tilde{\gamma} \perp \tilde{v}_{t_0}$.

Finally, since $\tilde{b}^+(x) + b^+(p) \leq b^+(x)$,
\[
-(\tilde{b}^+(x) + b^+(p)) \geq -b^+(x).
\]
But $\tilde{b}^+ = -\tilde{b}^-$ and $b^+ = -b^-$, so
\[
\tilde{b}^-(x) + b^-(p) \geq b^-(x).
\]
Since $\tilde{b}^-(x) + b^-(p) \leq b^-(x)$ as well,
\[
\tilde{b}^-(x) + b^-(p) = b^-(x).
\]
Thus the level sets of $b^+$ are the level sets of $\tilde{b}^+$, which proves the result.

Note that $b^+ : M \to \mathbb{R}$ is smooth. Since $b^+$ is linear on $\gamma$ with a Lipschitz constant 1, $\|\nabla b^+\| = 1$. Thus $v_0 = (b^+)^{-1}(0)$ is a smooth (n-1) submanifold of $M$.

**Proof of Splitting Theorem.** Let $\phi : \mathbb{R} \times v_0 \to M$ be given by $(t, p) \mapsto \gamma(t) = \exp_p t\gamma'(0)$, where $\gamma$ is the unique line passing through $p$, perpendicular to $v_0$. By the existence and uniqueness of $\gamma$, $\phi$ is bijective. Since $\exp_p$ is a local dierhomorphism and $\gamma'(0) = (\nabla b^+)(v)$ smooth, $\phi$ is a dierhomorphism.

To show that $\phi$ is an isometry, set $v_t = (b^+)^{-1}(t)$ and let $m(t)$ be the mean curvature of $v_t$. Then $m(t) = \Delta b^+ = 0$. In the proof of the Laplacian comparison, we derived
\[
\text{Ric}(N, N) + \|\text{Hess}(r)\|^2 = m'(r, \theta),
\]
where $N = \nabla \gamma$. Note that $\gamma$ is the integral curve of $\nabla b^+$ passing through $p$, so $\Delta \gamma = m(t)$ and $\nabla b^+ = N$.

In our case, $\text{Ric}(N, N) \geq 0$ and $m'(r, \theta) = 0$, so
\[
\|\text{Hess}(b^+)\| = \|\text{Hess}(r)\| \leq 0.
\]
Thus \( \|\text{Hess}(b^+)\| = 0 \), so that \( \nabla b^+ \) is a parallel vector field.

Now \( \phi \) is an isometry in the \( t \) direction since \( \exp_b \) is a radial isometry. Suppose \( X \) is a vector field on \( v_0 \). Then

\[
R(N,X)N = \nabla_N \nabla_X N - \nabla_X \nabla_N N - \nabla_{[X,N]} N.
\]

But \( \nabla_N N = 0 \), and we may extend \( X \) in the coordinate direction so that \( [X,N] = 0 \). Since

\[
\nabla_X N = \nabla_X \nabla b^+ = 0,
\]

we have \( R(N,X)N = 0 \).

Let \( J(t) = \phi_s(x) = \frac{d}{ds}(\phi(c(s)))\big|_{s=t} \), where \( c : (-\varepsilon, \varepsilon) \to v_0 \) has \( c'(t) = X \). Then \( J(t) \) is a Jacobi field, \( J''(t) = 0 \) and \( J \perp N \). Thus \( J(t) \) is constant. Hence \( \|\phi_s(X)\| = \|X\| \), so \( \phi \) is an isometry.

**Remark:** Since \( \|\text{Hess}(b^+)\| = 0 \), we have \( \nabla_X \nabla b^+ = 0 \) for all vector fields \( X \). By the de Rham decomposition, \( \phi \) is a locally isometric splitting.

**Summary of Proof of Splitting Theorem.**

1. Laplaceian Comparison in Barrier Sense
2. Maximal Principle
3. Bochner Formula: Generalizes \( \text{Ric}(N,N) + \|\text{Hess}(r)\| = m'(r,\theta) \)
4. de Rham Decomposition

Also, the Regularity Theorem was used.

### 3.5 Applications of the Splitting Theorem

**Theorem 3.5.1 (Cheeger-Gromoll 1971)** If \( M^n \) is compact with \( \text{Ric}_M \geq 0 \) then the universal cover \( \tilde{M} \cong N \times \mathbb{R}^k \), where \( N \) is a compact \((n-k)\)-manifold. Thus \( \pi_1(M) \) is almost \( \pi_1(\text{Flat Manifold}) \), i.e.

\[
0 \to F \to \pi_1(M) \to B_k \to 0,
\]

where \( F \) is a finite group and \( B_k \) is the fundamental group of some compact flat manifold.
B_k is called a Bieberbach group.

**Proof.** By the splitting theorem, \( \tilde{M} \simeq N \times \mathbb{R}^k \), where \( N \) has no line. We show \( N \) is compact.

Note that isometries map lines to lines. Thus, if \( \psi \in Iso(\tilde{M}) \), then \( \psi = (\psi_1, \psi_2) \), where \( \psi_1 : N \to N \) and \( \psi_2 : \mathbb{R}^k \to \mathbb{R}^k \) are isometries. Suppose \( N \) is not compact, so \( N \) contains a ray \( \gamma : [0, \infty) \to N \). Let \( F \) be a fundamental domain of \( M \), so \( \bar{F} \) is compact, and let \( p_1 \) be the projection \( \tilde{M} \to N \).

Pick \( t_i \to \infty \). For each \( i \) there is \( g_i \in \pi_1(M) \) such that \( g_i(\gamma(t_i)) \in p_1(F) \). But \( p_1(\bar{F}) \) is compact, so we may assume \( g_i(\gamma(t_i)) \to p \in N \). Set \( \gamma_i(t) = g_i(\gamma(t + t_i)) \). Then \( \gamma_i : [-t_i, \infty) \to N \) is minimal, and \( \{\gamma_i\} \) converges to a line \( \sigma \) in \( N \).

Thus \( N \) is compact. For the second statement, let \( p_2 : \pi_1(M) \to Iso(\mathbb{R}^k) \) be the map \( \psi = (\psi_1, \psi_2) \mapsto \psi_2 \). Then

\[
0 \to \text{Ker}(p_2) \to \pi_1(M) \to \text{Im}(p_2) \to 0
\]

is exact. Now \( \text{Ker}(p_2) = \{(\psi_1, 0)\} \), while \( \text{Im}(\psi_2) = \{(0, \psi_2)\} \). Since \( \text{Ker}(p_2) \) gives a properly discontinuous group action on a compact manifold, \( \text{Ker}(p_2) \) is finite. On the other hand, \( \text{Im}(p_2) \) is an isometry group on \( \mathbb{R}^k \), so \( \text{Im}(p_2) \) is a Bieberbach group.

**Remark:** The curvature condition is only used to obtain the splitting \( \tilde{M} \simeq N \times \mathbb{R}^k \). Thus, if the conclusion of the splitting theorem holds, the curvature condition is unnecessary.

**Corollary** If \( M^n \) is compact with \( \text{Ric}_M \geq 0 \) and \( \text{Ric}_M > 0 \) at one point, then \( \pi_1(M) \) is finite.

**Remark:** This corollary improves the theorem of Bonnet-Myers. The corollary can also be proven using the Bochner technique. In fact, Aubin’s deformation gives another metric that has \( \text{Ric}_M > 0 \) everywhere.

**Corollary** If \( M^n \) has \( \text{Ric}_M \geq 0 \) then \( b_1(M) \leq n \), with equality if and only if \( M^n \cong T^n \), where \( T^n \) is a flat torus.

**Definition 3.5.1** Suppose \( M^n \) is noncompact. Then \( M \) is said to have the geodesic loops to infinity property if for any ray \( \gamma \) in \( M \), any \( g \in \pi_1(M, \gamma(0)) \) and any compact \( K \subset M \) there is a geodesic loop \( c \) at \( \gamma_0 \) in \( M - K \) such that \( g = [c] = [(\gamma|^{\infty}_0)^{-1} \circ c \circ \gamma|^{\infty}_0] \).

**Example 3.5.1** \( M = N \times \mathbb{R} \) If the ray \( \gamma \) is in the splitting direction, then any \( g \in \pi_1(M, \gamma) \) is homotopic to a geodesic loop at infinity along \( \gamma \).
Theorem 3.5.2 (Sormani, 1999) If $M^n$ is complete and noncompact with $\text{Ric}_M > 0$ then $M$ has the geodesic loops to infinity property.

Theorem 3.5.3 (Line Theorem) If $M^n$ does not have the geodesic loops to infinity property then there is a line in $\tilde{M}$.

Application: (Shen-Sormani) If $M^n$ is noncompact with $\text{Ric}_M > 0$ then $H_{n-1}(M, \mathbb{Z}) = 0$.

3.6 Excess Estimate

Definition 3.6.1 Given $p, q \in M$, the excess function associated to $p$ and $q$ is

$$e_{p,q}(x) = d(p, x) + d(q, x) - d(p, q).$$

For fixed $p, q \in M$, write $e(x)$. If $\gamma$ is a minimal geodesic connecting $p$ and $q$ with $\gamma(0) = p$ and $\gamma(1) = q$, let $h(x) = \min_{0 \leq t \leq 1} d(x, \gamma(t))$. Then

$$0 \leq e(x) \leq 2h(x).$$

Let $y$ be the point along $\gamma$ between $p$ and $q$ with $d(x, y) = h(x)$.

Set

$$s_1 = d(p, x), \quad t_1 = d(p, y)$$

$$s_2 = d(q, x), \quad t_2 = d(q, y).$$

We consider triangles $pqx$ for which $h/t_1$ is small; such triangles are called thin.

Example 3.6.1 In $\mathbb{R}^n$,

$$s_1 = \sqrt{h^2 + t_1^2} = t_1 \sqrt{1 + (h/t_1)^2}.$$ 

For a thin triangle, we may use a Taylor expansion to obtain $s_1 \leq t_1(1 + (h/t_1)^2)$. Thus

$$e(x) = s_1 + s_2 - t_1 - t_2$$

$$\leq h^2/t_1 + h^2/t_2$$

$$= h(h/t_1 + h/t_2)$$

$$\leq 2h(h/t),$$

where $t = \min\{t_1, t_2\}$. Thus $e(x)$ is small is $h^2/t$ is small.
If \( M \) has \( K \geq 0 \) then the Toponogov comparison shows that \( s_1 \leq \sqrt{h^2 + t^2_1} \), so the same estimate holds.

**Lemma 3.6.1** \( e(x) \) has the following basic properties:

1. \( e(x) \geq 0 \).
2. \( e|_\gamma = 0 \).
3. \( |e(x) - e(y)| \leq 2d(x, y) \).
4. If \( M \) has \( \text{Ric} \geq 0 \),
   \[
   \Delta(e(x)) \leq (n - 1)(1/s_1 + 1/s_2) \leq (n - 1)(2/s),
   \]
   where \( s = \min\{s_1, s_2\} \).

**Proof.** (1), (2) and (3) are clear. (4) is a consequence of the following Laplacian comparison.

**Lemma 3.6.2** Suppose \( M \) has \( \text{Ric} \geq (n - 1)H \). Set \( r(x) = d(p, x) \), and let \( f : \mathbb{R} \to \mathbb{R} \). Then, in the barrier sense,

1. If \( f' \geq 0 \) then \( \Delta f(r(x)) \leq \Delta_H f|_{r=r(x)} \).
2. If \( f' \leq 0 \) then \( \Delta f(r(x)) \geq \Delta_H f|_{r=r(x)} \).

**Proof.** Recall that \( \Delta = \frac{\partial^2}{\partial r^2} + m(r, \theta) \frac{\partial}{\partial \theta} + \hat{\Delta} \), where \( \hat{\Delta} \) is the Laplacian on the geodesic sphere. Hence
   \[
   \Delta f(r(x)) = f'' + m(r, \theta)f' + \Deltarf',
   \]
so we need only show \( \Delta r \leq \Delta_H r \) in the barrier sense.

We have proved the result where \( r \) is smooth, so need only prove at cut points. Suppose \( q \) is a cut point of \( p \). Let \( \gamma \) be a minimal geodesic with \( \gamma(0) = p \) and \( \gamma(\ell) = q \). We claim that \( d(\gamma(\varepsilon), x) + \varepsilon \) is an upper barrier function of \( r(x) = d(p, x) \) at \( q \), as

1. \( d(\gamma(\varepsilon), x) + \varepsilon \geq d(p, x) \),
2. \( d(\gamma(\varepsilon), q) + \varepsilon = d(p, q) \) and

3. \( d(\gamma(\varepsilon), x) + \varepsilon \) is smooth near \( q \), since \( q \) is not a cut point of \( \gamma(\varepsilon) \) for \( \varepsilon > 0 \).

Since

\[
\Delta(d(\gamma(\varepsilon), x) + \varepsilon) \leq \Delta_H(d(\gamma(\varepsilon), x)) = m_H(d(\gamma(\varepsilon), x)) \leq m_H(d(p, x)) + c\varepsilon = \Delta_H(r(x)) + c\varepsilon,
\]

we have the result.

**Definition 3.6.2** The dilation of a function is

\[
\text{dil}(f) = \min_{x,y} \frac{|f(x) - f(y)|}{d(x,y)}.
\]

By property (2) of \( e(x) \), we have \( \text{dil}(e(x)) \leq 2 \).

**Theorem 3.6.1** Suppose \( U : B(y, R + \eta) \to \mathbb{R} \) is a Lipschitz function on \( M, \text{Ric}_M \geq (n - 1)H \) and

1. \( U \geq 0 \),
2. \( \text{dil}(U) \leq a \),
3. \( u(y_0) = 0 \) for some \( y_0 \in B(y, R) \) and
4. \( \Delta U \leq b \) in the barrier sense.

Then \( U(y) \leq ac + G(c) \) for all \( 0 < c < R \), where \( G(r(x)) \) is the unique function on \( M_H \) such that:

1. \( G(r) > 0 \) for \( 0 < r < R \).
2. \( G'(r) < 0 \) for \( 0 < r < R \).
3. \( G(R) = 0 \).
4. \( \Delta_H G \equiv b \).
Proof. Suppose $H = 0$, $n \geq 3$. We want $\Delta H G = b$. Since $\Delta H = \frac{\partial^2}{\partial r^2} + m_H(r, \theta) \frac{\partial}{\partial r} + \tilde{\Delta}$, we solve

$$G'' + (n - 1)G'/r = b$$

$$G''r^2 + (n - 1)G' = br^2,$$

which is an Euler type O.D.E. The solutions are $G = G_p + G_h$, where $G_p = b/2nr^2$ and $G_h = c_1 + c_2 r^{-(n-2)}$.

Now $G(R) = 0$ gives

$$\frac{b}{2n} R^2 + c_1 + c_2 R^{-(n-2)} = 0,$$

while $G' < 0$ gives

$$\frac{b}{n} r - (n - 2)c_2 r^{-(n-1)} > -0$$

for all $0 < r < R$. Thus $c_2 \geq \frac{b}{n(n-2)} R^n$.

Hence $G(r) = \frac{b}{2n} r^2 + \frac{2}{n-2} r^{-(n-2)} - \frac{n}{n-2} R^2$. Note that $G > 0$ follows from $G(R) = 0$ and $G' < 0$.

For general $H < 0$,

$$G(r) = b \int_r^R \int_r^t \left( \frac{\sinh \sqrt{-H}t}{\sinh \sqrt{-H}s} \right)^{n-1} ds dt.$$

Note that $\Delta H G \geq b$ by the Laplacian comparison.

To complete the proof, fix $0 < c < R$. If $d(y, y_0) \leq c$,

$$U(y) = U(y) - U(y_0) \leq ad(y, y_0) \leq ac \leq ac + G(c).$$

If $d(y, y_0) > c$ then consider $G$ defined on $B(y, R + \varepsilon)$, where $0 < \varepsilon < \eta$. Letting $\varepsilon \to 0$ gives the result.

Consider $V = G - U$. Then $\Delta V = \Delta G - \Delta U \geq 0$, $V|_{\partial B(R+\varepsilon)} \leq 0$ and $V(y_0) > 0$. Now $y_0$ is in the interior of $B(y, R + \varepsilon) - B(y, c)$, so $V(y') > 0$ for some $y' \in \partial B(y, c)$. Since

$$U(y) - U(y') \leq ad(y, y') = ac.$$
and
\[ G(c) - U(y') = V(y') > 0, \]
we have
\[ U(y) \leq ac + U(y') < ac + G(c). \]

We now apply this result to \( e(x) \). Here \( e(x) \geq 0 \), \( a = 2 \) and \( R = h(x) \).
We assume \( s(x) \geq 2h(x) \). On \( B(x, R) \),
\[ \Delta e \leq \frac{4(n-1)}{s(x)}, \]
so \( b = 4(n-1)/s(x) \). Thus
\[ e(x) \leq 2c + G(c) \]
\[ = 2c + \frac{2(n-1)}{ns}(c^2 + frac{2n}{2} - 2h^n e^{-(n-2)} + \frac{n}{n-2}h^2) \]
for all \( 0 < c < h \).

To find the minimal value for \( ar + G(r) \), \( 0 < r < R \), consider
\[ a + G'(r) = a + \frac{b}{2n}(2r - 2R^n r^{1-n}) = 0. \]
This gives \( r(R^n/r^n - 1) = an/b \). To get an estimate, choose \( r \) small. Then \( R^n/r^n \) is large, so \( R^n/r^n \approx an/b \). Hence
\[ r = \left( \frac{R^n b}{an} \right)^{\frac{1}{n-1}} \]
is close to a minimal point.

For the excess function, choose
\[ c = \left( \frac{2h^n}{s} \right)^{\frac{1}{n-1}} \approx \left( \frac{h^n 4(n-1)}{2n} \right)^{\frac{1}{n-1}}. \]
Then
\[ G(c) = \frac{2(n-1)}{ns} \left( \left( \frac{2h^n}{s} \right)^{2/(n-1)} \right) + \frac{2}{n-2}h^n \left( \frac{2h^n}{s} \right)^{\frac{n-2}{n-1}} - \frac{n}{n-2}h^2 \].
Now
\[
\left( \frac{2h^n}{s} \right)^{\frac{2}{n-1}} = h^2 \left( \frac{2h}{s} \right)^{\frac{2}{n-1}},
\]
and
\[
\frac{2h}{s} \leq 1
\]
so
\[
G(c) \leq \frac{2(n-1)}{n} \frac{2}{n-2} \frac{h^n}{s} \left( \frac{2h^n}{s} \right)^{\frac{1}{n-1}} \frac{1}{n-1}
\]
\[
\leq \frac{2(n-1)}{n(n-2)} \left( \frac{2h^n}{s} \right)^{\frac{1}{n-1}}
\]
\[
\leq 2c.
\]
Thus
\[
e(x) \leq 2c + G(c)
\]
\[
= 2c + 2c
\]
\[
= \left( \frac{2h^n}{s} \right)^{\frac{1}{n-1}}
\]
\[
\leq 8 \left( \frac{h^n}{s} \right)^{\frac{1}{n-1}}.
\]

Remarks:

1. A more careful estimate is
\[
e(x) \leq 2 \left( \frac{n-1}{n-2} \right) \left( \frac{c_3 h^n}{2} \right)^{\frac{1}{n-1}} = 8h \left( \frac{h}{s} \right)^{\frac{1}{n-1}},
\]
where \( c_3 = \frac{n-1}{n} \left( \frac{1}{s_1-h} + \frac{1}{s_2-h} \right) \) and \( h < \min(s_1, s_2) \).

2. In general, if \( \text{Ric}_M \geq (n-1)H \) then \( e(x) \leq hF \left( \frac{h}{s} \right) \) for some continuous \( F \) satisfying \( F(0) = 0 \). \( F \) is given by an integral; consider the proof of the estimate in the case \( \text{Ric}_M \geq 0 \).
3.7 Applications of the Excess Estimate

**Theorem 3.7.1 (Sormani, 1998)** Suppose $M^n$ complete and noncompact with $\text{Ric}_M \geq 0$. If, for some $p \in M$,

$$\limsup_{r \to \infty} \frac{\text{diam}(\partial B(p, r))}{r} < 4s_n,$$

where

$$s_n = \frac{1}{2} \frac{1}{3^n} \frac{1}{4^{n-1}},$$

then $\pi_1(M)$ is finitely generated.

Compare this result with:

**Theorem 3.7.2 (Abresch and Gromoll)** If $M$ is noncompact with $\text{Ric}_M \geq 0$, $K \geq -1$ and diameter growth $o(r^{\frac{d}{2}})$, then $M$ has finite topological type.

Note: Diameter growth is the growth of $\text{diam}(\partial B(p, r))$. When $\text{Ric} \geq 0$, $\text{diam}(\partial B(p, r)) \leq r$. To say $M$ has finite topological type is to say that each $H_i(M, \mathbb{Z})$ is finite.

To prove Sormani’s result we choose a desirable set of generators for $\pi_1(M)$.

**Lemma 3.7.1** For $M^n$ complete we may choose a set of generators $g_1, \ldots, g_n, \ldots$ of $\pi_1(M)$ such that:

1. $g_i \in \text{span}\{g_1, \ldots, g_{i-1}\}$.

2. Each $g_i$ can be represented by a minimal geodesic loop $\gamma_i$ based at $p$ such that if $\ell(\gamma_i) = d_i$ then $d(\gamma(0), \gamma(d_i/2)) = d_i/2$, and the lift $\tilde{\gamma}_i$ based at $\tilde{p}$ is a minimal geodesic.

**Proof.** Fix $\tilde{p} \in \tilde{M}$. Let $G = \pi_1(M)$. Choose $g_1 \in G$ such that $d(\tilde{p}, g_1(\tilde{p})) \leq d(\tilde{p}, g(\tilde{p}))$ for all $g \in G - \{e\}$. Note that since $G$ acts discretely on $\tilde{M}$, only finitely many elements of $G$ satisfy a given distance restraint.

Let $G_i = \langle g_1, \ldots, g_{i-1} \rangle$. Choose $g_i \in G - G_i$ such that $d(\tilde{p}, g_i(\tilde{p})) \leq d(\tilde{p}, g(\tilde{p}))$ for all $g \in G - G_i$. If $\pi_1(M)$ is finitely generated, we have a sequence $g_1, \ldots, g_n, \ldots$; otherwise we have a list. The $g_i$’s satisfy (1). Let $\gamma_i$ be the minimal geodesic connecting $\tilde{p}$ to $g_i(\tilde{p})$. Set $\gamma_i = \pi(\tilde{\gamma}_i)$, where is the covering $\pi : \tilde{M} \to M$. We claim that if $\ell(\gamma_i) = d_i$ then $d(\gamma(0), \gamma(d_i/2)) = d_i/2$.
Otherwise, for some $i$ and some $T < d_i/2$, $\gamma_i(T)$ is a cut point of $p$ along $\gamma_i$. Since $M$ and $\tilde{M}$ are locally isometric, and $\tilde{\gamma}_i(T)$ is not conjugate to $\tilde{p}$ along $\tilde{\gamma}_i$, $\gamma_i(T)$ is not conjugate to $p$ along $\gamma$. Hence we can connect $p$ to $\gamma_i(T)$ with a second minimal geodesic $\sigma$. Set

$$h_1 = \sigma^{-1} \circ \gamma_i|_{[0,T]}$$

and

$$h_2 = \gamma_i|_{[T,d_i]} \circ \sigma.$$ 

Now $h_1$ is not a geodesic, so

$$d(\tilde{p}, h_1(\tilde{p})) < 2T < d_i.$$ 

Similarly,

$$d(\tilde{p}, h_2(\tilde{p})) < T + d_i - T = d_i.$$ 

Hence $h_1, h_2 \in G_i$. But then $\gamma_i = h_2 \circ h_1 \in G_i$, which is a contradiction.

**Lemma 3.7.2** Suppose $M^n$ has $\text{Ric} \geq 0$, $n \geq 3$ and $\gamma$ is a geodesic loop based at $p$. Set $D = \ell(\gamma)$. Suppose

1. $\gamma|_{[0,D/2]}$, and $\gamma|_{[D/2,D]}$ are minimal.
2. $\ell(\gamma) \leq \ell(\sigma)$ for all $[\sigma] = [\gamma]$.

Then for $x \in \partial B(p,RD)$, $R \geq 1/2 + s_n$, we have $d(x, \gamma(D/2)) \geq (R - 1/2)D + 2s_nD$.

**Remark:** $\gamma(D/2)$ is a cut point of $p$ along $\gamma$. Since $d(p,x) > D/2$, any minimal geodesic connecting $p$ and $x$ cannot pass through $\gamma(D/2)$. Thus

$$d(\gamma(D/2), x) > d(p, x) - d(p, \gamma(D/2)) = RD - D/2 = (R - 1/2)D.$$ 

The lemma gives a bound on how much larger $d(\gamma(D/2), x)$ is.

**Proof.** It is enough to prove for $R = 1/2 + s_n$. For if $R > 1/2 + s_n$, we may choose $y \in \partial B(p,(1/2 + s_n))$ such that

$$d(x, \gamma(D/2)) = d(x, y) + d(y, \gamma(D/2)).$$
Then
\[ d(x, \gamma(D/2)) \geq d(x, y) + 3s_n D \]
\[ \geq (R - (1/2 + s_n))D + 3s_n D \]
\[ = (R - 1/2)D + s_n D. \]

Suppose there exists \( x \in \partial B(p, (1/2 + s_n)D) \) such that
\[ d(x, \gamma(D/2)) = H < 3s_n D. \]

Let \( c \) be a minimal geodesic connecting \( x \) and \( \gamma(D/2) \). Let \( \tilde{p} \) be a lift of \( p \), and lift \( \gamma \) to \( \tilde{\gamma} \) starting at \( \tilde{p} \). If \( g = [\gamma] \), then \( \tilde{\gamma} \) connects \( \tilde{p} \) and \( g(\tilde{p}) \).

Lift \( c \) to \( \tilde{c} \) starting at \( \tilde{\gamma}(D/2) \), and lift \( c \circ \gamma|_{[0,D/2]} \) to \( \tilde{c} \circ \tilde{\gamma}|_{[0,D/2]} \). Then
\[ d(\tilde{p}, \tilde{x}) \geq d(p, x) = (1/2 + s_n)D, \]
and
\[ d(g(\tilde{p}), \tilde{x}) \geq (1/2 + s_n)D. \]

Thus
\[ e_{\tilde{p}, g(\tilde{p})}(\tilde{x}) = d(\tilde{p}, \tilde{x}) + d(g(\tilde{p}), \tilde{x}) - d(\tilde{p}, g(\tilde{p})) \geq (1/2 + s_n)D + (1/2 + s_n)D - D = 2s_n D. \]

But, by the excess estimate, if \( s \geq 2h \),
\[ e(\tilde{x}) \leq 8 \left( \frac{h^n}{s} \right)^{\frac{1}{n-1}}. \]

In this case, \( h \leq H < 3s_n D \). Also,
\[ s \geq (1/2 + s_n)D > D/2. \]

Since \( s_n < 1/12 \) for \( n \geq 2 \), we have \( s \geq 2h \). Thus
\[ e(\tilde{x}) \leq 8 \left( \frac{(3s_n D)^n}{D/2} \right)^{\frac{1}{n-1}}. \]

But this gives
\[ 2s_n D \leq 8D (2(3s_n)^n)^{\frac{1}{n-1}}, \]

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whence
\[ s_n > \frac{1}{2} \frac{1}{3^n} \frac{1}{4^{n-1}}. \]

We may now prove Sormani’s result.

**Proof of Theorem.** Pick a set of generators \( \{g_k\} \) as in the lemma, where \( g_k \) is represented by \( \gamma_k \). If \( x_k \in \partial B(p, (1/2 + s_n)d_k) \), where \( d_k = \ell(\gamma_k) \to \infty \), we showed that
\[ d(x_k, \gamma(d_k/2)) \geq 3s_nd_k. \]

Let \( y_k \in \partial B(p, d_k/2) \) be the point on a minimal geodesic connecting \( p \) and \( x_k \). Then
\[
\limsup_{r \to \infty} \frac{\text{diam}(\partial B(p, r))}{r} \geq \lim_{k \to \infty} \frac{d(y_k, \gamma_k(d_k/2))}{d_k/2} \geq \lim_{k \to \infty} \frac{2s_nd_k}{d_k/2} = 4s_n,
\]
so we have a contradiction if there are infinitely many generators.

The excess estimate can also be used for compact manifolds.

**Lemma 3.7.3** Suppose \( M^n \) with \( \text{Ric}_M \geq (n - 1) \). Then given \( \delta > 0 \) there is \( \varepsilon(n, \delta) > 0 \) such that if \( d(p, q) \geq \pi - \varepsilon \) then \( e_{p,q}(x) \leq \delta \).

This lemma can be used to prove the following:

**Theorem 3.7.3** There is \( \varepsilon(n, H) \) such that if \( M^n \) has \( \text{Ric}_M \geq (n - 1) \), \( \text{diam}_M \geq \pi - \varepsilon \) and \( K_M \geq H \) then \( M \) is a twisted sphere.

**Proof of Lemma.** Fix \( x \) and set \( e = e_{p,q}(x) \). Then \( B(x, e/2), B(p, d(p, x) - e/2) \) and \( B(q, d(x, q) - e/2) \) are disjoint. Thus
\[
\text{vol}(M) \geq \text{vol}(B(x, e/2)) + \text{vol}(B(p, d(p, x) - e/2)) + \text{vol}(B(q, d(q, x) - e/2))
\geq \text{vol}(M) \left( \frac{\text{vol}(B(x, e/2))}{\text{vol}(B(x, \pi))} + \frac{\text{vol}(B(p, d(p, x) - e/2))}{\text{vol}(B(p, \pi))} + \frac{\text{vol}(B(q, d(q, x) - e/2))}{\text{vol}(B(q, \pi))} \right)
\geq \text{vol}(M) \left( \frac{v(n, 1, e/2) + v(n, 1, d(p, x) - e/2) + v(n, 1, d(q, x) - e/2)}{v(n, 1, \pi)} \right),
\]
where \( v(n, H, r) = \text{vol}(B(r)), B(r) \subset M^n_H \).
Now in $S^n(1)$, $\text{vol}(B(r)) = \int_0^r \sin^{n-1} t \, dt$ is a convex function of $r$. Thus we have

\[
v(n, 1, \pi) \geq v(n, 1, e/2) + v(n, 1, d(p, x) - e/2) + v(n, 1, d(q, x) - e/2)
\]

\[
\geq v(n, 1, e/2) + 2v\left(n, 1, \frac{d(p, x) + d(q, x) - e}{2}\right)
\]

\[
= v(n, 1, e/2) + 2v\left(n, 1, \frac{d(p, q)}{2}\right).
\]

Hence

\[
v(n, 1, e/2) \leq v(n, 1, \pi) - 2v\left(n, 1, \frac{d(p, q)}{2}\right),
\]

which tends to 0 as $\varepsilon \to 0$. Thus $e \to 0$. 

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