Math 241A

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The basic idea of this course is that curvature bounds give information about manifolds, which in turn gives topological results. A typical example is the Bonnet-Myers Theorem. Intuitively,

Bigger curvature $\rightsquigarrow$ Smaller manifold[1].

1 Volume Comparison Theorem

1.1 Volume of Riemannian Manifold

Recall: For $U \subset \mathbb{R}^n$,

$$\mathrm{vol}(U) = \int_U 1\, dv = \int_U 1\, dx_1 \cdots dx_n.$$

Note - $dx_1 \cdots dx_n$ is called the volume density element.

Change of variable formula: Suppose $\psi : V \to U$ is a diffeomorphism, with $U, V \subset \mathbb{R}^n$. Suppose $\psi(x) = y$. Then

$$\int_U dv = \int_U 1\, dy_1 \cdots dy_n = \int_V |Jac(\psi)|\, dx_1 \cdots dx_n.$$

On a Riemannian manifold M^n, let $\psi_\alpha : U_\alpha \to \mathbb{R}^n$ be a chart. Set $E_{ip} = (\psi_\alpha^{-1})_*(\frac{\partial}{\partial x_i})$. In general, the E_{ip}'s are not orthonormal. Let $\{e_k\}$ be an orthonormal basis of T_pM. Then $E_{ip} = \displaystyle\sum_{k=1}^{n} a_{ik} e_k$. The volume of

[1]This quarter we use that bigger curvature $\Rightarrow$ smaller volume

the parallelepiped spanned by $\{E_{ip}\}$ is $|\det(a_{ik})|$. Now $g_{ij} = \sum_{k=1}^{n} a_{ik}a_{kj}$, so $\det(g_{ij}) = \det(a_{ij})^2$. Thus

$$\mathrm{vol}(U_\alpha) = \int_{\psi(U_\alpha)} \sqrt{|\det(g_{ij})|} \circ (\psi_\alpha^{-1})\, dx_1 \cdots dx_n$$

Note - $dv = \sqrt{|\det(g_{ij})|} \circ (\psi_\alpha^{-1})\, dx_1 \cdots dx_n$ is called a volume density element, or volume form, on M.

We have our first result, whose proof is left as an exercise.

Lemma 1.1.1 *Volume is well defined.*

Definition 1.1.1 *Let M be a Riemannian manifold, and let $\{U_\alpha\}$ be a covering of M by domains of coordinate charts. Let $\{f_\alpha\}$ be a partition of unity subordinate to $\{U_\alpha\}$. The volume of M is*

$$\mathrm{vol}(M) = \int_M 1\, dv = \sum_\alpha \int_{\psi(U_\alpha)} f_\alpha\, dv.$$

Lemma 1.1.2 *The volume of a Riemannian manifold is well defined.*

1.2 Computing the volume of a Riemannian manifold

Partitions of unity are not practically effective. Instead we look for charts that cover all but a measure zero set.

Example 1.2.1 *For S^2, use stereographic projection.*

In general, we use the exponential map. We may choose normal coordinates or geodesic polar coordinates. Let $p \in M^n$. Then $\exp_p : T_pM \to M$ is a local diffeomorphism. Let $D_p \subset T_pM$ be the segment disk. Then if C_p is the cut locus of p, $\exp_p : D_p \to M - C_p$ is a diffeomorphism.

Lemma 1.2.1 *C_p has measure zero.*

Hence we may use $\exp_p$ to compute the volume element $dv = \sqrt{\det(g_{ij})}\, dx_1 \cdots dx_n$. Now polar coordinates are not defined at p, but $\{p\}$ has measure zero. We have

$$\exp_p : D_p - \{0\} \overset{diffeo}{\to} M - C_p \cup \{p\}.$$

Set $E_i = (\exp_p)_*(\frac{\partial}{\partial theta_i})$ and $E_n = (\exp_p)_*(\frac{\partial}{\partial r})$. To compute g_{ij}'s, we want E_i and E_n explicitly. Since $\exp_p$ is a radial isometry, $g_{nn} = 1$ and $g_{ni} = 0$ for $1 \leq i < n$. Let $J_i(r, \theta)$ be the Jacobi field with $J_i(0) = 0$ and ${J_i}'(0) = \frac{\partial}{\partial \theta_i}$. Then $E_i(\exp_p(r, \theta) = J_i(r, \theta)$.

If we write J_i and $\frac{\partial}{\partial r}$ in terms of an orthonormal basis $\{e_k\}$, we have $J_i = \sum_{k=1}^{n} a_{ik} e_k$. Thus

$$\sqrt{\det(g_{ij})(r, \theta)} = |\det(a_{ik})| \triangleq ||J_1 \wedge \cdots \wedge J_{n-1} \wedge \frac{\partial}{\partial r}||$$

The volume density, or volume element, of M is

$$dv = ||J_1 \wedge \cdots \wedge J_{n-1} \wedge \frac{\partial}{\partial r}||\, dr d\theta_{n-1} \triangleq \mathcal{A}(r, \theta)\, dr d\theta_{n-1}$$

Example 1.2.2 *$\mathbb{R}^n$ has Jacobi equation $J'' = R(T, J)T$.*

If $J(0) = 0$ and $J'(0) = \frac{\partial}{\partial \theta_i}$ then $J(r) = r \frac{\partial}{\partial \theta_i}$. Thus the volume element is $dv = r^{n-1}\, dr d\theta_{n-1}$.

Example 1.2.3 *S^n has $J_i(r) = \sin(r) \frac{\partial}{\partial \theta_i}$. Hence $dv = \sin^{n-1}(r)\, dr d\theta_{n-1}$.*

Example 1.2.4 *$\mathbb{H}^n$ has $J_i(r) = \sinh(r) \frac{\partial}{\partial \theta_i}$. Hence $dv = \sinh^{n-1}(r)\, dr d\theta_{n-1}$.*

Example 1.2.5 *Volume of unit disk in $\mathbb{R}^n$*

$$\omega_n = \int_{S^{n-1}} \int_0^1 r^{n-1}\, dr d\theta_{n-1} = \frac{1}{n} \int_{S^{n-1}} d\theta_{n-1}$$

Note -

$$\int_{S^{n-1}} d\theta_{n-1} = \frac{2(\pi)^{n/2}}{\Gamma(n/2)}.$$

1.3 Comparison of Volume Elements

Theorem 1.3.1 *Suppose M^n has $\mathrm{Ric}_M \geq (n-1)H$. Let $dv = \mathcal{A}(r,\theta)\, dr d\theta_{n-1}$ be the volume element of M and let $dv_H = \mathcal{A}_H(r,\theta)\, dr d\theta_{n-1}$ be the volume element of the model space (simply connected n-manifold with $K \equiv H$). Then*

$$\frac{\mathcal{A}(r,\theta)}{\mathcal{A}_H(r,\theta)}$$

is a nonincreasing function in r.

Proof[2]**.** We show that

$$\nabla_{\frac{\partial}{\partial r}} \left(\frac{\mathcal{A}(r,\theta)}{\mathcal{A}_H(r,\theta)}\right)^2 \leq 0.$$

Since

$$\mathcal{A}(r,\theta)^2 = \langle J_1 \wedge \cdots \wedge J_{n-1} \wedge \frac{\partial}{\partial r}, J_1 \wedge \cdots \wedge J_{n-1} \wedge \frac{\partial}{\partial r}\rangle,$$

we wish to show that

$$\begin{aligned} \langle J_1 \wedge \cdots \wedge J_{n-1} \wedge \frac{\partial}{\partial r}, J_1 \wedge \cdots \wedge J_{n-1} \wedge \frac{\partial}{\partial r}\rangle' \mathcal{A}_H(r,\theta)^2 - \\ \mathcal{A}(r,\theta)^2 \langle J_1^H \wedge \cdots \wedge J_{n-1}^H \wedge \frac{\partial}{\partial r}, J_1^H \wedge \cdots \wedge J_{n-1}^H \wedge \frac{\partial}{\partial r}\rangle' \quad \leq \quad 0. \end{aligned}$$

Thus we wish to show that

$$2\sum_{i=1}^{n-1} \frac{\langle J_1 \wedge \cdots \wedge J_i{}' \wedge \cdots \wedge J_{n-1} \wedge \frac{\partial}{\partial r}, J_1 \wedge \cdots \wedge J_{n-1} \wedge \frac{\partial}{\partial r}\rangle}{\langle J_1 \wedge \cdots \wedge J_{n-1} \wedge \frac{\partial}{\partial r}, J_1 \wedge \cdots \wedge J_{n-1} \wedge \frac{\partial}{\partial r}\rangle}$$

$$\leq 2\sum_{i=1}^{n-1} \frac{\langle J_1^H \wedge \cdots \wedge (J_i^H)' \wedge \cdots \wedge J_{n-1}^H \wedge \frac{\partial}{\partial r}, J_1^H \wedge \cdots \wedge J_{n-1}^H \wedge \frac{\partial}{\partial r}\rangle}{\langle J_1^H \wedge \cdots \wedge J_{n-1}^H \wedge \frac{\partial}{\partial r}, J_1^H \wedge \cdots \wedge J_{n-1}^H \wedge \frac{\partial}{\partial r}\rangle} \quad (1)$$

At $r = r_0$, let $\bar{J}_i(r_0)$ be orthonormal such that $\bar{J}_n(r_0) = \frac{\partial}{\partial r}|_{r=r_0}$. Then for $1 \leq i < n$,

$$\bar{J}_i(r_0) = \sum_{k=1}^{n-1} b_{ik} J_k(r_0).$$

[2] Compare to the proof of the Rauch Comparison Theorem.

Define $\bar{J}_i(r) = \sum_{k=1}^{n-1} b_{ik} J_k(r)$, where the b_{ik}'s are fixed. Then each $\bar{J}_i$ is a linear combination of Jacobi fields, and hence is a Jacobi field.

The left hand side of (1), evaluated at $r = r_0$ is

$$2 \sum_{i=1}^{n-1} \frac{\langle \bar{J}_1 \wedge \cdots \wedge \bar{J}_i{}' \wedge \cdots \wedge \bar{J}_{n-1} \wedge \frac{\partial}{\partial r}, \bar{J}_1 \wedge \cdots \wedge \bar{J}_{n-1} \wedge \frac{\partial}{\partial r} \rangle}{\langle \bar{J}_1 \wedge \cdots \wedge \bar{J}_{n-1} \wedge \frac{\partial}{\partial r}, \bar{J}_1 \wedge \cdots \wedge \bar{J}_{n-1} \wedge \frac{\partial}{\partial r} \rangle} \Bigg|_{r=r_0}$$

$$= 2 \sum_{i=1}^{n-1} \langle \bar{J}_i{}'(r_0), \bar{J}_i(r_0) \rangle = 2 \sum_{i=1}^{n-1} I(\bar{J}_i, \bar{J}_i),$$

where I is the index form $I(v,v) = \int_0^{r_0} \langle v', v' \rangle + \langle R(T,v)T, v \rangle dt$. Note that for a Jacobi field J,

$$\begin{aligned} I(J,J) &= \int_0^{r_0} \langle J', J' \rangle + \langle R(T,J)T, J \rangle dt \\ &= \int_0^{r_0} \langle J', J \rangle' - \langle J'', J \rangle + \langle R(T,J)T, J \rangle dt \\ &= \langle v', v \rangle |_{r=r_0}. \end{aligned}$$

Let E_i be a parallel field such that $E_i(r_0) = \bar{J}_i(r_0)$, and let $w_i = \frac{\sin \sqrt{H} r}{\sin \sqrt{H} r_0} E_i$. By the Index Lemma, Jacobi fields minimize the index form provided there are no conjugate points. Thus we have $2 \sum_{i=1}^{n-1} I(\bar{J}_i, \bar{J}_i) \leq 2 \sum_{i=1}^{n-1} I(w_i, w_i)$. By the curvature condition, $2 \sum_{i=1}^{n-1} I(w_i, w_i) \leq 2 \sum_{i=1}^{n-1} I(\bar{J}_i^H, \bar{J}_i^H)$, which is the right hand side of (1), evaluated at $r = r_0$.

Thus $\dfrac{\mathcal{A}(r,\theta)}{\mathcal{A}_H(r,\theta)}$ is nonincreasing in r.

Remarks:

1. $\lim_{r \to 0} \dfrac{\mathcal{A}(r,\theta)}{\mathcal{A}_H(r,\theta)} = 1$, so $\mathcal{A}(r,\theta) \leq \mathcal{A}_H(r,\theta)$.
2. (Rigidity) If $\mathcal{A}(r_0,\theta) = \mathcal{A}_H(r_0,\theta)$ for some r_0, then $\mathcal{A}(r,\theta) = \mathcal{A}_H(r,\theta)$ for all $0 \leq r \leq r_0$. But then $B(p, r_0)$ is isometric to $B(r_0) \subset S^n_H$, where S^n_H is the model space. But then the Jacobi fields in M correspond to the Jacobi fields in the model space, so that M is isometric to the model space.

3. We cannot use the Index Lemma to prove an analogous result for $\mathrm{Ric}_M \leq (n-1)H$. In fact, there is no such result. For example, consider Einstein manifolds with $\mathrm{Ric} \equiv (n-1)H$.

4. If $K_M \leq H$, we may use the Rauch Comparison Theorem to prove a similar result inside the injectivity radius.

5. (Lohkamp) $\mathrm{Ric}_M \leq (n-1)H$ has no topological implications. Any smooth manifold M^n, with $n \geq 3$, has a complete Riemannian metric with $\mathrm{Ric}_M \leq 0$.

6. $\mathrm{Ric}_M \leq (n-1)H$ may still have geometric implications. For example, if M is compact with $\mathrm{Ric}_M < 0$ then M has a finite isometry group.

1.4 Volume Comparison Theorem

Theorem 1.4.1 (Bishop-Gromov) *If M^n has $\mathrm{Ric}_M \geq (n-1)H$ then*

$$\frac{\mathrm{vol}(B(p,R))}{\mathrm{vol}(B^H(R))}$$

is nonincreasing in R.

Proof. We have

$$\begin{aligned}\mathrm{vol}B(p,R) &= \int_{B(p,R)} 1 dv \\ &= \int_0^R \int_{S_p(r)} \mathcal{A}(r,\theta) d\theta_{n-1} dr,\end{aligned}$$

where $S_p(r) = \{\theta \in S_p : r\theta \in D_p\}$. Note that $S_p(r_1) \subset S_p(r_2)$ if $r_1 \geq r_2$. The theorem now follows from two lemmas:

Lemma 1.4.1 *If $f(r)/g(r) \geq 0$ is nonincreasing in r, with $g(r) > 0$, then*

$$\frac{\int_0^R f(r)dr}{\int_0^R g(r)dr}$$

is nonincreasing in R.

Proof of Lemma.- The numerator of the derivative is

$$
\begin{aligned}
(\int_0^R g(r)dr)(\int_0^R f(r)dr)' - (\int_0^R f(r)dr)(\int_0^R g(r)dr)' \\
&= f(R)(\int_0^R g(r)dr) - g(R)(\int_0^R f(r)dr) \\
&= g(R)(\int_0^R g(r)dr)\left[\frac{f(R)}{g(R)} - \frac{\int_0^R f(r)dr}{\int_0^R g(r)dr}\right]
\end{aligned}
$$

Now

$$\frac{f(r)}{g(r)} \geq \frac{f(R)}{g(R)} \Rightarrow g(R)f(r) \geq f(R)g(r),$$

so

$$\int_0^R g(R)f(r)dr \geq \int_0^R f(R)g(r)dr.$$

Thus

$$\frac{f(R)}{g(R)} \leq \frac{\int_0^R f(r)dr}{\int_0^R g(r)dr},$$

so the derivative is nonpositive.

Lemma 1.4.2 (Comparison of Lower Area of Geodesic Sphere) *Suppose r lies inside the injectivity radius of the model space S^n_H, so that if $H > 0$, $r < \pi/\sqrt{H}$. Then*

$$\frac{\int_{S_p(r)} \mathcal{A}(r,\theta)\, d\theta_{n-1}}{\int_{S^{n-1}} \mathcal{A}^H(r)\, d\theta_{n-1}}$$

is nonincreasing in r.

Proof of Lemma. In the model space, $\mathcal{A}^H(r,\theta)$ does not depend on θ, so we write $\mathcal{A}^H(r)$. Note that if $r \leq R$,

$$
\begin{aligned}
\frac{\int_{S_p(R)} \mathcal{A}(R,\theta)\, d\theta_{n-1}}{\int_{S^{n-1}} \mathcal{A}^H(R)\, d\theta_{n-1}} &= \frac{1}{\int_{S^{n-1}}\ d\theta_{n-1}} \int_{S_p(R)} \frac{\mathcal{A}(R,\theta)}{\mathcal{A}^H(R)}\, d\theta_{n-1} \\
&\leq \frac{1}{\int_{S^{n-1}} d\theta_{n-1}} \int_{S_p(r)} \frac{\mathcal{A}(r,\theta)}{\mathcal{A}^H(r)}\, d\theta_{n-1} \\
&= \frac{\int_{S_p(r)} \mathcal{A}(r,\theta) d\theta_{n-1}}{\int_{S^{n-1}} \mathcal{A}^H(r)\, d\theta_{n-1}},
\end{aligned}
$$

since $S_p(r) \supset S_p(R)$ and $\frac{\mathcal{A}(r,\theta)}{\mathcal{A}^H(r)}$ is nonincreasing in r. The theorem now follows.

Note that if R is greater than the injectivity radius then $\mathrm{vol}B(p,R)$ decreases. Thus the volume comparison theorem holds for all R.

Corollaries:

1. (Bishop Absolute Volume Comparison) Under the same assumptions, $\mathrm{vol}B(p,r) \leq \mathrm{vol}B^H(r)$.

2. (Relative Volume Comparison) If $r \leq R$ then

$$\frac{\mathrm{vol}B(p,r)}{\mathrm{vol}B(p,R)} \geq \frac{\mathrm{vol}B^H(r)}{\mathrm{vol}B^H(R)}.$$

   If equality holds for some r_0 then equality holds for all $0 \leq r \leq r_0$, and $B(p,r_0)$ is isometric to $B^H(r_0)$.

Proofs:

(1) holds because $\lim_{r\to 0} \dfrac{\mathrm{vol}B(p,r)}{\mathrm{vol}B^H(r)} = 1$.

(2) is a restatement of the the volume comparison theorem.

Sometimes we let $R = 2r$ in (2). Then (2) gives a lower bound on the ratio $\dfrac{\mathrm{vol}B(p,r)}{\mathrm{vol}B(p,R)}$, called the doubling constant. If $\mathrm{vol}(M) \geq V$ then we obtain a lower bound on the volume of small balls.

Generalizations:

1. The same proof shows that the result holds for $\mathrm{vol}^\Gamma B(p,R)$, where $\Gamma \subset S_p = S^{n-1} \subset T_pM$. In particular, the result holds for annuli $(\int_{r_0}^{R_0} \cdots)$ and for cones.

2. Integral Curvature

3. Stronger curvature conditions give submanifold results.

2 Applications of Volume Comparison

2.1 Cheng's Maximal Diameter Rigidity Theorem

Theorem 2.1.1 (Cheng) *Suppose M^n has $\mathrm{Ric}_M \geq (n-1)H > 0$. By the Bonnet-Myers Theorem, $\mathrm{diam}_M \leq \pi/\sqrt{H}$. If $\mathrm{diam}_M = \pi/\sqrt{H}$, Cheng's result states that M is isometric to the sphere S^n_H with radius $1/\sqrt{H}$.*

Proof. (Shiohama) Let $p, q \in M$ have $d(p,q) = \pi/\sqrt{H}$. Then

$$\begin{aligned} \frac{\mathrm{vol}\, B(p, \pi/(2\sqrt{H}))}{\mathrm{vol}\, M} &= \frac{\mathrm{vol}\, B(p, \pi/(2\sqrt{H}))}{\mathrm{vol}\, B(p, \pi/\sqrt{H})} \\ &\geq \frac{\mathrm{vol}\, B_H(\pi(/2\sqrt{H}))}{\mathrm{vol}\, B_H(\pi/\sqrt{H})} = 1/2 \end{aligned}$$

Thus $\mathrm{vol}\, B(p, \pi/2\sqrt{H}) \geq (\mathrm{vol}\, M)/2$. Similarly for q. Hence $\mathrm{vol}\, B(p, \pi/(2\sqrt{H})) = (\mathrm{vol}\, M)/2$, so we have equality in the volume comparison. By rigidity, $B(p, \pi/(2\sqrt{H}))$ is isometric to the upper hemisphere of S^n_H. Similarly for $B(q, \pi/2\sqrt{H})$, so $\mathrm{vol}\, M = \mathrm{vol}, S^n_H$.

Question: What about perturbation? Suppose $\mathrm{Ric}_M \geq (n-1)H$ and $\mathrm{diam}_M \geq \pi/\sqrt{H} - \varepsilon$. In general there is no result for $\varepsilon > 0$. There are spaces not homeomorphic to S^n, provided $n \geq 4$, with $\mathrm{Ric} \geq (n-1)H$ and $\mathrm{diam} \geq \pi/\sqrt{H} - \varepsilon$. Still, if $\mathrm{Ric} \geq (n-1)H$ and $\mathrm{vol}\, M \geq \mathrm{vol}\, S^n_H - \varepsilon(n, H)$ then $M^n \overset{diffeo}{\simeq} S^n_H$.

2.2 Growth of Fundamental Group

Suppose Γ is a finitely generated group, say $\Gamma = \langle g_1, \ldots, g_k \rangle$. Any $g \in \Gamma$ can be written as a word $g = \prod_i g_{k_i}^{n_i}$, where $k_i \in \{1, \ldots, k\}$. Define the length of this word to be $\sum_i |n_i|$, and let $|g|$ be the mininimum of the lengths of all word representations of g. Note that $|\cdot|$ depends on the choice of generators.

Fix a set of generators for Γ. The growth function of Γ is

$$\Gamma(s) = \#\{g \in \Gamma : |g| \leq s\}.$$

Example 2.2.1 *If Γ is a finite group then $\Gamma(s) \leq |\Gamma|$.*

Example 2.2.2 $\Gamma = \mathbb{Z} \oplus \mathbb{Z}$. *Then* $\Gamma = \langle g_1, g_2 \rangle$, *where* $g_1 = (1,0)$ *and* $g_2 = (0,1)$. *Any* $g \in \Gamma$ *can be written as* $g = s_1 g_1 + s_2 g_2$. *To find* $\Gamma(s)$, *we want* $|s_1| + |s_2| \leq s$.

$$\begin{aligned}
\Gamma(s) &= 2s + 1 + \sum_{t=1}^{s} 2(2(s-t)+1) \\
&= 2s + 1 + \sum_{t=1}^{s} (4s - 4t + 2) \\
&= 2s + 1 + 4s^2 + 2s - 4\sum_{t=1}^{2} t \\
&= 4s^2 + 4s + 1 - 4(s(s+1)/2) \\
&= 4s^2 + 4s + 1 - 2(s^2 + s) \\
&= 2s^2 + 2s + 1
\end{aligned}$$

In this case we say Γ *has polynomial growth.*

Example 2.2.3 Γ *free abelian on* k *generators. Then* $\Gamma(s) = \sum_{i=0}^{k} \binom{k}{i}\binom{s}{i}$. Γ *has polynomial growth of degree* k.

Definition 2.2.1 Γ *is said to have polynomial growth of degree* $\leq n$ *if for each set of generators the growth function* $\Gamma(s) \leq as^n$ *for some* $a > 0$.

Γ *is said to have exponential growth if for each set of generators the growth function* $\Gamma(s) \geq e^{as}$ *for some* $a > 0$.

Lemma 2.2.1 *If for some set of generators,* $\Gamma(s) \leq as^n$ *for some* $a > 0$, *then* Γ *has polynomial growth of degree* $\leq n$. *If for some set of generators,* $\Gamma(s) \geq e^{as}$ *for some* $a > 0$, *then* Γ *has exponential growth.*

Example 2.2.4 $\mathbb{Z}^k$ *has polynomial growth of degree* k.

Example 2.2.5 $\mathbb{Z} * \mathbb{Z}$ *has exponential growth.*

Note that for each group Γ there always exists $a > 0$ so that $\Gamma(s) \leq e^{as}$.

Definition 2.2.2 *A group is called almost nilpotent if it has a nilpotent subgroup of finite index.*

Theorem 2.2.1 (Gromov) *A finitely generated group* Γ *has polynomial growth iff* Γ *is almost nilpotent.*

Theorem 2.2.2 (Milnor) *If* M^n *is complete with* $\mathrm{Ric}_M \geq 0$, *then any finitely generated subgroup of* $\pi_1(M)$ *has polynomial growth of degree* $\leq n$.

Proof. Let $\tilde{M}$ have the induced metric. Then $\mathrm{Ric}_{\tilde{M}} \geq 0$, and $\pi_1(M)$ acts isometrically on $\tilde{M}$. Suppose $\Gamma = \langle g_1, \ldots, g_k \rangle$ be a finitely generated subgroup of $\pi_1(M)$. Pick $p \in M$.

Let $\ell = \max_i d(g_i\tilde{p}, \tilde{p})$. Then if $g \in \pi_1(M)$ has $|g| \leq s$, $d(g\tilde{p}, \tilde{p}) \leq s\ell$.

On the other hand, for any cover there exists $\varepsilon > 0$ such that $B(g\tilde{p}, \varepsilon)$ are pairwise disjoint for all $g \in \pi_1(M)$. Note that $gB(\tilde{p}, \varepsilon) = B(g\tilde{p}, \varepsilon)$.

Now

$$\bigcup_{|g|\leq s} B(g\tilde{p}, \varepsilon) \subset B(\tilde{p}, s\ell + \varepsilon);$$

since the $B(g\tilde{p}, \varepsilon)$'s are disjoint and have the same volume,

$$\Gamma(s)\,\mathrm{vol}B(\tilde{p}, \varepsilon) \leq \mathrm{vol}B(\tilde{p}, s\ell + \varepsilon).$$

Thus

$$\begin{aligned} \Gamma(s) &\leq \frac{\mathrm{vol}B(\tilde{p}, s\ell + \varepsilon)}{\mathrm{vol}B(\tilde{p}, \varepsilon)} \\ &\leq \frac{\mathrm{vol}B_{\mathbb{R}^n}(0, s\ell + \varepsilon)}{\mathrm{vol}B_{\mathbb{R}^n}(0, \varepsilon)}. \end{aligned}$$

Now

$$\begin{aligned} \mathrm{vol}(B_{\mathbb{R}^n}(0, s) &= \int_{S^{n-1}} \int_0^s r^{n-1} dr d\theta_{n-1} \\ &= \frac{1}{n} s^n \int_{S^{n-1}} d\theta_{n-1} \\ &= s^n \omega_n, \end{aligned}$$

so $\Gamma(s) \leq \dfrac{s\ell + \varepsilon^n}{\varepsilon^n}$.

Since ℓ and ε are fixed, we may choose a so that $\Gamma(s) \leq as^n$.

Example 2.2.6 *Let H be the Heisenberg group*

$$\left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, x \in \mathbb{R} \right\},$$

and let

$$H_{\mathbb{Z}} = \left\{ \begin{pmatrix} 1 & n_1 & n_2 \\ 0 & 1 & n_3 \\ 0 & 0 & 1 \end{pmatrix} : n_i \in \mathbb{Z} \right\}.$$

Then $H/H_{\mathbb{Z}}$ is a compact 3-manifold with $\pi_1(H/H_{\mathbb{Z}}) = H_{\mathbb{Z}}$. The growth of $H_{\mathbb{Z}}$ is polynomial of degree 4, so $H/H_{\mathbb{Z}}$ has no metric with $\mathrm{Ric} \geq 0$.

Remarks:

1. If $\mathrm{Ric}_M \geq 1/k^2 > 0$, then M is compact. Thus $\pi_1(M)$ is finitely generated. It is unknown whether $\pi_1(M)$ is finitely generated if M is noncompact.

2. Ricci curvature gives control on $\pi(M)$, while sectional curvature gives control on the higher homology groups. For example, if $K \geq 0$ then the Betti numbers of M are bounded by dimension.

3. If M is compact then growth of $\pi_1(M) \leftrightarrow$ volume growth of $\tilde{M}$.

Related Results:

1. (Gromov) If $\mathrm{Ric}_M \geq 0$ then any finitely generated subgroup of $\pi_1(M)$ is almost nilpotent.

2. (Cheeger-Gromoll, 1972) If M is compact with $\mathrm{Ric} \geq 0$ then $\pi_1(M)$ is abelian up to finite index.

3. (Wei, 1988; Wilking 1999) Any finitely generated almost nilpotent group can be realized as $\pi_1(M)$ for some M with $\mathrm{Ric} \geq 0$.

4. Milnor's Conjecture (Open) If M^n has $\mathrm{Ric}_M \geq 0$ then $\pi_1(M)$ is finitely generated.

In 1999, Wilking used algebraic methods to show that $\pi_1(M)$ is finitely generated iff any abelian subgroup of $\pi_1(M)$ is finitely generated (provided $\mathrm{Ric}_M \geq 0$).

Sormani showed in 1998 that if M^n has small linear diameter growth, i.e. if

$$\limsup_{r\to\infty} \frac{\mathrm{diam}\partial B(p,r)}{r} < s_n = \frac{n}{(n-1)3^n}\left(\frac{n-1}{n-2}\right)^{n-1},$$

then $\pi_1(M)$ is finitely generated.

2.2.1 Basic Properties of Covering Space

Suppose $\tilde{M} \to M$ has the covering metric.

1. M compact $\Rightarrow$ there is a compact set $K \subset \tilde{M}$ such that $\{\gamma K\}_{\gamma\in\pi_1(M)}$ covers $\tilde{M}$. K is the closure of a fundamental domain.
2. $\{\gamma K\}_{\gamma\in\pi_1(M)}$ is locally finite.

Definition 2.2.3 *Suppose $\delta > 0$. Set $S = \{\gamma : d(K, \gamma K) \leq \delta\}$. Note that S is finite.*

Lemma 2.2.2 *If $\delta > D = \mathrm{diam}_M$ then S generates $\pi_1(M)$. In fact, for any $a \in K$, if $d(a, \gamma K) \leq (\delta - D)s + D$, then $|\gamma| \leq s$.*

Proof. There exists $y \in \gamma K$ such that $d(a, y) = d(a, \gamma K)$. Connect a and y a minimal geodesic σ. Divide σ by $y_1, \ldots, y_{s+1} = y$, where $d(y_i, y_{i+1}) \leq \delta - D$ and $d(a, y_i) < D$.

Now $\{\gamma K\}_{\gamma\in\pi_1(M)}$ covers $\tilde{M}$, so there exist $\gamma_i \in \pi_1(M)$ and $x_i \in K$ such that $\gamma_i(x_i) = y_i$. Choose $\gamma_{s+1} = \gamma$ and $\gamma_1 = Id$. Then $\gamma = \gamma_1^{-1}\gamma_2 \cdots \gamma_s^{-1}\gamma_{s+1}$. But $\gamma_i^{-1}\gamma_{i+1} \in S$, since

$$\begin{aligned} d(x_i, \gamma_i^{-1}\gamma_{i+1}x_i) &= d(\gamma_i x_i, \gamma_{i+1}x_i) \\ &= d(y_i, \gamma_{i+1}x_i) \\ &\leq d(y_i, y_{i+1}) + d(y_{i+1}, \gamma_{i+1}x_i) \\ &= d(y_i, y_{i+1}) + d(x_{i+1}, x_i) \\ &\leq \delta. \end{aligned}$$

Thus $|\gamma| \leq s$.

Theorem 2.2.3 (Milnor 1968) *Suppose M is compact with $K_M < 0$. Then $\pi_1(M)$ has exponential growth.*

Note that $K_M \leq -H < 0$ since M is compact. The volume comparison holds for $K \leq -H$, but only for balls inside the injectivity radius. Since $K_M < 0$, though, the injectivity radius is infinite.

Proof of Theorem. By the lemma,

$$\bigcup_{|\gamma|\leq s} \gamma K \supset B(a, (\delta - D)s + D),$$

so $\Gamma(s)\mathrm{vol}(K) \geq \mathrm{vol}B(a, (\delta - d)s + D)$. Note that

$$\mathrm{vol}B(a, (\delta - D)s + D) \geq \mathrm{vol}B^{-H}(a, (\delta - D)s + D),$$

since $K_{\tilde{M}} \leq -H < 0$.

Now

$$\begin{aligned}
\mathrm{vol}B^{-H}(r) &= \int_{S^{n-1}} \int_0^r \left(\frac{\sinh \sqrt{H} r}{\sqrt{H}} \right)^{n-1} dr d\theta_{n-1} \\
&= n\omega_n \int_0^r \left(\frac{\sinh \sqrt{H} r}{\sqrt{H}} \right)^{n-1} dr d\theta_{n-1} \\
&\geq \frac{n\omega_n}{2(2\sqrt{H})^{n-1}(n-1)\sqrt{H}} e^{\sqrt{H}r},
\end{aligned}$$

for r large.

Thus

$$\Gamma(s) \geq \frac{\mathrm{vol}B(a, (\delta - D)s + D)}{\mathrm{vol}(K)} \geq C(n, H) e^{(\delta - D)\sqrt{H}s},$$

where $C(n, H)$ is constant.

Corollary The torus does not admit a metric with negative sectional curvature.

2.3 First Betti Number Estimate

Suppose M is a manifold. The first Betti number of M is

$$b_1(M) = \dim H_1(M, \mathbb{R}).$$

Now $H_1(M,\mathbb{Z}) = \pi_1(M)/[\pi_1(M), \pi_1(M)]$, which is the fundamental group of M made abelian. Let T be the group of torsion elements in $H_1(M,\mathbb{Z})$. Then $T \lhd H_1(M,\mathbb{Z})$ and $\Gamma = H_1(M,\mathbb{Z})/T$ is a free abelian group. Moreover,

$$b_1(M) = \text{rank}(\Gamma) = \text{rank}(\Gamma'),$$

where Γ' is any subgroup of Γ with finite index.

Theorem 2.3.1 (Gromov, Gallot) *Suppose M^n is a compact manifold with $\text{Ric}_M \geq (n-1)H$ and $\text{diam}_M \leq D$. There is a function $C(n, HD^2)$ such that $b_1(M) \leq C(n, HD^2)$ and $\lim_{x\to 0^-} C(n,x) = n$ and $C(n,x) = 0$ for $x > 0$. In particular, if HD^2 is small, $b_1(M) \leq n$.*

Proof. First note that if M is compact and $\text{Ric}_M > 0$ then $\pi_1(M)$ is finite. In this case $b_1(M) = 0$. Also, by Milnor's result, if M is compact with $\text{Ric}_M \geq 0$ then $b_1(M) \leq n$.

As above, $b_1(M) = \text{rank}(\Gamma)$, where $\Gamma = \pi_1(M)/[\pi_1(M), \pi_1(M)]/T$. Set $\bar{M} = \tilde{M}/[\pi_1(M), \pi_1(M)]/T$ be the covering space of M corresponding to Γ. Then Γ acts isometrically as deck transformations on $\bar{M}$.

Lemma 2.3.1 *For fixed $\tilde{x} \in \bar{M}$ there is a subgroup $\Gamma' \leq \Gamma$, $[\Gamma : \Gamma']$ finite, such that $\Gamma' = \langle \gamma_1, \ldots, \gamma_2 \rangle$, where:*

1. *$d(x, \gamma_i(x)) \leq 2\text{diam}_M$ and*
2. *For any $\gamma \in \Gamma' - \{e\}, d(x, \gamma(x)) > \text{diam}_M$.*

Proof of Lemma. For each $\varepsilon \geq 0$ let $\Gamma_\varepsilon \leq \Gamma$ be generated by

$$\{\gamma \in \Gamma : d(x, \gamma(x)) \leq 2\text{diam}_M + \varepsilon\}.$$

Then Γ_ε has finite index. For if $\bar{M}/\Gamma_\varepsilon$ is a covering space corresponding to $\Gamma/\Gamma_\varepsilon$. Then $[\Gamma : \Gamma_\varepsilon]$ is the number of copies of M in $\bar{M}/\Gamma_\varepsilon$. We show that $\text{diam}(\bar{M}/\Gamma_\varepsilon) \leq 2\text{diam}_M + 2\varepsilon$ so that $\bar{M}/\Gamma_\varepsilon$.

Suppose not, so there is $z \in \bar{M}$ such that $d(x,z) = \text{diam}_M + \varepsilon$. Then there is $\gamma \in \Gamma$ that $d(\gamma(x), z) \leq \text{diam}_M$. Then if π_ε is the covering $\bar{M} \to \bar{M}/\Gamma_\varepsilon$,

$$\begin{aligned} d(\pi_\varepsilon x, \pi_\varepsilon \gamma(x)) &\geq d(\pi_\varepsilon x, \pi_\varepsilon z) - d(\pi_\varepsilon z, \pi_\varepsilon \gamma(x)) \\ &\geq \text{diam}_M + \varepsilon - \text{diam}_M \\ &= \varepsilon. \end{aligned}$$

Thus $\gamma \notin \Gamma_\varepsilon$. But

$$\begin{aligned} d(x,\gamma(x)) &\leq d(x,z) + d(z,\gamma(x)) \\ &\leq 2\mathrm{diam}_M + \varepsilon \end{aligned}$$

Thus $\bar{M}/\Gamma_\varepsilon$ is compact, so Γ_ε has finite index. Moreover,

$$\{\gamma \in \Gamma : d(x,\gamma(x)) \leq 3\mathrm{diam}_M\}$$

is finite, Γ_ε is finitely generated. Also, note that for ε small,

$$\{\gamma \in \Gamma : d(x,\gamma(x)) \leq 2\mathrm{diam}_M\} = \{\gamma \in \Gamma : d(x,\gamma(x)) \leq 2\mathrm{diam}_M + \varepsilon\}.$$

Pick such an $\varepsilon > 0$.

Since $\Gamma_\varepsilon \leq \Gamma$ has finite index, $b_1(M) = \mathrm{rank}(\Gamma_\varepsilon)$. Now Γ_ε is finitely generated, say $\Gamma_\varepsilon = \langle \gamma_1, \ldots, \gamma_m \rangle$; pick linearly independent generators $\gamma_1, \ldots, \gamma_{b_1}$ so that $\Gamma'' = \langle \gamma_1, \ldots, \gamma_{b_1} \rangle$ has finite index in Γ_ε.

Let $\Gamma' = \langle \tilde{\gamma}_1, \ldots, \tilde{\gamma}_{b_1} \rangle$, where $\tilde{\gamma}_k = \ell_{k1}\gamma_1 + \cdots + \ell_{kk}\gamma_k$ and the coefficients ℓ_{ki} are chosen so that ℓ_{kk} is maximal with respect to the constraints:

1. $\tilde{\gamma}_k \in \Gamma'' \cap \{\gamma \in \Gamma : d(x,\gamma(x)) \leq 2\mathrm{diam}_M\}$ and
2. $\mathrm{span}\{\tilde{\gamma}_1, \ldots, \tilde{\gamma}_k\} \leq \mathrm{span}\{\gamma_1, \ldots, \gamma_k\}$ with finite index.

Then $\Gamma' \leq \Gamma''$ has finite index, and $d(x, \tilde{\gamma}_i(x)) \leq 2\mathrm{diam}_M$ for each i.

Finally, suppose there exists $\gamma \in \Gamma' - \{e\}$ with $d(x,\gamma(x)) \leq \mathrm{diam}_M$, write

$$\gamma = m_1\tilde{\gamma}_1 + \cdots + \tilde{\gamma}_k,$$

with $m_k \neq 0$. Then $d(x,\gamma^2(x)) \leq 2d(x,\gamma(x)) \leq 2\mathrm{diam}_M$, but

$$\begin{aligned} \gamma^2 &= 2m_1\tilde{\gamma}_1 + \cdots + 2m_k\tilde{\gamma}_k \\ &= (\text{terms involving } \gamma_i, i < k) + 2m_k\ell_{kk}\gamma_k, \end{aligned}$$

which contradicts the choice of the coefficients ℓ_{ki}.

Proof of Theorem. Let $\Gamma' = \langle \gamma_1, \ldots, \gamma_{b_1} \rangle$ be as in the lemma. Then $d(\gamma_i(x), \gamma_j(x)) = d(x, \gamma_i^{-1}\gamma_j(x)) > D = \mathrm{diam}_M$, where $i \neq j$. Thus

$$B(\gamma_i(x), D/2) \cap B(\gamma_j(x), D/2) = \emptyset$$

for $i \neq j$. Also

$$B(\gamma_i(x), D/2) \subset B(x, 2D + D/2)$$

for all i, so that

$$\bigcup_{i=1}^{b_1} B(\gamma_i(x), D/2) \subset B(x, 2D + D/2).$$

Hence

$$b_1 \leq \frac{\mathrm{vol}B(x, 2D + D/2)}{\mathrm{vol}B(x, D/2)} \leq \frac{\mathrm{vol}B^H(2D + D/2)}{\mathrm{vol}B^H(D/2)}.$$

Since the result holds for $H \geq 0$, assume $H < 0$. Then

$$\begin{aligned}
\frac{\mathrm{vol}B^H(2D + D/2)}{\mathrm{vol}B^H(D/2)} &= \frac{\int_{S^{n-1}} \int_0^{5D/2} (\frac{\sinh\sqrt{-H}t}{\sqrt{-H}})^{n-1} dt d\theta}{\int_{S^{n-1}} \int_0^{D/2} (\frac{\sinh\sqrt{-H}t}{\sqrt{-H}})^{n-1} dt d\theta} \\
&= \frac{\int_0^{5D/2} (\sinh\sqrt{-H}t)^{n-1} dt}{\int_0^{D/2} (\sinh\sqrt{-H}t)^{n-1} dt} \\
&= \frac{\int_0^{5D\sqrt{-H}/2} (\sinh r)^{n-1} dr}{\int_0^{D\sqrt{-H}/2} (\sinh r)^{n-1} dr}
\end{aligned}$$

Let $U(s) = \{\gamma \in \Gamma' : |\gamma| \leq s\}$. Then

$$\bigcup_{\gamma \in U(s)} B(\gamma x, D/2) \subset B(x, 2Ds + D/2),$$

whence

$$\begin{aligned}
\#U(s) &\leq \frac{\mathrm{vol}B(x, 2Ds + D/2)}{\mathrm{vol}B(x, D/2)} \\
&\leq \frac{\mathrm{vol}B^H(2Ds + D/2)}{\mathrm{vol}B^H(D/2)} \\
&= \frac{\int_0^{(2s+\frac{1}{2})D\sqrt{-H}} (\sinh r)^{n-1} dr}{\int_0^{D\sqrt{-H}/2} (\sinh r)^{n-1} dr} \\
&\leq \frac{2(2s + \frac{1}{2})^n (D\sqrt{-H})^n}{(\frac{1}{2})^n (D\sqrt{-H})^n} \\
&= 2^{n+1}(2s + \frac{1}{2})^n
\end{aligned}$$

Thus $b_1(M) = \mathrm{rank}(\Gamma') \leq n$, so that for HD^2 small, $b_1(M) \leq n$.

Conjecture: For M^n with $\mathrm{Ric}_M \geq (n-1)H$ and $\mathrm{diam}_M \leq D$, the number of generators of $\pi_1(M)$ is uniformly bounded by $C(n, H, D)$.

2.4 Finiteness of Fundamental Groups

Lemma 2.4.1 (Gromov, 1980) *For any compact M^n and each $\tilde{x} \in \tilde{M}$ there are generators $\gamma_1, \ldots, \gamma_k$ of $\pi_1(M)$ such that $d(\tilde{x}, \gamma_i \tilde{x}) \leq 2\mathrm{diam}_M$ and all relations of $\pi_1(M)$ are of the form $\gamma_i \gamma_j = \gamma_\ell$.*

Proof. Let $0 < \varepsilon <$ injectivty radius. Triangulate M so that the length of each adjacent edge is less than ε. Let $x_1, \ldots, x_k$ be the vertices of the triangulation, and let e_{ij} be minimal geodesics connecting x_i and x_j.

Connect x to each x_i by a minimal geodesic σ_i, and set $\sigma_{ij} = \sigma_j^{-1} e_{ij} \sigma_i$. Then $\ell(\sigma_{ij}) < 2\mathrm{diam}_M + \varepsilon$, so $d(\tilde{x}, \sigma_{ij}\tilde{x}) < 2\mathrm{diam}_M + \varepsilon$.

We claim that $\{\sigma_{ij}\}$ generates $\pi_1(M)$. For any loop at x is homotopic to a 1-skeleton, while $\sigma_{jk}\sigma_{ij} = \sigma_{ik}$ as adjacent vertices span a 2-simplex. In addition, if $1 = \sigma \in \pi_1(M)$, σ is trivial in some 2-simplex. Thus $\sigma = 1$ can be expressed as a product of the above relations.

Theorem 2.4.1 (Anderson, 1990)) *In the class of manifolds M with $\mathrm{Ric}_M \geq (n-1)H$, $\mathrm{vol}_M \geq V$ and $\mathrm{diam}_M \leq D$ there are only finitely many isomorphism types of $\pi_1(M)$.*

Remark: The volume condition is necessary. For example, $S^3/\mathbb{Z}_n$ has $K \equiv 1$ and diam $= \pi/2$, but $\pi_1(S^3/\mathbb{Z}_n) = \mathbb{Z}_n$. In this case, $\mathrm{vol}(S^3/\mathbb{Z}_n) \to 0$ as $n \to \infty$.

Proof of Theorem. Choose generators for $\pi_1(M)$ as in the lemma; it is sufficient to bound the number of generators.

Let F be a fundamental domain in $\tilde{M}$ that contains $\tilde{x}$. Then

$$\bigcup_{i=1}^{k} \gamma_i(F) \subset B(\tilde{x}, 3D).$$

Also, $\mathrm{vol}(F) = \mathrm{vol}(M)$, so

$$k \leq \frac{\mathrm{vol}\, B(\tilde{x}, 3D)}{\mathrm{vol} M} \leq \frac{\mathrm{vol} B^H(3D)}{V}.$$

This is a uniform bound depending on H, D and V.

Theorem 2.4.2 (Anderson, 1990) *For the class of manifolds M with $\mathrm{Ric}_M \geq (n-1)H$, $\mathrm{vol}_M \geq V$ and $\mathrm{diam}_M \leq D$ there are $L = L(n, H, V, D)$ and $N = N(n, H, V, D)$ such that if $\Gamma \subset \pi_1(M)$ is generated by $\{\gamma_i\}$ with each $\ell(\gamma_i) \leq L$ then the order of Γ is at most N.*

Proof. Let $\Gamma = \langle \gamma_1, \ldots, \gamma_k \rangle \subset \pi_1(M)$, where each $\ell(\gamma_i) \leq L$. Set

$$U(s) = \{\gamma \in \Gamma : |\gamma| \leq s\},$$

and let $F \subset \tilde{M}$ be a fundamental domain of M. Then $\gamma_i(F) \cap \gamma_j(F)$ has measure zero for $i \neq j$. Now

$$\bigcup_{\gamma \in U(s)} \gamma(F) \subset B(\tilde{x}, sL + D),$$

so

$$\#U(s) \leq \frac{\mathrm{vol} B^H(sL + D)}{V}.$$

Note that if $U(s) = U(s+1)$, then $U(s) = \Gamma$. Also, $U(1) \geq 1$. Thus, if Γ has order greater than N, then $U(N) \geq N$.

Set $L = D/N$ and $s = N$. Then

$$N \leq U(N) \leq \frac{\mathrm{vol} B^H(2D)}{V}.$$

Hence $|\Gamma| \leq N = \dfrac{\mathrm{vol} B^H(2D)}{V} + 1$, so Γ is finite.

3 Laplacian Comparison

3.1 What is the Laplacian?

We restrict our attention to functions, so the Laplacian is a function

$$\Delta : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M).$$

3.1.1 Invariant definition of the Laplacian

Suppose $f \in \mathcal{C}^\infty(M)$. The gradient of f is defined by $\langle \nabla f, X \rangle = Xf$. Note that the gradient depends on the metric. We may also define the Hessian of f to be the symmetric bilinear form $\mathrm{Hess}\, f : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{C}^\infty(M)$ by

$$\mathrm{Hess}\, f(X, Y) = \nabla^2_{X,Y} f = X(Yf) - (\nabla_X Y) f = \langle \nabla_X \nabla f, Y \rangle.$$

The Laplacian of f is the trace of Hess f, $\Delta f = \mathrm{tr}(\mathrm{Hess}\, f)$. Note that if $\{e_i\}$ is an orthonormal basis, we have

$$\begin{aligned}\Delta f &= \mathrm{tr}\,\langle \nabla_X \nabla f, Y\rangle \\ &= \sum_{i=1}^n \langle \nabla_{e_i}\nabla f, e_i\rangle \\ &= \mathrm{div}\,\nabla f.\end{aligned}$$

3.1.2 Laplacian in terms of geodesic polar coordinates

Fix $p \in M$ and use geodesic polar coordinates about p. For any $x \in M - C_p$, $x \neq p$, connect p to x by a normalized minimal geodesic γ so $\gamma(0) = p$ and $\gamma(r) = x$. Set $N = \gamma'(r)$, the outward pointing unit normal of the geodesic sphere. Let $e_2, \ldots, e_n$ be an orthonormal basis tangent to the geodesic sphere, and extend $N, e_2, \ldots, e_n$ to an orthonormal frame in a neighborhood of x. Then if $e_1 = N$,

$$\Delta f = \sum_{i=1}^n \langle \nabla_{e_i}\nabla f, e_i\rangle = \sum_{i=1}^n (e_i(e_i f) - (\nabla_{e_i} e_i) f).$$

Note that

$$\begin{aligned}\nabla_{e_i} e_i &= \langle \nabla_{e_i} e_i, N\rangle N + (\nabla_{e_i} e_i)^T \\ &= \langle \nabla_{e_i} e_i, N\rangle N + (\bar{\nabla}_{e_i} e_i),\end{aligned}$$

where $\bar{\nabla}$ is the induced connection on $\partial B(p, r)$. Thus

$$\begin{aligned}\Delta f &= N(Nf) - (\nabla_N N) f + \sum_{i=2}^n (e_i(e_i f) - (\nabla_{e_i} e_i) f) \\ &= \frac{\partial^2 f}{\partial r^2} + \sum_{i=2}^n (e_i(e_i f) - (\bar{\nabla}_{e_i} e_i) f) - (\sum_{i=2}^n \langle \nabla_{e_i} e_i, N\rangle N) f \\ &= \bar{\Delta} f + m(r,\theta)\frac{\partial}{\partial r} f + \frac{\partial^2 f}{\partial r^2},\end{aligned}$$

where $\bar{\Delta}$ is the induced Laplacian on the sphere and $m(r,\theta) = -\sum_{i=2}^n \langle \nabla_{e_i} e_i, N\rangle$ is the mean curvature of the geodesic sphere in the inner normal direction.

3.1.3 Laplacian in local coordinates

Let $\varphi : U \subset M^n \to \mathbb{R}^n$ be a chart, and let $e_i = (\varphi^{-1})_*(\frac{\partial}{\partial x_i})$ be the corresponding coordinate frame on U. Then

$$\Delta f = \sum_{k,\ell} \frac{1}{\sqrt{\det g_{ij}}} \partial_k(\sqrt{\det g_{ij}}\; g^{k\ell}\, \partial_\ell) f,$$

where $g_{ij} = \langle e_i, e_j \rangle$ and $(g^{ij}) = (g_{ij})^{-1}$.

Notes:

1. $\Delta f = \frac{\partial^2}{\partial r^2} f + m(r,\theta)\frac{\partial}{\partial r} + \bar{\Delta} f$. Let $m_H(r)$ be the mean curvature in the inner normal direction of $\partial B_H(x,r)$. Then

$$m_H(r) = (n-1) \begin{cases} \frac{1}{r} & \text{if } H = 0 \\ \sqrt{H}\cot\sqrt{H}r & \text{if } H > 0 \\ \sqrt{-H}\coth\sqrt{-H}r & \text{if } H < 0 \end{cases}.$$

2. We have

$$m(r,\theta) = \frac{\mathcal{A}'(r,\theta)}{\mathcal{A}(r,\theta)},$$

where $\mathcal{A}(r,\theta) dr d\theta$ is the volume element.

3. We also have

$$m(r,\theta) = -\sum_{k=0}^{n} \langle \nabla_{e_i} e_i, N \rangle.$$

In

$$\begin{array}{ll} \mathbb{R}^n, & g = dr^2 + r^2 d\theta_{n-1}^2 \\ S_H^n, & g = dr^2 + \left(\frac{\sin\sqrt{H}r}{\sqrt{H}}\right)^2 d\theta_{n-1}^2 \\ \mathbb{H}_H^n, & g = dr^2 + \left(\frac{\sinh\sqrt{-H}r}{\sqrt{-H}}\right)^2 d\theta_{n-1}^2. \end{array}$$

By Koszul's formula,

$$\langle \nabla_{e_i} e_i, N \rangle = -\langle e_i, [e_i, N] \rangle.$$

In Euclidean space, $N = \frac{\partial}{\partial r}, \frac{1}{r} e_i$ are orthonormal. In S_H^n,

$$N = \frac{\partial}{\partial r}, \frac{\sqrt{H}}{\sin\sqrt{H}r}\, e_i$$

are orthonormal, while

$$N = \frac{\partial}{\partial r}, \frac{\sqrt{-H}}{\sinh \sqrt{-H} r}\, e_i$$

are orthonormal in $\mathbb{H}^n_H$.

3.2 Laplacian Comparison

On a Riemannian manifold M^n, the most natural function to consider is the distance function $r(x) = d(x,p)$ with $p \in M$ fixed. Then $r(x)$ is continuous, and is smooth on $M - (\{p\} \cup C_p)$. We consider Δr where r is smooth.

If $x \in M - (\{p\} \cup C_p)$, connect p and x with a normalized, minimal geodesic γ. Then $\gamma(0) = p$, $\gamma(r(x)) = x$ and $\nabla r = \gamma'(r)$. In polar coordinates,

$$\Delta = \frac{\partial^2}{\partial r^2} + m(r,\theta)\frac{\partial}{\partial r} + \bar{\Delta}.$$

Thus $\Delta r = m(r,\theta)$.

Theorem 3.2.1 (Laplacian Comparison, Mean Curvature Comparison) *Suppose M^n has $\mathrm{Ric}_M \geq (n-1)H$. Let Δ_H be the Laplacian of S^n_H and $m_H(r)$ be the mean curvature of $\partial B_H(r) \subset M^n_H$. Then:*

1. *$\Delta r \leq \Delta_H r$ (Laplacian Comparison)*
2. *$m(r,\theta) \leq m_H(r)$ (Mean Curvature Comparison)*

Proof. We first derive an equation. Let $N, e_2, \ldots, e_n$ be an orthonormal basis at p, and extend to an orthonormal frame $N, e_2, \ldots, e_n$ by parallel translation along N. Then $\nabla_N e_i = 0$, so $\langle \nabla_N \nabla_{e_i} N, e_i \rangle = N \langle \nabla_{e_i} N, e_i \rangle$. Also, $\nabla_{e_i} \nabla_N N = 0$. Thus

$$\begin{aligned}
\mathrm{Ric}(N,N) &= \sum_{i=2}^{n} \langle R(e_i,N)N, e_i \rangle \\
&= \sum_{i=2}^{n} \langle \nabla_{e_i}\nabla_N N - \nabla_N \nabla_{e_i} N - \nabla_{[e_i,N]} N, e_i \rangle \\
&= -\sum_{i=2}^{n} N \langle \nabla_{e_i} N, e_i \rangle - \sum_{i=2}^{n} \langle \nabla_{[e_i,N]} N, e_i \rangle.
\end{aligned}$$

Now

$$\begin{aligned}\sum_{i=2}^{n}\langle\nabla_{e_i}N,e_i\rangle &= \sum_{i=2}^{n} e_i\langle N,e_i\rangle-\langle N,\nabla_{e_i}e_i\rangle \\ &= -\sum_{i=2}^{n}\langle N,\nabla_{e_i}e_i\rangle \\ &= m(r,\theta),\end{aligned}$$

so

$$\mathrm{Ric}(N,N)=-m'(r,\theta)-\sum_{i=2}^{n}\langle\nabla_{[e_i,N]}N,e_i\rangle.$$

In addition,

$$\nabla_{e_i}N=\sum_j\langle\nabla_{e_i}N,e_j\rangle e_j+\langle\nabla_{e_i}N,N\rangle N.$$

But

$$2\langle\nabla_{e_i}N,N\rangle=e_i\langle N,N\rangle=0,$$

so

$$\nabla_{e_i}N=\sum_j\langle\nabla_{e_i}N,e_j\rangle e_j.$$

Thus

$$\begin{aligned}\sum_{i=2}^{n}\langle\nabla_{[e_i,N]}N,e_i\rangle &= \sum_{i=2}^{n}\sum_{j=2}^{n}\langle\nabla_{e_i}N,E_j\rangle\langle\nabla_{e_j}N,e_i\rangle \\ &= \|\mathrm{Hess}\,(r)\|^2,\end{aligned}$$

where $\|A\|^2=\mathrm{tr}(AA^t)$. Hence $\mathrm{Ric}(N,N)=-m'(r,\theta)-\|\mathrm{Hess}\,(r)\|^2$.

Now $\|A\|^2=\lambda_1^2+\cdots+\lambda_n^2$, where the λ_i's are the eigenvalues of A. Let $\lambda_1,\ldots,\lambda_n$ be the eigenvalues of $\mathrm{Hess}\,r$; since $\nabla_N N=0$ we may assume $\lambda_1=0$. Then

$$\begin{aligned}\|\mathrm{Hess}\,(r)\|^2 &= \lambda_2^2+\cdots+\lambda_n^2 \\ &\geq (\lambda_2+\cdots+\lambda_n)^2/(n-1),\end{aligned}$$

since $\langle A,I\rangle^2\leq\|A\|^2\|I\|^2$ with A diagonal and $\langle A,B\rangle=tr(AB^t)$. But $\lambda_2+\cdots+\lambda_n=m(r,\theta)$, so

$$\|\mathrm{Hess}\,(r)\|^2\geq\frac{m(r,\theta)^2}{(n-1)}.$$

Since $\mathrm{Ric}(N, N) \geq (n-1)H$, we have

$$(n-1)H + m(r,\theta)^2/(n-1) \leq -m'(r,\theta).$$

Set $u = (n-1)/m(r,\theta)$, so $m(r,\theta) = (n-1)/u$. Then

$$H + 1/u^2 \leq (1/u^2)u',$$

so

$$Hu^2 + 1 \leq u',$$

which is

$$\frac{u'}{Hu^2+1} \geq 1.$$

Thus

$$\int_0^r \frac{u'}{Hu^2+1} \geq \int_0^r 1 = r.$$

If $H = 0$, we have $u \geq r$. In this case $(n-1)/r \geq m(r,\theta)$, so

$$m(r,\theta) \leq m_H(r).$$

If $H > 0$, $(\tan^{-1}(\sqrt{H}u))/\sqrt{h} \geq r$. Now as $r \to 0$, $m(r,\theta) \to (n-1)/r$. Thus $u \to 0$ as $r \to 0$. Hence $\sqrt{H}u \geq \tan(\sqrt{h}r)$. Thus $\sqrt{H}(n-1)/m(r,\theta) \geq \tan(\sqrt{H}r)$, so

$$m(r,\theta) \leq \frac{\sqrt{H}(n-1)}{\tan(\sqrt{H}r)} = m_H(r).$$

Note that inside the cut locus of M, the mean curvature is positive, so the inequality is unchanged when we may multiply by $m(r,\theta)$.

If $H < 0$, similar arguments show that

$$m(r,\theta \leq \frac{(n-1)\sqrt{-H}}{\coth(\sqrt{-H}r)} = m_H(r).$$

3.3 Maximal Principle

We first define the Laplacian for continuous functions, and then relate the Laplacian to local extrema.

Lemma 3.3.1 *Suppose $f, h \in \mathcal{C}^2(M)$ and $p \in M$. Then if*

1. $f(p) = h(p)$

2. $f(x) \geq h(x)$ *for all* x *in some neighborhood of* p

then

1. $\nabla f(p) = \nabla h(p)$

2. $\text{Hess}\,(f)(p) \geq \text{Hess}\,(h)(p)$

3. $\Delta f(p) \geq \Delta h(p)$.

Proof. Suppose $v \in T_pM$. Pick $\gamma : (-\varepsilon, \varepsilon) \to M$ so that $\gamma(0) = p$ and $\gamma'(0) = v$. Then $(f - h) \circ \gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}$, so the result follows from the real case.

Definition 3.3.1 *Suppose* $f \in \mathcal{C}^0(M)$. *We say that* $\Delta f(p) \geq a$ *in the barrier sense if for any* $\varepsilon > 0$ *there exists a function* f_ε, *called a support function, such that*

1. $f_\varepsilon \in \mathcal{C}^2(U)$ *for some neighborhood* U *of* p

2. $f_\varepsilon(p) = f(p)$ *and* $f(x) \geq f_\varepsilon(x)$ *for all* $x \in U$

3. $\Delta f_\varepsilon(p) \geq a - \varepsilon$.

Note that f_ε is also called support from below, or a lower barrier, for f at p. A similar definition holds for upper barrier.

Theorem 3.3.1 (Maximal Principle) *If* $f \in \mathcal{C}^0(M)$ *and* $\Delta f \geq 0$ *then* f *is constant is a neighborhood of each local maximum. In particular, if* f *has a global maximum, then* f *is constant.*

Proof. If $\Delta f > 0$ then f cannot have a local maximum. Suppose $\Delta f \geq 0$, f has a local maximum at p, but f is not constant at p. We perturb f so that $\Delta F > 0$.

Consider the geodesic sphere $\partial B(p, r)$. For r sufficiently small, there is $z \in \partial B(p, r)$ with $f(z) < f(p)$. We define h in a neighborhood of p such that

1. $\Delta h > 0$

2. $h < 0$ on $V = \{x : f(x) = p\} \cap \partial B(p, r)$

3. $h(p) = 0$

To this end, set $h = e^{\alpha\psi} - 1$. Then

$$\begin{aligned}\nabla h &= \alpha e^{\alpha\psi}\nabla\psi \\ \Delta h &= \alpha^2 e^{\alpha\psi}\langle\nabla\psi, \nabla\psi\rangle + \alpha e^{\alpha\psi}\Delta\psi \\ &= \alpha e^{\alpha\phi}(\alpha|\nabla\psi|^2 + \Delta\psi\end{aligned}$$

We want ψ such that

1. $\psi(p) = 0$
2. $\psi(x) < 0$ on some neighborhood containing V
3. $\nabla\psi \neq 0$

Choose coordinates so $p \mapsto 0$ and $z \mapsto (r, 0, \ldots, 0)$. Set

$$\psi = x_1 - \beta(x_2^2 + \cdots + x_n^2),$$

where β is chosen large enough that $\psi < 0$ on some open set in $S_r^{n-1} - z$. Then ψ satisfies the above conditions.

Since $|\nabla\psi|^2 \geq 1$ and $\Delta\psi$ is continuous, we may choose α large enough that $\Delta h > 0$. Now consider $f_\delta = f + \delta h$ on $B(p, r)$. For δ small,

$$f_\delta(p) = f(p) > \max_{\partial B(p,r)} f_\delta(x).$$

Thus, for δ small, f_δ has a local maximum in the interior of $B(p, r)$. Call this point q, and set $N = \Delta h(q) > 0$. Since $\Delta f(q) \geq 0$, there is a lower barrier function for f at q, say g, with $\Delta g > -\delta N/2$. Then

$$\Delta(g + \delta h)(q) = \Delta g + \delta\Delta h > \delta N/2$$

and $g + \delta h$ is a lower barrier function for f_δ at q. Thus $\Delta f_\delta(q) > 0$, which is a contradiction.

Theorem 3.3.2 (Regularity) *If $f \in \mathcal{C}^0(M)$ and $\Delta f \equiv 0$ in the barrier sense, then f is $\mathcal{C}^\infty$.*

If $\Delta f \equiv 0$, f is called harmonic.

3.4 Splitting Theorem

Definition 3.4.1 *A normalized geodesic* $\gamma : [0, \infty) \to M$ *is called a ray if* $d(\gamma(0), \gamma(t)) = t$ *for all* t. *A normalized geodesic* $\gamma : (-\infty, \infty)$ *is called a line if* $d(\gamma(t), \gamma(s)) = s - t$ *for all* $s \geq t$.

Definition 3.4.2 M *is called connected at infinity if for all* $K \subset M$, K *compact, there is a compact* $\tilde{K} \supset K$ *such that every two points in* $M - \tilde{K}$ *can be connected in* $M - K$.

Lemma 3.4.1 *If* M *is noncompact then for each* $p \in M$ *there is a ray* γ *with* $\gamma(0) = p$.

If M *is disconnected at infinity then* M *has a line.*

Example 3.4.1 *A Paraboloid has rays but no lines.*

Example 3.4.2 $\mathbb{R}^2$ *has lines.*

Example 3.4.3 *A cylinder has lines.*

Example 3.4.4 *A surface of revolution has lines.*

The theorem that we seek to prove is:

Theorem 3.4.1 (Splitting Theorem: Cheeger Gromoll 1971) *Suppose that* M^n *is noncompact,* $\mathrm{Ric}_M \geq 0$, *and* M *contains a line. Then* M *is isometric to* $N \times \mathbb{R}$, *with the product metric, where* N *is a smooth* $(n-1)$*-manifold with* $\mathrm{Ric}_N \geq 0$. *Thus, if* N *contains a line we may apply the result to* N.

To prove this theorem, we introduce Busemann functions.

Definition 3.4.3 *If* $\gamma : [0, \infty) \to M$ *is a ray, set* $b_t^\gamma(x) = t - d(x, \gamma(t))$.

Lemma 3.4.2 *We have*

1. $|b_t^\gamma(x)| \leq d(x, \gamma(0))$.
2. *For* x *fixed,* $b_t^\gamma(x)$ *is nondecreasing in* t.
3. $b_t^\gamma(x) - b_t^\gamma(y)| \leq d(x, y)$.

Proof. (1) and (3) are the triangle inequality. For (2), suppose $s < t$. Then

$$\begin{aligned} b_s^\gamma(x) - b_t^\gamma(x) &= (s-t) - d(x,\gamma(s)) + d(x,\gamma(t)) \\ &= d(x,\gamma(t)) - d(x,\gamma(s)) - d(\gamma(s),\gamma(t)) \\ &\leq 0 \end{aligned}$$

Definition 3.4.4 *If $\gamma : [0,\infty) \to M$ is a ray, the Busemann function associated to γ is*

$$\begin{aligned} b^\gamma(x) &= \lim_{t\to\infty} b_t^\gamma(x) \\ &= \lim_{t\to\infty} t - d(x,\gamma(t)). \end{aligned}$$

By the above, Busemann functions are well defined and Lipschitz continuous. Intuitively, $b^\gamma(x)$ is the distance from $\gamma(\infty)$. Also, since

$$\begin{aligned} b^\gamma(\gamma(s)) &= \lim_{t\to\infty} t - d(\gamma(s),\gamma(t)) \\ &= \lim_{t\to\infty} t - (t-s) = s, \end{aligned}$$

$b^\gamma(x)$ is linear along $\gamma(t)$.

Example 3.4.5 *In $\mathbb{R}^n$, the rays are $\gamma(t) = \gamma(0)+\gamma'(0)t$. In this case, $b^\gamma(x) = \langle x-\gamma(0), \gamma'(0)\rangle$. The level sets of b^γ are hyperplanes.*

Lemma 3.4.3 *If M has $\mathrm{Ric}_M \geq 0$ and γ is a ray on M then $\Delta(b^\gamma) \geq 0$ in the barrier sense.*

Proof. For each $p \in M$, we construct a support function of b^γ at p. We first construct asymptotic rays of γ at p.

Pick $t_i \to \infty$. For each i, connect p and $\gamma(t_i)$ by a minimal geodesic σ_i. Then $\{\sigma_i'(0)\} \subset S^{n-1}$, so there is a subsequential limit $\tilde{\gamma}'(0)$. The geodesic $\tilde{\gamma}$ is called an asymptotic ray of γ at p; note that $\tilde{\gamma}$ need not be unique.

We claim that $b^{\tilde{\gamma}}(x)+b^\gamma(p)$ is a support function of b^γ at p. For $b^{\tilde{\gamma}}(p) = 0$, so the functions agree at p. In addition, $\tilde{\gamma}$ is a ray, so $\tilde{\gamma}(t)$ is not a cut point of $\tilde{\gamma}$ along p. Hence $d(\tilde{\gamma}(t), *)$ is smooth at p, so $b^{\tilde{\gamma}}(x)$ is smooth in a neighborhood of p.

Now

$$\begin{aligned} b^{\tilde{\gamma}}(x) &= \lim_{t\to\infty} t - d(x,\tilde{\gamma}(t)) \\ &\leq \lim_{t\to\infty} t - d(x,\gamma(s)) + d(\tilde{\gamma}(t),\gamma(s)) \\ &= \lim_{t\to\infty} t + s - d(x,\gamma(s)) - s + d(\tilde{\gamma}(t),\gamma(s)) \\ &= \lim_{t\to\infty} t + b_s^{\gamma}(x) - b_s^{\gamma}(\tilde{\gamma}(t)); \end{aligned}$$

letting $s \to \infty$, we obtain

$$b^{\tilde{\gamma}}(x) \leq \lim_{t\to\infty} t + b^{\gamma}(x) - b^{\gamma}(\tilde{\gamma}(t)).$$

We also have

$$\begin{aligned} b^{\gamma}(p) &= \lim_{i\to\infty} t_i - d(p,\gamma(t_i)) \\ &= \lim_{i\to\infty} t_i - d(p,\sigma_i(t)) - d(\sigma_i(t),\gamma(t_i)) \\ &= -d(p,\tilde{\gamma}(t)) + \lim_{i\to\infty} t_i - d(\sigma_i(t),\gamma(t_i)) \\ &= -d(p,\tilde{\gamma}(t)) + b^{\gamma}(\tilde{\gamma}(t)) \\ &= -t + b^{\gamma}(\tilde{\gamma}(t)). \end{aligned}$$

Thus

$$\begin{aligned} b^{\tilde{\gamma}}(x) + b^{\gamma}(p) &\leq \lim_{t\to\infty} t + b^{\gamma}(x) - b^{\gamma}(\tilde{\gamma}(t)) - t + b^{\gamma}(\tilde{\gamma}(t)) \\ &= b^{\gamma}(x), \end{aligned}$$

so $b^{\tilde{\gamma}}(x) + b^{\gamma}(p)$ is a support function for b^{γ} at p. By a similar argument, each $b_t^{\gamma}(x) + b^{\gamma}(p)$ is a support function for b^{γ} at p.

Finally, since $\mathrm{Ric}_M \geq 0$,

$$\begin{aligned} \Delta(b_t^{\tilde{\gamma}}(x) + b^{\gamma}(p)) &= \Delta(t - d(x,\tilde{\gamma}(t))) \\ &= -\Delta(x,\tilde{\gamma}(t)) \\ &\geq -\frac{n-1}{d(x,\tilde{\gamma}(t))}, \end{aligned}$$

which tends to 0 as $t \to \infty$. Thus $\Delta(b^{\gamma}) \geq 0$ in the barrier sense.

The level sets of b_t^{γ} are geodesic spheres at $\gamma(t)$. The level sets of b_t^{γ} are geodesic spheres at $\gamma(\infty)$.

Lemma 3.4.4 *Suppose γ is a line in M, $\mathrm{Ric}_M \geq 0$. Then γ defines two rays, γ^+ and γ^-. Let b^+ and b^- be the associated Busemann functions. Then:*

1. *$b^+ + b^- \equiv 0$ on M.*
2. *b^+ and b^- are smooth.*
3. *Given any point $p \in M$ there is a unique line passing through p that is perpendicular to $v_0 = \{x : b^+(x) = 0\}$ and consists of asymptotic rays.*

Proof. For

1. Observe that

$$\begin{aligned} b^+(x) + b^-(x) &= \lim_{t\to\infty}(t - d(x, \gamma^+(t))) + \lim_{t\to\infty}(t - d(x, \gamma^-(t))) \\ &= \lim_{t\to\infty} 2t - (d(x, \gamma^+(t)) - d(x, \gamma^-(t))) \\ &\leq 2t - d(\gamma^+(t), \gamma^-(t)) = 0. \end{aligned}$$

Since $b^+(\gamma(0)) + b^-(\gamma(0)) = 0$, 0 is a global maximum. But

$$\Delta(b^+ + b^-) = \Delta b^+ + \Delta b^- \geq 0,$$

so $b^+ + b^- \equiv 0$.

2. We have $b^+ = -b^-$. Thus

$$0 \leq \Delta b^+ = -\Delta b^- \leq 0,$$

so both b^+ and b^- are smooth by regularity.

3. At p there are asymptotic rays $\tilde{\gamma}^+$ and $\tilde{\gamma}^-$. We first show that $\tilde{\gamma}^+ + \tilde{\gamma}^-$ is a line. Since

$$\begin{aligned} d(\tilde{\gamma}^+(s_1), \tilde{\gamma}^-(s_2)) &\geq d(\tilde{\gamma}^-(s_2), \gamma^+(t)) - d(\tilde{\gamma}^+(s_1), \gamma^+(t)) \\ &= (t - d(\tilde{\gamma}^+(s_1), \gamma^+(t))) - (t - d(\tilde{\gamma}^-(s_2), \gamma^+(t))) \end{aligned}$$

holds for all t, we have

$$\begin{aligned} d(\tilde{\gamma}^+(s_1), \tilde{\gamma}^-(s_2)) &\geq b^+(\tilde{\gamma}^+(s_1)) - b^+(\tilde{\gamma}^-(s_2)) \\ &= b^+(\tilde{\gamma}^+(s_1)) + b^-(\tilde{\gamma}^-(s_2)) \\ &\geq \tilde{b}^+(\tilde{\gamma}^+(s_1)) + b^+(p) + \tilde{b}^-(\tilde{\gamma}^-(s_2)) + b^-(p) \\ &= s_1 + s_2. \end{aligned}$$

Thus $\tilde{\gamma}^+ + \tilde{\gamma}^-$ is a line. But our argument shows that any two asymptotic rays form a line, so the line is unique.

Set $\tilde{v}_{t_0} = (\tilde{b}^+)^{-1}(t_0)$. Then if $y \in \tilde{v}_{t_0}$ we have

$$\begin{aligned} d(y, \tilde{\gamma}^+(t)) &\geq |\tilde{b}^+(y) - \tilde{b}^+(\tilde{\gamma}^+(t))| \\ &= |t_0 - t| \\ &= d(\tilde{\gamma}^+(t_0), \tilde{\gamma}^+(t)), \end{aligned}$$

which shows $\tilde{\gamma} \perp \tilde{v}_{t_0}$.

Finally, since $\tilde{b}^+(x) + b^+(p) \leq b^+(x)$,

$$-(\tilde{b}^+(x) + b^+(p)) \geq -b^+(x).$$

But $\tilde{b}^+ = -\tilde{b}^-$ and $b^+ = -b^-$, so

$$\tilde{b}^-(x) + b^-(p) \geq b^-(x).$$

Since $\tilde{b}^-(x) + b^-(p) \leq b^-(x)$ as well,

$$\tilde{b}^-(x) + b^-(p) = b^-(x).$$

Thus the level sets of b^+ are the level sets of $\tilde{b}^+$, which proves the result.

Note that $b^+ : M \to \mathbb{R}$ is smooth. Since b^+ is linear on γ with a Lipschitz constant 1, $\|\nabla b^+\| = 1$. Thus $v_0 = (b^+)^{-1}(0)$ is a smooth (n-1) submanifold of M.

Proof of Splitting Theorem. Let $\phi : \mathbb{R} \times v_0 \to M$ be given by $(t, p) \mapsto \gamma(t) = \exp_p t\gamma'(0)$, where γ is the unique line passing through p, perpendicular to v_0. By the existence and uniqueness of γ, ϕ is bijective. Since $\exp_p$ is a local diffeomorphism and $\gamma'(0) = (\nabla b^+)(v)$ smooth, ϕ is a diffeomorphism.

To show that ϕ is an isometry, set $v_t = (b^+)^{-1}(t)$ and let $m(t)$ be the mean curvature of v_t. Then $m(t) = \Delta b^+ = 0$. In the proof of the Laplacian comparison, we derived

$$\mathrm{Ric}(N, N) + \|\mathrm{Hess}\,(r)\|^2 = m'(r, \theta),$$

where $N = \nabla\gamma$. Note that γ is the integral curve of ∇b^+ passing through p, so $\Delta\gamma = m(t)$ and $\nabla b^+ = N$.

In our case, $\mathrm{Ric}(N, N) \geq 0$ and $m'(r, \theta) = 0$, so

$$\|\mathrm{Hess}\,(b^+)\| = \|\mathrm{Hess}\,(r)\| \leq 0.$$

Thus $\|\text{Hess}\,(b^+)\| = 0$, so that ∇b^+ is a parallel vector field.

Now ϕ is an isometry in the t direction since $\exp_p$ is a radial isometry. Suppose X is a vector field on v_0. Then

$$R(N, X)N = \nabla_N \nabla_X N - \nabla_X \nabla_N N - \nabla_{[X,N]} N.$$

But $\nabla_N N = 0$, and we may extend X in the coordinate direction so that $[X, N] = 0$. Since

$$\nabla_X N = \nabla_X \nabla b^+ = 0,$$

we have $R(N, X)N = 0$.

Let $J(t) = \phi_*(x) = \frac{d}{ds}(\phi(c(s)))\big|_{s=t}$, where $c : (-\varepsilon, \varepsilon) \to v_0$ has $c'(t) = X$. Then $J(t)$ is a Jacobi field, $J''(t) = 0$ and $J \perp N$. Thus $J(t)$ is constant. Hence $\|\phi_*(X)\| = \|X\|$, so ϕ is an isometry.

Remark: Since $\|\text{Hess}\,(b^+)\| = 0$, we have $\nabla_X \nabla b^+ = 0$ for all vector fields X. By the de Rham decomposition, ϕ is a locally isometric splitting.

Summary of Proof of Splitting Theorem.

1. Laplcaian Comparison in Barrier Sense
2. Maximal Principle
3. Bochner Formula: Generalizes $\text{Ric}(N, N) + \|\text{Hess}\,(r)\| = m'(r, \theta)$
4. de Rham Decomposition

Also, the Regularity Theorem was used.

3.5 Applications of the Splitting Theorem

Theorem 3.5.1 (Cheeger-Gromoll 1971) *If M^n is compact with $\text{Ric}_M \geq 0$ then the universal cover $\tilde{M} \overset{iso}{\simeq} N \times \mathbb{R}^k$, where N is a compact $(n-k)$-manifold. Thus $\pi_1(M)$ is almost π_1(Flat Manifold), i.e.*

$$0 \to F \to \pi_1(M) \to B_k \to 0,$$

where F is a finite group and B_k is the fundamental group of some compact flat manifold.

B_k is called a Bieberbach group.

Proof. By the splitting theorem, $\tilde{M} \simeq N \times \mathbb{R}^k$, where N has no line. We show N is compact.

Note that isometries map lines to lines. Thus, if $\psi \in Iso(\tilde{M})$, then $\psi = (\psi_1, \psi_2)$, where $\psi_1 : N \to N$ and $\psi_2 : \mathbb{R}^k \to \mathbb{R}^k$ are isometries. Suppose N is not compact, so N contains a ray $\gamma : [0, \infty) \to N$. Let F be a fundamental domain of M, so $\bar{F}$ is compact, and let p_1 be the projection $\tilde{M} \to N$.

Pick $t_i \to \infty$. For each i there is $g_i \in \pi_1(M)$ such that $g_i(\gamma(t_i)) \in p_1(F)$. But $p_1(\bar{F})$ is compact, so we may assume $g_i(\gamma(t_i)) \to p \in N$. Set $\gamma_i(t) = g_i(\gamma(t+t_i))$. Then $\gamma_i : [-t_i, \infty) \to N$ is minimal, and $\{\gamma_i\}$ converges to a line σ in N.

Thus N is compact. For the second statement, let $p_2 : \pi_1(M) \to Iso(\mathbb{R}^k)$ be the map $\psi = (\psi_1, \psi_2) \mapsto \psi_2$. Then

$$0 \to Ker(p_2) \to \pi_1(M) \to Im(p_2) \to 0$$

is exact. Now $Ker(p_2) = \{(\psi_1, 0)\}$, while $Im(\psi_2) = \{(0, \psi_2)\}$. Since $Ker(p_2)$ gives a properly discontinuous group action on a compact manifold, $Ker(p_2)$ is finite. On the other hand, $Im(p_2)$ is an isometry group on $\mathbb{R}^k$, so $Im(p_2)$ is a Bieberbach group.

Remark: The curvature condition is only used to obtain the splitting $\tilde{M} \simeq N \times \mathbb{R}^k$. Thus, if the conclusion of the splitting theorem holds, the curvature condition is unnecessary.

Corollary If M^n is compact with $\mathrm{Ric}_M \geq 0$ and $\mathrm{Ric}_M > 0$ at one point, then $\pi_1(M)$ is finite.

Remark: This corollary improves the theorem of Bonnet-Myers. The corollary can also be proven using the Bochner technique. In fact, Aubin's deformation gives another metric that has $\mathrm{Ric}_M > 0$ everywhere.

Corollary If M^n has $\mathrm{Ric}_M \geq 0$ then $b_1(M) \leq n$, with equality if and only if $M^n \overset{iso}{\simeq} T^n$, where T^n is a flat torus.

Definition 3.5.1 *Suppose M^n is noncompact. Then M is said to have the geodesic loops to infinity property if for any ray γ in M, any $g \in \pi_1(M, \gamma(0))$ and any compact $K \subset M$ there is a geodesic loop c at γ_{t_0} in $M - K$ such that $g = [c] = [(\gamma|_0^{t_0})^{-1} \circ c \circ \gamma|_0^{t_0}]$.*

Example 3.5.1 $M = N \times \mathbb{R}$ *If the ray γ is in the splitting direction, then any $g \in \pi_1(M, \gamma)$ is homotopic to a geodesic loop at infinity along γ.*

Theorem 3.5.2 (Sormani, 1999) *If M^n is complete and noncompact with* $\mathrm{Ric}_M > 0$ *then M has the geodesic loops to infinity property.*

Theorem 3.5.3 (Line Theorem) *If M^n does not have the geodesic loops to infinity property then there is a line in $\tilde{M}$.*

Application:(Shen-Sormani) If M^n is noncompact with $\mathrm{Ric}_M > 0$ then $H_{n-1}(M, \mathbb{Z}) = 0$.

3.6 Excess Estimate

Definition 3.6.1 *Given $p, q \in M$, the excess function associated to p and q is*

$$e_{p,q}(x) = d(p, x) + d(q, x) - d(p, q).$$

For fixed $p, q \in M$, write $e(x)$.If γ is a minimal geodesic connecting p and q with $\gamma(0) = p$ and $\gamma(1) = q$, let $h(x) = \min_{0 \le t \le 1} d(x, \gamma(t))$. Then

$$0 \le e(x) \le 2h(x).$$

Let y be the point along γ between p and q with $d(x, y) = h(x)$.

Set

$$\begin{array}{ll} s_1 = d(p, x), & t_1 = d(p, y) \\ s_2 = d(q, x), & t_2 = d(q, y). \end{array}$$

We consider triangles pqx for which h/t_1 is small; such triangles are called thin.

Example 3.6.1 *In $\mathbb{R}^n$,*

$$s_1 = \sqrt{h^2 + t_1^2} = t_1 \sqrt{1 + (h/t_1)^2}.$$

For a thin triangle, we may use a Taylor expansion to obtain $s_1 \le t_1(1 + (h/t_1)^2)$. Thus

$$\begin{aligned} e(x) &= s_1 + s_2 - t_1 - t_2 \\ &\le h^2/t_1 + h^2/t_2 \\ &= h(h/t_1 + h/t_2) \\ &\le 2h(h/t), \end{aligned}$$

where $t = \min\{t_1, t_2\}$. Thus $e(x)$ is small is h^2/t is small.

If M has $K \geq 0$ then the Toponogov comparison shows that $s_1 \leq \sqrt{h^2 + t_1^2}$, so the same estimate holds.

Lemma 3.6.1 *$e(x)$ has the following basic properties:*

1. *$e(x) \geq 0$.*

2. *$e|_\gamma = 0$.*

3. *$|e(x) - e(y)| \leq 2d(x, y)$.*

4. *If M has $\mathrm{Ric} \geq 0$,*

$$\begin{aligned} \Delta(e(x)) &\leq (n-1)(1/s_1 + 1/s_2) \\ &\leq (n-1)(2/s), \end{aligned}$$

   where $s = \min\{s_1, s_2\}$.

Proof. (1), (2) and (3) are clear. (4) is a consequence of the following Laplacian comparison.

Lemma 3.6.2 *Suppose M has $\mathrm{Ric} \geq (n-1)H$. Set $r(x) = d(p, x)$, and let $f : \mathbb{R} \to \mathbb{R}$. Then, in the barrier sense,*

1. *If $f' \geq 0$ then $\Delta f(r(x)) \leq \Delta_H f|_{r=r(x)}$.*

2. *If $f' \leq 0$ then $\Delta f(r(x)) \geq \Delta_H f|_{r=r(x)}$.*

Proof. Recall that $\Delta = \frac{\partial^2}{\partial r^2} + m(r,\theta)\frac{\partial}{\partial r} + \tilde{\Delta}$, where $\tilde{\Delta}$ is the Laplacian on the geodesic sphere. Hence

$$\begin{aligned} \Delta f(r(x)) &= f'' + m(r,\theta) f' \\ &= f'' + \Delta r f', \end{aligned}$$

so we need only show $\Delta r \leq \Delta_H r$ in the barrier sense.

We have proved the result where r is smooth, so need only prove at cut points. Suppose q is a cut point of p. Let γ be a minimal geodesic with $\gamma(0) = p$ and $\gamma(\ell) = q$. We claim that $d(\gamma(\varepsilon), x) + \varepsilon$ is an upper barrier function of $r(x) = d(p, x)$ at q, as

1. $d(\gamma(\varepsilon), x) + \varepsilon \geq d(p, x)$,

2. $d(\gamma(\varepsilon), q) + \varepsilon = d(p, q)$ and

3. $d(\gamma(\varepsilon), x) + \varepsilon$ is smooth near q, since q is not a cut point of $\gamma(\varepsilon)$ for $\varepsilon > 0$.

Since

$$\begin{aligned}\Delta(d(\gamma(\varepsilon), x) + \varepsilon) &\leq \Delta_H(d(\gamma(\varepsilon), x)) \\ &= m_H(d(\gamma(\varepsilon), x)) \\ &\leq m_H(d(p, x)) + c\varepsilon = \Delta_H(r(x)) + c\varepsilon,\end{aligned}$$

we have the result.

Definition 3.6.2 *The dilation of a function is*

$$\operatorname{dil}(f) = \min_{x,y} \frac{|f(x) - f(y)|}{d(x, y)}.$$

By property (2) of $e(x)$, we have $\operatorname{dil}(e(x)) \leq 2$.

Theorem 3.6.1 *Suppose $U : B(y, R + \eta) \to \mathbb{R}$ is a Lipschitz function on M, $\operatorname{Ric}_M \geq (n-1)H$ and*

1. $U \geq 0$,

2. $\operatorname{dil}(U) \leq a$,

3. $u(y_0) = 0$ *for some* $y_0 \in B(\bar{y}, R)$ *and*

4. $\Delta U \leq b$ *in the barrier sense.*

Then $U(y) \leq ac + G(c)$ for all $0 < c < R$, where $G(r(x))$ is the unique function on M_H such that:

1. $G(r) > 0$ *for* $0 < r < R$.

2. $G'(r) < 0$ *for* $0 < r < R$.

3. $G(R) = 0$.

4. $\Delta_H G \equiv b$.

Proof. Suppose $H = 0$, $n \geq 3$. We want $\Delta_H G = b$. Since $\Delta_H = \frac{\partial^2}{\partial r^2} + m_H(r,\theta)\frac{\partial}{\partial r} + \tilde{\Delta}$, we solve

$$\begin{aligned} G'' + (n-1)G'/r &= b \\ G''r^2 + (n-1)G'r &= br^2, \end{aligned}$$

which is an Euler type O.D.E. The solutions are $G = G_p + G_h$, where $G_p = b/2nr^2$ and $G_h = c_1 + c_2 r^{-(n-2)}$.

Now $G(R) = 0$ gives

$$\frac{b}{2n}R^2 + c_1 + c_2 R^{-(n-2)} = 0,$$

while $G' < 0$ gives

$$\frac{b}{n}r - (n-2)c_2 r^{-(n-1)} > -0$$

for all $0 < r < R$. Thus $c_2 \geq \frac{b}{n(n-2)}R^n$.

Hence $G(r) = \frac{b}{2n}(r^2 + \frac{2}{n-2}r^{-(n-2)} - \frac{n}{n-2}R^2$. Note that $G > 0$ follows from $G(R) = 0$ and $G' < 0$.

For general $H < 0$,

$$G(r) = b\int_r^R \int_r^t \left(\frac{\sinh\sqrt{-H}t}{\sinh\sqrt{-H}s}\right)^{n-1} dsdt.$$

Note that $\Delta_H G \geq b$ by the Laplacian comparison.

To complete the proof, fix $0 < c < R$. If $d(y, y_0) \leq c$,

$$\begin{aligned} U(y) &= U(y) - U(y_0) \\ &\leq ad(y, y_0) \\ &\leq ac \\ &\leq ac + G(c). \end{aligned}$$

If $d(y, y_0) > c$ then consider G defined on $B(y, R+\varepsilon)$, where $0 < \varepsilon < \eta$. Letting $\varepsilon \to 0$ gives the result.

Consider $V = G - U$. Then $\Delta V = \Delta G - \Delta U \geq 0$, $V|_{\partial B(R+\varepsilon)} \leq 0$ and $V(y_0) > 0$. Now y_0 is in the interior of $\bar{B}(y, R+\varepsilon) - B(y, c)$, so $V(y') > 0$ for some $y' \in \partial B(y, c)$. Since

$$U(y) - U(y') \leq ad(y, y') = ac$$

and

$$G(c) - U(y') = V(y') > 0,$$

we have

$$U(y) \leq ac + U(y') < ac + G(c).$$

We now apply this result to $e(x)$. Here $e(x) \geq 0$, $a = 2$ and $R = h(x)$. We assume $s(x) \geq 2h(x)$. On $B(x, R)$,

$$\Delta e \leq \frac{4(n-1)}{s(x)},$$

so $b = 4(n-1)/s(x)$. Thus

$$\begin{aligned} e(x) &\leq 2c + G(c) \\ &= 2c + \frac{2(n-1)}{ns}(c^2 + frac2n - 2h^n c^{-(n-2)} + \frac{n}{n-2}h^2) \end{aligned}$$

for all $0 < c < h$.

To find the minimal value for $ar + G(r)$, $0 < r < R$, consider

$$a + G'(r) = a + \frac{b}{2n}(2r - 2R^n r^{1-n}) = 0.$$

This gives $r(R^n/r^n - 1) = an/b$. To get an estimate, choose r small. Then R^n/r^n is large, so $R^n/r^{n-1} \approx an/b$. Hence

$$r = \left(\frac{R^n b}{an}\right)^{\frac{1}{n-1}}$$

is close to a minimal point.

For the excess function, choose

$$c = \left(\frac{2h^n}{s}\right)^{\frac{1}{n-1}} \approx \left(\frac{h^n \frac{4(n-1)}{s}}{2n}\right)^{\frac{1}{n-1}}.$$

Then

$$G(c) = \frac{2(n-1)}{ns}\left(\left(\frac{2h^n}{s}\right)^{2/(n-1)} + \frac{2}{n-2}h^n\left(\frac{2h^n}{s}\right)^{-\frac{n-2}{n-1}} - \frac{n}{n-2}h^2\right).$$

Now

$$\left(\frac{2h^n}{s}\right)^{\frac{2}{n-1}} = h^2\left(\frac{2h}{s}\right)^{\frac{2}{n-1}},$$

$$\frac{2h}{s} \leq 1$$

and

$$\frac{n}{n-2} > 1,$$

so

$$\begin{aligned} G(c) &\leq \frac{2(n-1)}{n}\frac{2}{n-2}\frac{h^n}{s}\left(\frac{2h^n}{s}\right)^{\frac{1}{n-1}-1} \\ &\leq \frac{2(n-1)}{n(n-2)}\left(\frac{2h^n}{s}\right)^{\frac{1}{n-1}} \\ &\leq 2c. \end{aligned}$$

Thus

$$\begin{aligned} e(x) &\leq 2c + G(c) \\ &= 2c + 2c \\ &= \left(\frac{2h^n}{s}\right)^{\frac{1}{n-1}} \\ &\leq 8\left(\frac{h^n}{s}\right)^{\frac{1}{n-1}}. \end{aligned}$$

Remarks:

1. A more careful estimate is

$$e(x) \leq 2\left(\frac{n-1}{n-2}\right)\left(\frac{c_3h^n}{2}\right)^{\frac{1}{n-1}} = 8h\left(\frac{h}{s}\right)^{\frac{1}{n-1}},$$

where $c_3 = \frac{n-1}{n}\left(\frac{1}{s_1-h} + \frac{1}{s_2-h}\right)$ and $h < \min(s_1, s_2)$.

2. In general, if $\mathrm{Ric}_M \geq (n-1)H$ then $e(x) \leq hF(\frac{h/s}{)}$ for some continuous F satisfying $F(0) = 0$. F is given by an integral; consider the proof of the estimate in the case $\mathrm{Ric}_M \geq 0$.

3.7 Applications of the Excess Estimate

Theorem 3.7.1 (Sormani, 1998) *Suppose M^n complete and noncompact with $\mathrm{Ric}_M \geq 0$. If, for some $p \in M$,*

$$\limsup_{r\to\infty} \frac{\mathrm{diam}(\partial B(p,r))}{r} < 4s_n,$$

where

$$s_n = \frac{1}{2}\frac{1}{3^n}\frac{1}{4^{n-1}},$$

then $\pi_1(M)$ is finitely generated.

Compare this result with:

Theorem 3.7.2 (Abresch and Gromoll) *If M is noncompact with $\mathrm{Ric}_M \geq 0$, $K \geq -1$ and diameter growth $o(r^{\frac{1}{n}})$, then M has finite topological type.*

Note: Diameter growth is the growth of $\mathrm{diam}\,\partial B(p,r)$. When $\mathrm{Ric} \geq 0$, $\mathrm{diam}\,\partial B(p,r) \leq r$. To say M has finite topological type is to say that each $H_i(M,\mathbb{Z})$ is finite.

To prove Sormani's result we choose a desirable set of generators for $\pi_1(M)$.

Lemma 3.7.1 *For M^n complete we may choose a set of generators $g_1, \ldots, g_n, \ldots$ of $\pi_1(M)$ such that:*

1. *$g_i \in \mathrm{span}\{g_1, \ldots, g_{i-1}\}$.*
2. *Each g_i can be represented by a minimal geodesic loop γ_i based at p such that if $\ell(\gamma_i) = d_i$ then $d(\gamma(0), \gamma(d_i/2)) = d_i/2$, and the lift $\tilde{\gamma}_i$ based at $\tilde{p}$ is a minimal geodesic.*

Proof. Fix $\tilde{p} \in \tilde{M}$. Let $G = \pi_1(M)$. Choose $g_1 \in G$ such that $d(\tilde{p}, g_1(\tilde{p})) \leq d(\tilde{p}, g(\tilde{p}))$ for all $g \in G - \{e\}$. Note that since G acts discretely on $\tilde{M}$, only finitely many elements of G satisfy a given distance restraint.

Let $G_i = \langle g_1, \ldots, g_{i-1}\rangle$. Choose $g_i \in G - G_i$ such that $d(\tilde{p}, g_i(\tilde{p})) \leq d(\tilde{p}, g(\tilde{p}))$ for all $g \in G - G_i$. If $\pi_1(M)$ is finitely generated, we have a sequence $g_1, \ldots, g_n, \ldots$; otherwise we have a list. The g_i's satisfy (1). Let $\tilde{\gamma}_i$ be the minimal geodesic connecting $\tilde{p}$ to $g_i(\tilde{p})$. Set $\gamma_i = \pi(\tilde{\gamma}_i)$, where is the covering $\pi : \tilde{M} \to M$. We claim that if $\ell(\gamma_i) = d_i$ then $d(\gamma(0), \gamma(d_i/2)) = d_i/2$.

Otherwise, for some i and some $T < d_i/2$, $\gamma_i(T)$ is a cut point of p along γ_i. Since M and $\tilde{M}$ are locally isometric, and $\tilde{\gamma}_i(T)$ is not conjugate to $\tilde{p}$ along $\tilde{\gamma}$, $\gamma_i(T)$ is not conjugate to p along γ. Hence we can connect p to $\gamma_i(T)$ with a second minimal geodesic σ. Set

$$h_1 = \sigma^{-1} \circ \gamma_i|_{[0,T]}$$

and

$$h_2 = \gamma_i|_{[T,d_i]} \circ \sigma.$$

Now h_1 is not a geodesic, so

$$d(\tilde{p}, h_1(\tilde{p})) < 2T < d_i.$$

Similarly,

$$d(\tilde{p}, h_2(\tilde{p})) < T + d_i - T = d_i.$$

Hence $h_1, h_2 \in G_i$. But then $\gamma_i = h_2 \circ h_1 \in G_i$, which is a contradiction.

Lemma 3.7.2 *Suppose M^n has* Ric ≥ 0, $n \geq 3$ *and γ is a geodesic loop based at p. Set $D = \ell(\gamma)$. Suppose*

1. $\gamma|_{[0,D/2]}$ *, and* $\gamma|_{[D/2,D]}$ *are minimal.*
2. $\ell(\gamma) \leq \ell(\sigma)$ *for all* $[\sigma] = [\gamma]$.

Then for $x \in \partial B(p, RD)$, $R \geq 1/2 + s_n$, we have $d(x, \gamma(D/2)) \geq (R - 1/2)D + 2s_n D$.

Remark: $\gamma(D/2)$ is a cut point of p along γ. Since $d(p,x) > D/2$, any minimal geodesic connecting p and x cannot pass through $\gamma(D/2)$. Thus

$$\begin{aligned} d(\gamma(D/2), x) &> d(p,x) - d(p, \gamma(D/2)) \\ &= RD - D/2 = (R - 1/2)D. \end{aligned}$$

The lemma gives a bound on how much larger $d(\gamma(D/2), x)$ is.

Proof. It is enough to prove for $R = 1/2 + s_n$. For if $R > 1/2 + s_n$, we may choose $y \in \partial B(p, (1/2 + s_n))$ such that

$$d(x, \gamma(D/2)) = d(x,y) + d(y, \gamma(D/2)).$$

Then

$$\begin{aligned} d(x, \gamma(D/2)) &\geq d(x,y) + 3s_n D \\ &\geq (R - (1/2 + s_n))D + 3s_n D \\ &= (R - 1/2)D + s_n D. \end{aligned}$$

Suppose there exists $x \in \partial B(p, (1/2 + s_n)D)$ such that

$$d(x, \gamma(D/2)) = H < 3s_n D.$$

Let c be a minimal geodesic connecting x and $\gamma(D/2)$. Let $\tilde{p}$ be a lift of p, and lift γ to $\tilde{\gamma}$ starting at $\tilde{p}$. If $g = [\gamma]$, then $\tilde{\gamma}$ connects $\tilde{p}$ and $g(\tilde{p})$.

Lift c to $\tilde{c}$ starting at $\tilde{\gamma}(D/2)$, and lift $c \circ \gamma|_{[0,D/2]}$ to $\tilde{c} \circ \tilde{\gamma}|_{[0,D/2]}$. Then

$$\begin{aligned} d(\tilde{p}, \tilde{x}) &\geq d(p, x) \\ &= (1/2 + s_n)D, \end{aligned}$$

and

$$d(g(\tilde{p}), \tilde{x}) \geq (1/2 + s_n)D.$$

Thus

$$\begin{aligned} e_{\tilde{p}, g(\tilde{p})}(\tilde{x}) &= d(\tilde{p}, \tilde{x}) + d(g(\tilde{p}), \tilde{x}) - d(\tilde{p}, g(\tilde{p})) \\ &\geq (1/2 + s_n)D + (1/2 + s_n)D - D = 2s_n D. \end{aligned}$$

But, by the excess estimate, if $s \geq 2h$,

$$e(\tilde{x}) \leq 8 \left(\frac{h^n}{s} \right)^{\frac{1}{n-1}}.$$

In this case, $h \leq H < 3s_n D$. Also,

$$s \geq (1/2 + s_n)D > D/2.$$

Since $s_n < 1/12$ for $n \geq 2$, we have $s \geq 2h$. Thus

$$e(\tilde{x}) \leq 8 \left(\frac{(3s_n D)^n}{D/2} \right)^{\frac{1}{n-1}}.$$

But this gives

$$2s_n D \leq 8D \left(2(3s_n)^n\right)^{\frac{1}{n-1}},$$

whence

$$s_n > \frac{1}{2}\frac{1}{3^n}\frac{1}{4^{n-1}}.$$

We may now prove Sormani's result.

Proof of Theorem. Pick a set of generators $\{g_k\}$ as in the lemma, where g_k is represented by γ_k. If $x_k \in \partial B(p, (1/2 + s_n)d_k)$, where $d_k = \ell(\gamma_k) \to \infty$, we showed that $d(x_k, \gamma(d_k/2)) \geq 3s_n d_k$.

Let $y_k \in \partial B(p, d_k/2)$ be the point on a minimal geodesic connecting p and x_k. Then

$$\begin{aligned}
\limsup_{r\to\infty} \frac{\operatorname{diam}(\partial B(p,r))}{r} &\geq \lim_{k\to\infty} \frac{d(y_k, \gamma_k(d_k/2))}{d_k/2} \\
&\geq \lim_{k\to\infty} \frac{2s_n d_k}{d_k/2} = 4s_n,
\end{aligned}$$

so we have a contradiction if there are infinitely many generators.

The excess estimate can also be used for compact manifolds.

Lemma 3.7.3 *Suppose M^n with $\operatorname{Ric}_M \geq (n-1)$. Then given $\delta > 0$ there is $\varepsilon(n,\delta) > 0$ such that if $d(p,q) \geq \pi - \varepsilon$ then $e_{p,q}(x) \leq \delta$.*

This lemma can be used to prove the following:

Theorem 3.7.3 *There is $\varepsilon(n, H)$ such that if M^n has $\operatorname{Ric}_M \geq (n-1)$, $\operatorname{diam}_M \geq \pi - \varepsilon$ and $K_M \geq H$ then M is a twisted sphere.*

Proof of Lemma. Fix x and set $e = e_{p,q}(x)$. Then $B(x, e/2)$, $B(p, d(p,x) - e/2)$ and $B(q, d(x,q) - e/2)$ are disjoint. Thus

$$\begin{aligned}
\operatorname{vol}(M) &\geq \operatorname{vol}(B(x,e/2)) + \operatorname{vol}(B(p, d(p,x) - e/2)) + \operatorname{vol}(B(q, d(q,x) - e/2)) \\
&= \operatorname{vol}(M)\left(\frac{\operatorname{vol}(B(x,e/2))}{\operatorname{vol}(B(x,\pi))} + \frac{\operatorname{vol}(B(p,d(p,x)-e/2))}{\operatorname{vol}(B(p,\pi))} + \frac{\operatorname{vol}(B(q,d(q,x)-e/2))}{\operatorname{vol}(B(q,\pi))}\right) \\
&\geq \operatorname{vol}(M)\left(\frac{v(n,1,e/2) + v(n,1,d(p,x)-e/2) + v(n,1,d(q,x)-e/2)}{v(n,1,\pi)}\right),
\end{aligned}$$

where $v(n, H, r) = \operatorname{vol}(B(r))$, $B(r) \subset M^n_H$.

Now in $S^n(1)$, $\mathrm{vol}(B(r)) = \int_0^r \sin^{n-1} t\, dt$ is a convex function of r. Thus we have

$$\begin{aligned} v(n,1,\pi) &\geq v(n,1,e/2) + v(n,1,d(p,x)-e/2) + v(n,1,d(q,x)-e/2) \\ &\geq v(n,1,e/2) + 2v\left(n,1,\frac{d(p,x)+d(q,x)-e}{2}\right) \\ &= v(n,1,e/2) + 2v\left(n,1,\frac{d(p,q)}{2}\right). \end{aligned}$$

Hence

$$v(n,1,e/2) \leq v(n,1,\pi) - 2v\left(n,1,\frac{d(p,q)}{2}\right),$$

which tends to 0 as $\varepsilon \to 0$. Thus $e \to 0$.