Math 241B

Instructor: Guofang Wei

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Last quarter we studied the relationship between Ricci curvature and topology. The basic tools were:

1. Volume Comparison
2. Laplacian Comparison
3. Excess Estimate

This quarter we study the relationship between sectional curvature and topology. Our tools will be:

1. Critical Point Theory for Distance Functions
2. Toponogov Comparison

As applications we will have:

1. Generalized Sphere Theorem
2. Soul Theorem
3. Gromov's Betti Number Estimate
4. Homotopy Finiteness Theorem

1 Introduction to Critical Point Theory of Distance Functions

Morse Theory is an important tool in differential topology. Suppose M is a manifold and $f : M \to \mathbb{R}$ is a (smooth) Morse function. When $\nabla f(x) = 0$, so x is a critical point, and $\text{Hess}_f(x)$ has no 0 eigenvalues, the index of f at x is the number of negative eigenvalues of $\text{Hess}_f(x)$. Note, when $\text{Hess}_f(x)$ has no 0 eigenvalues, we say $\text{Hess}_f(x)$ is nondegenerate.

The idea of Morse Theory is to relate the critical points of f to the topology of M. For geometric purposes, we will apply Morse Theory to distance functions $\rho_p(x) = d(p,x)$. Note that ρ_p is not smooth at the cut locus C_p; ρ_p is only smooth on $M - (\{p\} \cup C_p)$.

On $M - (\{p\} \cup C_p)$, $(\nabla \rho_p)q$ is the tangent vector to the geodesic connecting q and p. At smooth points, $|\nabla \rho_p| \equiv 1$.

In 1977, Grove-Shiohama defined critical points for ρ_p so as to prove a generalized sphere theorem.

Definition 1.0.1 *A point $q \neq p$ is a critical point of ρ_p if for all $v \in T_qM$ there is a minimal geodesic γ connecting q to p such that $\measuredangle(\gamma'(0), v) \leq \pi/2$. In this case, q is called a critical point of p.*

Remarks

1. If q is not a critical point of p then there is $w \in T_qM$ with $\measuredangle(w, \gamma'(0)) < \pi/2$ for all minimal geodesics γ connecting q to p. In other words, the tangent vectors of all minimal geodesics γ connecting q to p lie in an open half space of T_qM.

2. If $q \neq p$ is a critical point of p then $q \in C_p$.

Example 1.0.1 *M a cylinder. The only critical point of (x, y) is $(-x, y)$.*

Example 1.0.2 *Suppose γ is a geodesic loop with length ℓ. If $\gamma|_{[0,\ell/2]}$ and $\gamma|_{[\ell/2,\ell/2]}$ are minimal then $\gamma(\ell/2)$ is a critical point of $\gamma(0)$.*

Example 1.0.3 *If M is compact, $p \in M$ and q a furthest point to p then q is a critical point of p. This result is a consequence of the following lemma:*

Lemma 1.0.1 (Isotopy Lemma) *If* $r_1 < r_2 \leq \infty$, *and if* $B(p, r_1) - B(p, r_2)$ *is free of critical points of* p, *then the region is homeomorphic to* $\partial B(p, r_1) \times [r_1, r_2]$.

Hence no critical points implies no change in topology.

Proof If x is not a critical point of p there is $w_x \in T_x M$ such that $\measuredangle(w_x, \gamma'(0)) < \pi/2$ for all minimal geodesics γ connecting x to p. Extend w_x to a vector field defined in a neighborhood U_x of x so that

$$\measuredangle(w_x(y), \gamma'(0)) < \pi/2$$

for all minimal geodesics connecting y to p.

Since $\overline{B(p, r_2) - B(p, r_1)}$ is paracompact, we may choose a locally finite covering U_{x_i} of $\overline{B(p, r_2) - B(p, r_1)}$ consisting of open sets U_x as above.

Let $\sum \phi_i \equiv 1$ be a partition of unity subordinate to $\{U_{x_i}\}$. Set

$$W = \sum \phi_i w_{x_i}.$$

Then W is nonvanishing. Let $\psi_x(t)$ be the integral curve of W passing through x. Now

$$\rho_p(\psi_x(t_2)) - \rho_p(\psi_x(t_1)) = \int_{t_1}^{t_2} \frac{d}{dt}(\rho_p(\psi_x(t)))dt.$$

Connect $\psi_x(t)$ to p by a minimal geodesic $\sigma_t(s)$. The first variation formula is

$$\frac{d}{dt} L(s,t) = \langle v(s), \sigma_t'(s)\rangle|_0^{s_1} - \int_0^{s_1} \langle \nabla \sigma_t'(s), v(s)\rangle ds.$$

Since σ_t is a geodesic, $\nabla \sigma_t' = 0$. Thus

$$\begin{aligned} \rho_p(\psi_x(t_2)) - \rho_p(\psi_x(t_1)) &= -\int_{t_1}^{t_2} \cos \measuredangle(w_x(\psi_x(t)), \sigma_t'(0))dt \\ &\leq -\cos(\pi/2 - \epsilon)(t_2 - t_1). \end{aligned}$$

Thus ρ_p is strictly decreasing as t increases along the integral curves of W, so the integral flow gives the homeomorphism

$$\partial B(p, r_1) \times [r_1, r_2] \to B(p, r_1) - B(p, r_2)$$

by $\varphi(x, t) = \psi_x(t_x)$, where t_x is chosen appropriately.

2 Toponogov Comparison Theorem

Definition 2.0.2 *$\{\gamma_0, \gamma_1, \gamma_2\}$ is called a geodesic triangle in M if each γ_i is a normalized geodesic and $\gamma_i(\ell_i) = \gamma_{i+1}(0)$, where $\ell_i = \ell(\gamma_i)$. Set $\alpha_i = \measuredangle(\gamma'_{i-1}(0), \gamma'_{i+1}(0))$, so each α_i is the interior angle opposite γ_i. Any pair of edges with a common vertex is called a hinge.*

The Toponogov Comparison Theorem has equivalent versions; there is the angle comparison and the hinge comparison (See also Burago-Gromov-Perelman, 'Alexandrov Spaces with Curvature Bounded Below,' 1992).

Theorem 2.0.1 (Toponogov Comparison Theorem 1959) *Suppose M^n is complete with $K_M \geq H$.*

Angle Version *Suppose $\{\gamma_0, \gamma_1, \gamma_2\}$ is a geodesic triangle in M^n. Assume γ_0, γ_2 are minimal and $\ell_1 + \ell_2 \geq \ell_0$. If $H > 0$ assume $\ell_0 \leq \pi/\sqrt{H}$. There is a geodesic triangle $\{\bar{\gamma}_0, \bar{\gamma}_1, \bar{\gamma}_2\}$ in the model space M^2_H such that each $\bar{\gamma}_i$ is minimal and $\ell(\bar{\gamma}_i) = \ell(\gamma_i)$. Moreover, $\bar{\alpha}_1 \leq \alpha_1$ and $\bar{\alpha}_2 \leq \alpha_2$.*

Note *If we assume that each γ_i is minimal we have $\bar{\alpha}_i \leq \alpha_i$ for all i.*

Hinge Version *Suppose we have a hinge $\{\gamma_0, \gamma_2\}$ with angle α_1. Assume γ_2 is minimal and, if $H > 0$, that $\ell(\gamma_0) \leq \pi/\sqrt{H}$. Then there is a hinge $\{\bar{\gamma}_0, \bar{\gamma}_2\}$ in M^2_H with $\ell(\bar{\gamma}_i) = \ell(\gamma_i)$ and angle α_1. Moreover,*

$$d(\bar{\gamma}_0(\ell_0), \bar{\gamma}_2(0)) \geq d(\gamma_0(\ell_0), \gamma_2(0)).$$

Remarks

1. When all edges lie inside the injectivity radius of the vertices, the result holds by the Rauch Comparison Theorem. But Toponogov holds globally, so it is a sectional curvature analogue to Volume Comparison.

2. If $K_M \leq H$ we only have a local version. Consider T^2 with $K \equiv 0$.

Proof We first show that such triangles exist in the model space; we can create such hinges. To construct such a triangle, let $\bar{\gamma}_0$ be a minimal geodesic with $\ell(\bar{\gamma}) = \ell(\gamma_0)$. Consider the sets $\partial B(\bar{\gamma}_0(0), \ell_2)$ and $\partial B(\bar{\gamma}_0(\ell_0), \ell_1)$. These sets intersect since $\ell_1 + \ell_2 \geq \ell_0$. Connect any point in their intersection to $\bar{\gamma}_0(0)$ and $\bar{\gamma}_0(\ell_0)$ with minimal geodesics.

We next show the equivalence of the versions.

Lemma 2.0.2 *If* $\{\gamma_0, \gamma_2\}$ *is a hinge in* M_H^2*, with* γ_0 *and* γ_2 *minimal, then if* $\ell(\gamma_0)$ *and* $\ell(\gamma_2)$ *are fixed while* α_1 *varies then* $d(\gamma_0(\ell_0), \gamma_2(0))$ *is strictly increasing.*

This lemma follows from the cosine law in the model space.

Lemma 2.0.3 (Cosine Law) *) If* $\{\gamma_0, \gamma_1, \gamma_2\}$ *is a geodesics triangle in* M_H^2 *with each* γ_i *minimal, then*

$$\cos\sqrt{H}\ell_2 = \cos\sqrt{H}\ell_0 \cos\sqrt{H}\ell_1 + \sin\sqrt{H}\ell_0 \sin\sqrt{H}\ell_1 \cos\alpha_2.$$

In particular,

$$\begin{aligned} H = 0 &\Rightarrow \ell_2^2 = \ell_0^2 + \ell_1^2 - 2\ell_0\ell_1\cos\alpha_2 \\ H = -1 &\Rightarrow \cosh\ell_2 = \cosh\ell_0\cosh\ell_1 - \sinh\ell_0\sinh\ell_1\cos\alpha_2 \\ H = 1 &\Rightarrow \cos\ell_2 = \cos\ell_0\cos\ell_1 + \sin\ell_0\sin\ell_1\cos\alpha_2. \end{aligned}$$

Note that the Cosine law implies that ℓ_2 is an increasing function of α_2.

Proof Let $p = \gamma_0(\ell_0)$. Consider the distance function $\rho_p \circ \sigma(t)$. We will use a modified distance function. Set

$$md_H(r) = \begin{cases} r^2/2 & \text{if } H = 0 \\ 1 - \cos r & \text{if } H = 1 \\ \cosh r - 1 & \text{if } H = -1 \end{cases}$$

Locally, these functions are $r^2/2$.

In general, let s_H and c_H be solutions of

$$f'' + Hf = 0$$

with $s_H(0) = 0$, $c_H(0) = 1$, $s'_H(0) = 1$ and $c'_H(0) = 0$. For $H > 0$,

$$s_H = \frac{\sin\sqrt{H}t}{\sqrt{H}} \text{ and } c_H = \cos\sqrt{H}t.$$

Then

$$\begin{aligned} md_H(r) &= \int_0^r s_H(t)dt \\ &= (1 - c_H)/H. \end{aligned}$$

In M^n_H, $\mathrm{Hess}\,\rho_p$ has matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & c_H/s_H & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & c_H/s_H \end{pmatrix},$$

with respect to the basis $\{N, \partial/\partial\theta_i\}$.

Claim The eigenvalues of the Hessian of $md_H \circ \rho_p$ in M^n_H are identical.

We have

$$\begin{aligned} \nabla md_M \circ \rho &= s_H(\rho)\nabla\rho \\ &= s_H(\rho)N, \end{aligned}$$

so

$$\begin{aligned} \mathrm{Hess}\,(md_H \circ \rho) &= s_H(\rho)\mathrm{Hess}\,\rho + c_H(\rho)N \otimes N \\ &= c_H(\rho)Id. \end{aligned}$$

In the above, we used that

$$\nabla(h \circ f) = h'(f)\nabla f,$$

so

$$\mathrm{Hess}\,(h \circ f) = h''(f)\nabla f \otimes \nabla f + h'(f)\mathrm{Hess}\,f.$$

Now $\rho \circ \sigma(t)$ gives the distance along $\sigma(t)$. Set $\varphi(t) = md_H \circ \rho \circ \sigma(t)$. Since $\sigma(t)$ is normalized,

$$\begin{aligned} \varphi''(t) &= \mathrm{Hess}\,(md_H \circ \rho)(\sigma'(t), \sigma'(t)) \\ &= c_H(\rho \circ \sigma(t)) \\ &= 1 - H\varphi \end{aligned}$$

Thus $\varphi'' + H\varphi = 1$. Let $\sigma = \gamma_2$ and $p = \gamma_0(\ell_0)$. Then

$$\varphi(0) = md_H(\ell_1) = \begin{cases} \ell_1^2/2, & H = 0 \\ 1 - \cos\ell_1, & H = 1 \\ \cosh\ell_1 - 1, & H = -1 \end{cases}$$

and

$$\begin{aligned}\varphi'(0) &= \langle \nabla md_H \circ \rho, \gamma_2'(0)\rangle \\ &= \gamma_2'(0)(md_H \circ \rho) \\ &= s_H \cos(\pi - \alpha_0) \\ &= -s_H \cos\alpha_0.\end{aligned}$$

When $H = 0$, $\varphi''(t) = 1$. Integrating both sides gives

$$\begin{aligned}\varphi(t) &= \varphi(0) + \varphi'(0)t + (1/2)t^2 \\ &= (1/2)\ell_1^2 - \ell_1 \cos\alpha_0 t + (1/2)t^2,\end{aligned}$$

whence

$$(1/2)\ell_0^2 = \varphi(\ell_2) = (1/2)\ell_1^2 - \ell_1 \cos\alpha_0 \ell_2 + (1/2)\ell_2^2.$$

Similarly for $H = -1$ and $H = 1$.

By the lemmas, the versions of Toponogov are equivalent. We give a proof of the hinge version. The proof uses an estimate for the Hessian of the distance function ρ. For Ricci curvature, we considered $\Delta\rho$. The integral version of the Laplacian comparison implied the volume comparison. For sectional curvature, we consider $\mathrm{Hess}\,\rho$. The integral version of the Hessian comparison will imply the Toponogov comparison.

We give an estimate on the Hessian of distance functions.

$$\begin{aligned}\mathrm{Hess}\,(\rho_p)(X,Y) &= \langle \nabla_X grad\rho, Y\rangle \\ &= \langle \nabla_X N, Y\rangle\end{aligned}$$

Let $S(X) = \nabla_X N$; this is the Hessian of ρ as a linear operator. S is called the shape operator, or the second fundamental form of the geodesic sphere. Let $R_N(X) = R(X,N)N$ be the curvature tensor in N. Then $\mathrm{Hess}\,\rho_p$ satisfies the following Riccati equation.

Lemma 2.0.4 *S satisfies* $S' = \nabla_N S = -R_N - S^2$.

Proof We show true for every X.

$$\begin{aligned}R_N(X) &= R(X,N)N \\ &= \nabla_X\nabla_N N - \nabla_N\nabla_X N - \nabla_{[X,N]}N \\ &= -\nabla_N\nabla_X N - \nabla_{\nabla_X N}N + \nabla_{\nabla_N X}N,\end{aligned}$$

while $\nabla_{\nabla_X N} N = S^2(X)$ and

$$\begin{aligned}(\nabla_N S)X &= \nabla_N(S(X)) - S(\nabla_N X) \\ &= \nabla_N(\nabla_X N) - \nabla_{\nabla_N X} N.\end{aligned}$$

Remark By taking traces we obtain $M' = -\mathrm{Ric} - \|\mathrm{Hess}\,\rho\|^2$.

Proof of Toponogov Theorem (Hinge Version) Let $\{\gamma_0, \gamma_1, \alpha_2\}$ be a hinge in M^n, and let $\{\bar\gamma_0, \bar\gamma_1, \alpha_2\}$ be a corresponding hinge in M^n_H. Consider

$$\begin{aligned}\varphi(t) &= md_H \circ \rho_p \circ \gamma_1(t) \\ \bar\varphi(t) &= md_H \circ \bar\rho_{\bar p} \circ \bar\gamma_1(t)\end{aligned}$$

Then $\bar\varphi''(t) + H\bar\varphi(t) = 1$, while

$$\begin{aligned}\varphi''(t) &= \mathrm{Hess}\,(md_H \circ \rho_p)(\gamma_1'(t), \gamma_1'(t)) \\ &\leq c_H = 1 - H\varphi(t).\end{aligned}$$

Thus $\varphi''(t) + H\varphi(t) \leq 1$. Since md_H is increasing, we want to show that $\bar\varphi(\ell_1) \geq \varphi(\ell_1)$.

We have $\bar\varphi(0) = md_H(\ell_0) = \varphi(0)$ and

$$\begin{aligned}\bar\varphi'(0) &= \langle grad(md_H) \circ \rho, \gamma_1'(0)\rangle \\ &= s_H \langle \gamma_0'(\ell_0), \gamma_1'(0)\rangle \\ &= s_H \cos(\pi \cdot \alpha_2) \\ &= \varphi'(0)\end{aligned}$$

Let $\psi(t) = \bar\varphi(t) - \varphi(t)$. Then $\psi''(t) + H\psi(t) \geq 0$ in the support sense. Also, $\psi(0) = 0$ and $\psi'(0) = 0$. We want to show that $\psi(\ell_1) \geq 0$.

At smooth points,

$$H = 0 : \psi''(t) \geq 0 \Rightarrow \psi(t) \geq 0,$$

$$H = -1 : \psi''(t) \geq \psi(t) \Rightarrow \psi(t) \geq 0 \text{ for } t > 0.$$

At non-smooth points, we use support. In this case, by continuity, we may assume $\psi(0) \geq \varepsilon > 0$ and $\psi'(0) \geq \delta > 0$.

If $H = 1$, $\psi''(t) \geq -\psi(t)$. Compare with $\zeta''(t) = -(1+\eta)\zeta(t), \eta > 0$. If $\zeta > 0$, we have $\psi(t) \geq \zeta(t)$ by the Sturm-Liouville Comparison.

Note that

$$\begin{aligned}\zeta(t) &= \left(\frac{\delta}{\sqrt{1+\eta}}\right)\sin(t\sqrt{1+\eta}) + \varepsilon\cos(t\sqrt{1+\eta}) \\ &= \left(\sqrt{\frac{\delta^2}{1+\eta}+\varepsilon^2}\right)\sin\left(t\sqrt{1+\eta}+\arctan\left(\frac{\varepsilon\sqrt{1+\eta}}{\delta}\right)\right)\end{aligned}$$

Thus $\zeta(t) > 0$ if

$$t < \frac{(\pi - \arctan(\frac{\varepsilon\sqrt{1+\eta}}{\delta}))}{\sqrt{1+\eta}}.$$

By continuity, $\zeta(t) > 0$ if $t < \pi$. In this case, $\zeta(t) \geq 0$ for $t \leq \pi$, which proves the result.

3 Main Results

3.1 Sphere Theorem

Theorem 3.1.1 (Rauch-Berger-Klingenberg ($\frac{1}{4}$)-pinching sphere theorem) *If M is simply connected and $1 \leq K_M < 4$ then M^n is homeomorphic to S^n.*

By Klingenberg's injectivity radius estimate, the ($\frac{1}{4}$)-pinching sphere theorem follows from the diameter sphere theorem.

Theorem 3.1.2 (Klingenberg 1961) *If M^n is simply connected and $1 \leq K_M < 4$ then* $\mathrm{inj}(M) > \pi/2$[1].

Theorem 3.1.3 (Diameter Sphere Theorem (Grove-Shiohama 1977)) *If M^n has $K_M \geq 1$ and* $\mathrm{diam}(M) > \pi/2$ *then* $M^n \overset{\text{homeo}}{\simeq} S^n$.

Proof Note that M is compact by Bonnet-Myers Theorem. Let $p, q \in M^n$ with $d(p,q) = \mathrm{diam}(M)$.

Claim If $x \neq p, q$ then x is not a critical point with respect to p. (x is not a critical point of the distance function at p.)

Proof of Claim Connect x, q by a minimal geodesic γ_2 If x is a critical point of p there is a minimal geodesic γ_1 connecting x and p with $\angle(\gamma_1'(0), \gamma_2'(0)) = \alpha \leq \pi/2$.

[1] implies $\mathrm{diam}(M) > \pi/2$.

Set $\ell(\gamma_i) = \ell_i$, and let $\ell = d(p,q)$. Note that $\ell > \pi/2$. Apply Toponogov comparison to the hinge $\{\gamma_1, \gamma_2, \alpha\}$ to obtain $\ell \leq \bar{\ell}$.

By the cosine law for S^2,

$$\cos\bar{\ell} = \cos\ell_1 \cos\ell_2 + \sin\ell_1 \sin\ell_2 \cos\alpha.$$

By Bonnet Myers Theorem, $\mathrm{diam}(M) \leq \pi$. Thus $\pi/2 < \ell \leq \pi$, whence

$$\cos\ell \geq \cos\ell_1 \cos\ell_2 + \sin\ell_1 \sin\ell_2 \cos\alpha.$$

Since $\sin(\ell_1)\sin(\ell_2), \cos(\alpha) \geq 0$, $\cos(\ell_1)\cos(\ell_2) \leq \cos(\ell) < 0$. Now if $d(p,x) = \pi$ then by maximal diameter rigidity result, $M^n \simeq S^n$. Otherwise, we have $-1 < \cos\ell_2 < 0$. Now $\cos\ell_1$ and $\cos\ell_2$ have opposite sign by the above, so in this case we have $0 < \cos\ell_1 < 1$. But then $\cos\ell > \cos\ell_2$, which implies $\ell < \ell_2$. This contradiction shows that x is not a critical point of p.

As in the proof of the isotropy lemma, we have $M^n \overset{homeo}{\simeq} S^n$.

Remark The $(\frac{1}{4})$-pinching sphere theorem and diameter sphere theorem are optimal: $\mathbb{C}\mathrm{P}^n$ has $1 \leq K_M \leq 4$, while $\mathbb{R}\mathrm{P}^n$ has $K_M \equiv 1$ and diam $= \pi/2$.

Open question Can we prove M is diffeomorphic to S^n in Sphere Theorem? If $0.68 \leq K_M < 1$ then yes; maybe weaker hypotheses are sufficient.

Regarding the previous remark, we have:

Theorem 3.1.4 (Berger) *If M^n is simply connected, $1 \leq K_M \leq 4$ then M^n is homeomorphic to S^n or is isometric to $\mathbb{C}\mathrm{P}^{n/2}$, $\mathbb{H}\mathrm{P}^{n/4}$ or the Cayley plane* CaP^2.

The last three spaces are called compact rank one symmetric spaces, or CROSS.

Theorem 3.1.5 (Gromov-Grove 1987) *If M^n has $K_M \geq 1$ and* $\mathrm{diam}(M) = \pi/2$ *then $M^n \overset{homeo}{\simeq} S^n$, or M is locally isometric to a finite quotient of a CROSS.*

Wilking finished the proof in 1999.

What if $1 \leq K_M \leq 4+\epsilon$? In this case we have a below $(\frac{1}{4})$-pinching sphere theorem:

Suppose M^n is simply connected and $1 \leq K_M \leq 4+\epsilon$. If n is even (Berger) and $\epsilon < \epsilon(n) > 0$ then M^n is homeomorphic to S^n or is diffeomorphic to a CROSS. If n is odd (Aberesch-Myers 1994) and $\epsilon < \epsilon_0 \approx 10^{-6}$ then M is homeomorphic to S^n.

3.2 Soul Theorem (Cheeger-Gromoll)

We start with a mini Soul theorem, due to Gromov:

Prop 3.2.1 *Let M^n be complete, noncompact with $K_M \geq 0$. Then for each $p \in M$ there is $R > 0$ such that p has no critical points outside $B(p, R)$. In particular, M is homeomorphic to the interior of a compact manifold with boundary.*

This proposition is not true for Ric ≥ 0. For example (Sha-Yang),

$$(S^2 \times S^2)\#(S^2 \times S^2)\# \cdots$$

has Ric > 0.

Proof To prove the proposition we first prove a lemma.

Lemma 3.2.1 *Suppose q_1 is a critical point of p and q_2 satisfies $d(p, q_2) \geq vd(p, q_1)$, $v > 1$. Let γ_i be a minimal geodesic connecting p, q_i, $\theta = \angle(\gamma_1'(0), \gamma_2'(0))$. Then*

1. *If $K_M \geq 0$ then $\theta \geq \arccos(1/v)$.*
2. *If $K_M \geq H < 0$ and $d(p, q_2) \leq D$ then*

$$\theta \geq \arccos\left(\frac{\tanh(\sqrt{-H}D/v)}{\tanh(\sqrt{-H}D}\right)$$

Proof As usual, set $\ell_i = d(p, q_i)$. We have $\ell_2 \geq v\ell_1$. Connect q_1 and q_2 with a minimal geodesic γ and let $\ell = L(\gamma)$.

Since q_1 is a critical point of p, we can choose a second minimal geodesic σ connecting q_1 and p with $\angle(\sigma'(0), \gamma_1'(0)) = \alpha \leq \pi/2$. Now apply Toponogov comparison to the two hinges $\{\gamma_1, \gamma_2, \theta\}$ and $\{\sigma, \gamma_1, \alpha\}$.

Case 1 $K_M \geq 0$

$$\begin{aligned} \ell_2^2 \leq \bar{\ell_2}^2 &= \ell_1^2 + \ell^2 - 2\ell\ell_1 \cos\alpha \\ &\leq \ell_1^2 + \ell^2 \qquad \text{[since } \alpha \leq \pi/2\text{]} \end{aligned}$$

We also have

$$\ell^2 \leq \bar{\ell}^2 = \ell_1^2 + \ell_2^2 - 2\ell_1\ell_2\cos\theta.$$

Thus

$$\ell_2^2 \leq \ell_1^2 + \ell_2^2 - 2\ell_1\ell_2\cos\theta,$$

so

$$\ell_1\ell_2\cos\theta \leq \ell_1^2$$

and

$$\cos\theta \leq \ell_1/\ell_2 \leq 1/v.$$

Case 2 $K_M \geq H < 0$.

Suppose $H = -1$. In this case,

$$\begin{aligned} \cosh\ell_2 \leq \cosh\bar{\ell}_2 &= \cosh\ell_1\cosh\ell - \sinh\ell_1\sinh\ell\cos\alpha \\ &\leq \cosh\ell_1\cosh\ell \qquad\qquad [\text{since } \alpha \leq \pi/2] \end{aligned}$$

Also,

$$\cosh\ell \leq \cosh\bar{\ell} = \cosh\ell_1\cosh\ell_2 - \sinh\ell_1\sinh\ell_2\cos\theta,$$

so

$$\cosh\ell_2 \leq \cosh\ell_1(\cosh\ell_1\cosh\ell_2 - \sinh\ell_1\sinh\ell_2\cos\theta).$$

Thus

$$\begin{aligned} \sinh\ell_1\sinh\ell_2\cos\theta &\leq \cosh\ell_1\cosh\ell_2 - \frac{\cosh\ell_2}{\cosh\ell_1} \\ &= \cosh\ell_2\left[\frac{\cosh\ell_1 - 1}{\cosh\ell_1}\right], \end{aligned}$$

$$\sinh\ell_2\cos\theta \leq \cosh\ell_2\tanh\ell_1$$

and

$$\cos\theta \leq \frac{\tanh\ell_1}{\tanh\ell_2}.$$

Now $\ell_2 \leq D$, so $\ell_1 \leq \ell_2/v \leq D/V$. Since $\ell_2 \geq v\ell_1$,

$$\cos\theta \leq \frac{\tanh\ell_1}{\tanh(v\ell_2)} \leq \frac{\tanh(D/v)}{\tanh D},$$

as $\dfrac{\tanh x}{\tanh(vx)}$ is increasing when $v > 1$.

Corollary Let $q_1, \ldots, q_N$ be a sequence of critical points of p with

$$d(p, q_{i+1}) \geq vd(p, q_i). \qquad (v > 1)$$

Then

1. If $K_M \geq 0$, $N \leq \mathcal{N}(n, v)$.
2. If $K_M \geq H < 0$ and $d(p, q_N) \leq D$ then $N =\leq \mathcal{N}(n, v, HD^2)$.

Proof Connect p, q_i with minimal geodesics γ_i. Set $\angle(\gamma_i'(0), \gamma_j'(0)) = \theta_{ij}$. By the lemma, if $K_M \geq 0, \theta_{ij} \geq \arccos(1/v)$ for all $i \neq j$. But $\gamma_i'(0) \in S^{n-1} \subset T_pM$, and $d_{S^{n-1}}(\gamma_i'(0), \gamma_j'(0)) = \theta_{ij}$. Hence the balls $B(\gamma_i'(0), \arccos(1/v)/2) \subset S^{n-1}$ are disjoint. But there can only be $\mathcal{N}(n, v)$ such balls. Similarly for $K_M \geq H < 0$.

Corollary Mini version of soul theorem (by isotopy lemma).

We can now prove the structure theorem for noncompact manifolds.

Theorem 3.2.1 (Cheeger-Gromoll 1972) *If M^n is a complete, noncompact Riemannian manifold with $K_M \geq 0$ then M contains a soul S that is a compact, totally convex submanifold such that M is diffeomorphic to the normal bundle of S. Moreover, if $K_M > 0$ then S is a point, so $M \overset{diffeo}{\simeq} \mathbb{R}^n$.*

Remark The second result was first proved by Gromoll-Myers.

Definition 3.2.1 *A subset $A \subset M$ is called totally convex if for any $p, q \in A$ and $\gamma(t)$ a geodesic in M connecting p and q we have $\gamma(t) \in A$ for all t.*

Remarks

1. A point can only be totally convex if there are no geodesic loops in M based at x. When M is closed, a point is never totally convex. In fact, if M is closed there are no totally convex there are no totally convex proper subsets.
2. Totally convex $\Rightarrow$ Totally geodesic, but not conversely. For example, great circles in S^2 are not totally convex.
3. Totally convex sets are interesting for Morse theory. If $A \subset M$ is totally convex, pick $p, q \in A$. Consider the energy function E on $\Lambda(p, q)$ (= path space). The critical points of E all lie in A. Thus the topology of M is similar to the topology of A. In fact, M is homotopic to A.

Example 3.2.1 *South pole of a paraboloid is the unique totally convex set.*

Example 3.2.2 *Meridinal circles on a cylinder (not unique).*

Question How do we find totally convex sets?

Lemma 3.2.2 *If $f : (M, g) \to \mathbb{R}$ is concave (*Hess f *is weakly nonpositive) then*

$$A = \{x \in M : f(x) \geq a\}$$

is totally convex in M.

Proof If $\gamma(t)$ is a geodesic in M, $(f \circ \gamma)(t)$ is concave. Thus the minimum of $(f \circ \gamma)(t)$ is realized at the end points.

Note that intersections of totally convex sets are totally convex.

Prop 3.2.2 *Suppose M^n is complete, noncompact with $K_M \geq 0$ and $p \in M$. Let $\{\gamma_\alpha\}$ be all rays beginning at p. Set*

$$f = \inf_\alpha b_{\gamma_\alpha},$$

where b_{γ_α} is the Busemann function associated with γ_α :

$$b_{\gamma_\alpha}(x) = \lim_{t\to\infty}(d(x, \gamma_\alpha(t)) - t) = \lim_{t\to\infty} b^t_{\gamma_\alpha}(x).$$

Then f is both proper and concave.

Proof Similar to $\mathrm{Ric}_M \geq 0$, $\Delta b_{\gamma_\alpha} \leq 0$. We want to show that $K_M \geq 0$ implies Hess $b_{\gamma_\alpha} \leq 0$ in the support sense. Note that

$$\begin{aligned} \text{Hess } b^t_{\gamma_\alpha} &= \text{Hess } \rho_{\gamma_\alpha(t)} \\ &\leq \text{Hess } \bar{\rho} \\ &= \begin{pmatrix} 0 & & & \\ & 1/\rho & & \\ & & \ddots & \\ & & & 1/\rho \end{pmatrix} \leq (1/\rho)Id. \end{aligned}$$

Let $t \to \infty$. Then $\rho \to \infty$, so Hess $b_{\gamma_\alpha}(x) \leq 0$.

Thus b_{γ_α} is concave. Hence f is concave, as the infimum of concave functions is concave.

Next we show $f^{-1}[a, \infty)$ is compact for $a < 0$. Since $b_{\gamma_\alpha}(p) = 0, f(p) = 0$. Thus $f^{-1}[a, \infty) = A$ is nonempty. Also A is closed. Thus if A is not compact, A is not bounded. Suppose there is $\{p_n\} \subset A$ with $d(p, p_n) \to \infty$.

Connect p and p_n with a minimal geodesic $\gamma_n(t)$. Since A is totally convex, these geodesics lie in A. Some subsequence of $\gamma_n(t)$ converges to a ray $\gamma(t)$ in A. Consider $f(\gamma(t))$.

$$f(\gamma(t)) \leq b_\gamma(\gamma(t)) = -t \to -\infty$$

But $\gamma \subset A$ since A is closed. This contradiction shows that A is compact. Thus f is proper.

Remark Soul Theorem does not hold for $\mathrm{Ric}_M \geq 0$. As mentioned above,

$$(S^2 \times S^2)\#(S^2 \times S^2)\# \cdots$$

has $\mathrm{Ric} > 0$ (Sha-Yang).

Question If M has $\mathrm{Ric} \geq 0$ is $f = \inf_\alpha b_{\gamma_\alpha}$ a proper function?

Returning to the Soul theorem, the above lemma and proposition show that there is a compact, totally convex subset $A \subset M$. Still, A may not be a submanifold. We will try to find the smallest totally convex set. First we prove some properties of convex sets.

Prop 3.2.3 *If $A \subset M$ is a totally convex set then A has interior* $\mathrm{int}(A)$ *and boundary ∂A in the sense that the boundary has the supporting hyperplane property and the interior lies strictly on the open half space cut out by each supporting hyperplane. Moreover,* $\mathrm{int}(A)$ *is a totally convex submanifold. For each $x \in \partial A$ there is $v \in T_xM$ such that $\angle(\gamma'(0), v) < \pi/2$ for all geodesics $\gamma : [0, a] \to A$ with $\gamma(0) = x$, $\gamma(a) \in$* $\mathrm{int}(A)$. *Thus ρ_q, $q \in$* $\mathrm{int}(A)$, *has no critical points on ∂A.*

Proof First we identify the interior and the boundary points. Find a maximal integer k such that A contains a k-dimensional submanifold of M. Note that if $K = 0$ then A is a point. Let $N \subset A$ be the union of all k-dimensional submanifolds of M contained in A. Set $\mathrm{int}(A) = N$ and $\partial A = A - N$. For all $p \in N$ there is $N_p \subset A$, a k-dimensional submanifold containing p.

There is a chart $\phi : B(p, \delta) \to \mathbb{R}^n$ such that

$$\phi(B(p, \delta) \cap N_p) = \phi(B(p, \delta)) \cap \mathbb{R}^k.$$

Claim $B(p,\delta)\cap N_p = B(p,\delta)\cap N$ for δ small.

Proof of Claim Suppose $q \in B(p,\delta)\cap N - B(p,\delta)\cap N_p$. Connect p and q with a unique minimal geodesic $\gamma : [0,a] \to N$. Consider $B(p,\delta)\cap N_p$ as a subset of $\mathbb{R}^n$. Connect q with each point in $B(p,\delta)\cap N_p$; this forms a cone over a k-dimensional submanifold, and gives a $(k+1)$-dimensional submanifold away from q. By convexity, this submanifold lies in A, which contradicts the maximality of k.

Thus $B(p,\delta)\cap N_p = B(p,\delta)\cap N$ for δ small. Note, we choose

$$\delta < \frac{\mathrm{inj}(p)}{2},$$

so q is connected uniquely (and hence smoothly) to $B(p,\delta)\cap N_p$.

Next, N is dense in A : For any $p \in A$ take $q \in N$. Let $\gamma : [0,a] \to A$ be a minimal geodesic connecting p and q. Then $\gamma[0,a) \subset N$.

Now we establish the supporting hyperplane property. For $p \in \partial A$, let C_p be the cone of vectors $v \in T_pM$ such that $exp_p(tv) \in \mathrm{int}(A)$ for some $t > 0$. Then C_p is open in $\mathrm{span}\, C_p$ and $\dim(\mathrm{span}\, C_p) = k$.

Claim If $q \in \mathrm{int}(A)$, $p \in \partial A$ such that $d(p,q) = d(q,\partial A)$ then

$$C_p = H = \{v \in \mathrm{span}\, C_p : \angle(\gamma'(0), v) < \pi/2\},$$

where γ is a minimal geodesic connecting p and q. In this case we say there is a unique supporting hyperplane at p.

Proof of Claim Suppose $\gamma : [0,d] \to A$. Choose $0 < s < d$ so

$$d(\gamma(s), p) < \frac{\mathrm{inj}(p)}{2}.$$

Then $\gamma|_{[0,s]}$ realizes the distance from $\gamma(s)$ to ∂A, so $B(\gamma(\bar{s}), s)\cap \partial A = p$ and $H \subset C_p$. Suppose there is $v \in C_p - H$. Since C_p is open, we may assume $\angle(\gamma'(0), v) > \pi/2$ Thus $-v \in H \subset C_p$. But then $q \in \mathrm{int}(A)$. This contradiction proves the claim.

Prop 3.2.4 *Let (M,g) be complete with $K_M \geq 0$ and $A \subset M$ totally convex. Then $d : A \to \mathbb{R}$ defined by $d(x) = d(x,\partial A)$ is concave in A. Moreover, d is strictly concave if $K_M > 0$.*

Proof We show $\mathrm{Hess}\,(d) \leq 0$ in support sense. For any $q \in int(A)$, let $p \in \partial A$ realize $d(q)$. Note that we work with A compact. Connect p, q with a minimal geodesic $\sigma(t)$. Also, there is a supporting hyperplane H at p.

Set $\bar{H} = \{exp_p(tv) : v \in H, t \leq \epsilon\}$. Then $\bar{H}$ is a submanifold that is totally geodesic at p. Also, $\bar{H} \cap \mathrm{int}(A) = \emptyset$ and $d(x, \bar{H})$ is a support function of $d(x, \partial A)$ at q from above.

The Hessian of distance functions satisfies the Ricatti equation $S' = -R_N - S^2$, so

$$\begin{aligned} \nabla_{\sigma'(0)}(\nabla^2 d(\cdot, \bar{H}) &= -R_{\sigma'(0)} - (\nabla^2 d(\cdot, \bar{H}))^2 \\ &\leq -(\nabla^2 d(\cdot, \bar{H}))^2. \end{aligned}$$

Observe that we have used that fact that $K_M \geq 0$ to obtain this inequality.

Since $\bar{H}$ is totally geodesic, the 2^{nd} fundamental form is 0, so $(\nabla^2 d(\cdot, \bar{H}))|_{\sigma(0)} = 0$.

Thus $\nabla^2 d(\cdot, \bar{H}) \leq 0$; this inequality is strict if $K_M > 0$.

If $d(\cdot, \bar{H})$ is smooth at q, the proof is complete. If $d(\cdot, \bar{H})$ is not smooth at q, consider $t + d(\cdot, \bar{H}_t)$. This is a support function for $d(\cdot, \bar{H})$ at q. (Consider $\sigma(t)$ instead of $\sigma(0) = p$, $H_t = exp_{\sigma(t)} H$)

Proof of Soul Theorem M^n, $K_M \geq 0$, $f = \inf_\alpha b_{\gamma_\alpha}$, $A = f^{-1}[a, \infty)$, $a < 0$.

We showed that A is a compact totally convex set. If A is a point or has no boundary, we have constructed the soul. Otherwise, $\partial A \neq \emptyset$. Set

$$C_1 = \{x \in A : d(x, \partial A) \text{ is maximal on } A\}.$$

By the previous result, C_1 is totally convex ($C_1 = f^{-1}[\max, \infty)$, where $f(x) = d(x, \partial A)$).

If C_1 is a submanifold, we are done. Otherwise repeat and get a sequence of totally convex compact sets

$$A \supset C_1 \supset \cdots$$

Claim This sequence has at most $(n-1)$ steps.

Proof of Claim We prove that $\dim C_i > \dim C_{i+1}$. Suppose $\dim C_i = \dim C_{i+1}$. Then there exists $B(q, \delta)$ such that

$$B(q, \delta) \cap \mathrm{int}(C_i) = B(q, \delta \cap \mathrm{int}(C_{i+1}).$$

Let $\sigma(t)$ be a minimal geodesic connecting q and $p \in \partial C_i$, $d(q, p) = d(q, \partial C_i)$. Then $d(\cdot, \partial C_i)$ is strictly increasing along $\sigma(t)$. But $sigma(t)$ passes through $p \in \partial C_i$, $d(q, p) = d(q, \partial C_i)$, which is a contradiction.

To see that M^n is diffeomorphic to the normal bundle over its soul, we use

Theorem 3.2.2 (Perelman 1994) *The map* $\mathrm{Sh} : M \to S$ *is a submersion and* $K(H,V) = 0$ *(*H *= Horizontal,* V *= Vertical). In particular, if* $K_M(p) > 0$ *at some point* $p \in M$ *the soul is a point. In this case* $M \overset{diffeo}{\simeq} \mathbb{R}^n$.

Question Given a closed S with $K_S \geq 0$ what vector bundles over S admit a metric with $K \geq 0$?

Theorem 3.2.3 (Ozaydu-Walschap 1994) *If* $S = T^2$ *only the trivial bundle has* $K \geq 0$.

Theorem 3.2.4 (Belegradek-Kapovitch) *Similar results for* T^k, $C \times T^k$ *only finitely many bundles have* $K \geq 0$.

The question is open $|\pi_1(S)| < \infty$. No obstructions known.

3.3 Finiteness Theorem

We start with an overview.

Theorem 3.3.1 (Cheeger 1970) *The class of manifolds* M^n *with* $|K_M| \leq H$, $\mathrm{vol}_M \geq V$, $\mathrm{diam}(M) \leq D$ *has only finitely many diffeomorphism types. In fact, this class is* $C^{1,\alpha}$ *compact in the Gromov-Hausdorff topology.*

Theorem 3.3.2 (Grove-Petersen 1988) M^n *with* $K_M \geq H$, $\mathrm{vol}_M \geq V$, $\mathrm{diam}(M) \leq D$ *has only finitely many homotopy types.*

Remark Perelman showed this result for homeomorphism types.

Theorem 3.3.3 (Gromov 1981) M^n, $K_M \geq H$, $\mathrm{diam}(M) \leq D$ *has finite Betti numbers:*

$$\sum_{i=0}^{n} b_i(M;F) \leq C(n, HD^2).$$

Theorem 3.3.4 (Gromov) *Suppose* M^n *is complete with* $K_M \geq 0$*. Then* $\pi_1(M)$ *can be generated by* $C(n)$ *generators, where* $C(n)$ *depends only on* n.

Proof (c.f. Sormani's result that small linear diameter growth and $\mathrm{Ric}_M \geq 0$ implies $\pi_1(M)$ finitely generated).

Fix $p \in \tilde{M}$. Select generators $g_1, g_2, \ldots$ of $\pi_1(M)$ as follows:

i) $d(p, g_1(p)) \leq d(p, g(p))$ for all $g \in \pi_1(M)$

ii) $d(p, g_k(p)) \leq d(p, g(p))$ for $g \in \pi_1(M) - \langle g_1, \ldots, g_{k-1} \rangle$

Join p and $g_k(p)$ by minimal geodesics σ_k.
Claim $\gamma_{ij} = \angle(\sigma_i'(0), \sigma_j'(0)) \geq \pi/3$ for all $i \neq j$.
To prove the claim, note that for $i < j$, $g_i^{-1} g_j \in G - \langle g_1, \ldots, g_{j-1} \rangle$. Hence

$$d(p, g_i p) \leq d(p, g_i^{-1} g_j p) = d(g_i p, g_j p).$$

We now use the hinge version of Toponogov comparison $\{\sigma_i, \sigma_j, \gamma_{ij}\}$. Since $K_{\tilde{M}} \geq 0$, we compare with $\mathbb{R}^n$:

$$\begin{aligned} d(g_i p, g_j p) &\leq d(p, g_i p)^2 + d(p, g_j p)^2 - 2d(p, g_i p) d(p, g_j p) \cos \gamma_{ij} \\ \ell_{ij} &\leq \ell_i^2 + \ell_j^2 - 2\ell_i \ell_j \cos \gamma_{ij} \end{aligned}$$

If $\gamma_{ij} < \pi/3$, we have

$$\begin{aligned} \ell_{ij}^2 &< \ell_i^2 + \ell_j^2 - \ell_i \ell_j \\ &< \ell_j^2, \end{aligned}$$

since $\ell_j \geq \ell_i$ implies $-\ell_i \ell_j \leq -\ell_i^2$. But then $d(g_i p, g_j p) < d(p, g_j p)$, which is a contradiction.

Thus $\angle(\sigma_i'(0), \sigma_j'(0)) \geq \pi/3$ for $i \neq j$, so each generator can be placed in a ball of radius $pi/6$ which is disjoint from the like balls about the other generators. Hence there are at most

$$C(n) = \frac{\text{vol}(n-1, 1, \pi)}{\text{vol}(n-1, 1, \pi/6)}$$

generators.

Remark A similar result holds for M^n with $K_M \geq H$ and $\text{diam}(M) < D/2$; the number of generators of $\pi_1(M)$ is at most $2(3 + 2\cosh\sqrt{-H}D)$. The proof is similar, but compares with hyperbolic space.

Theorem 3.3.5 (Gromov 1981) *Suppose M^n is complete with $K_M \geq H$,* $\text{diam}(M) \leq D$. *Then*

$$\sum_{i=0}^{n} b^i(M, F) \leq C(n, HD^2).$$

Moreover, if $K_M \geq 0$ *then*

$$\sum_{i=0}^{n} b^i(M,F) \leq C(n).$$

Consequence $\mathbb{C}P^2\#\cdots\#\mathbb{C}P^2$ has no metric with $K_M \geq 0$. (But it does have a metric with $\mathrm{Ric} \geq 0$).

Question If $K_M \geq 0$ is $\sum_{i=0}^{n} b^i(M,F) \leq 2^n$?

Mayer-Vietoris gives a way to estimate Betti numbers of union of balls in terms of each single ball and the intersections. If the balls are small, we know Betti numbers (since we work with manifolds). To this end we define the content of a ball as

$$\mathrm{Cont}(p,r) = \mathrm{Cont}(B(p,r)) = \sum_{i=0}^{n} \mathrm{rank}(H_i(B(p,r)) \to H_i(B(p,5r))).$$

One would expect to define $\mathrm{Cont}(p,r)$ as $\sum_{i=0}^{n} b_i(B(p,r))$. Content, as we have defined it, is related to $\sum b_i$, though; consider $r = \mathrm{diam}(M)$. For r small (inside the injectivity radius), $\mathrm{Cont}(p,r) = 1$. $\mathrm{Cont}(p,r)$ is better than $\sum b_i$.

Prop 3.3.1 *If* $K_M \geq 0$ *we have* $\mathrm{Cont}(p,r) \leq (n+1)2^{N(r/10^{n+1},r)}$, *where* $N(r/10^{n+1},r)$ *is the number of balls of radius* $r/10^{n+1}$ *required to cover the ball* $B(p,r)$.

Remark Using $\mathrm{Ric} \geq 0$ we have a bound on N.

We say that $B(p,r)$ compresses to $B(q,s)$, and write $B(p,r) \to B(q,s)$, provided $5s + d(p,q) \leq 5r$ and there is a homotopy $f_t : B(p,r) \to B(p,5r)$ with f_0 the inclusion, $f_1(B(p,r)) \subset B(q,s)$. Thus $B(q,5s) \subset B(p,5r)$ and $B(p,r)$ is deformable to $B(q,s)$ inside $B(p,5r)$. Note that in this case

$$\mathrm{Cont}(p,r) \leq \mathrm{Cont}(q,s).$$

$B(p,r)$ is incompressible if $B(p,r) \to B(q,s)$ implies $s > r/2$. We start with $B(p,r) = M$, and compress until incompressible. Then we reduce to balls of one tenth the radius, and repeat. The number of steps needed from $B(p,r)$ to contractible ball is $\mathrm{rank}(p,r)$. Then

$$\mathrm{Cont}(p,r) \leq [(n+1)2^{N(r/10^{n+1},r)}]^{\mathrm{rank}(p,r)}.$$

Lemma 3.3.1 *Suppose $q_1, q_2, \dots q_K$ are critical points of p. Then $K \leq N(n, DH^2)$. (c.f. Baby Soul Theorem).*

Theorem 3.3.6 (Mayer-Vietoris) - *Suppose U_1, U_2 open in M. There is a long exact sequence*

$$\cdots \to H_i(U_1 \cap U_2) \to H_i(U_1) \oplus H_i(U_2) \to H_i(U_1 \cup U_2) \stackrel{\Delta}{\to} H_{i-1}(U_1 \cap U_2) \to \cdots$$

where

$$\begin{array}{rcl} H_i(U_1 \cap U_2) & \to & H_i(U_1) \oplus H_i(U_2) \\ \alpha & \mapsto & (\alpha, \alpha) \end{array}$$

and

$$\begin{array}{rcl} H_i(U_1) \oplus H_i(U_2) & \to & H_i(U_1 \cup U_2) \\ (\alpha, \beta) & \mapsto & \alpha - \beta \end{array}$$

We have short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_i(U_1 \cap U_2) & \longrightarrow & C_i(U_1) \oplus C_i(U_2) & \longrightarrow & C_i(U_1 \cup U_2) & \longrightarrow & 0 \\ & & \downarrow \partial & & \downarrow \partial & & \downarrow \partial & & \\ 0 & \longrightarrow & C_{i-1}(U_1 \cap U_2) & \longrightarrow & C_{i-1}(U_1) \oplus C_{i-1}(U_2) & \longrightarrow & C_{i-1}(U_1 \cup U_2) & \longrightarrow & 0 \end{array}$$

which give a well defined $\Delta : H_i(U_1 \cup U_2) \to H_{i-1}(U_1 \cap U_2)$.

Thus

$$\begin{array}{rcl} b_i(U_1 \cup U_2) & = & dim(H_i(U_1 \cup U_2)) \\ & = & dim(Ker(\Delta)) + dim(Im(\Delta)) \\ & \leq & dim(Im(\Phi)) + b_{i-1}(U_1 \cap U_2) \\ & \leq & b_i(U_1) + b_i(U_2) + b_{i-1}(U_1 \cap U_2) \end{array}$$

Prop 3.3.2 $b_i(U_1 \cup \cdots \cup U_N) \leq \sum_{\substack{U(j) \\ j \leq i}} b_j(U_{i-j})$, *where $U(j)$ is the $(j+1)$-fold intersection of U_i's.*

Proof Case $N = 2$ is above. Assume true for N.

$$\begin{aligned} b_i(U_1 \cup \cdots \cup U_{N+1}) &\leq b_i(U_1 \cup \cdots \cup U_N) + b_i(U_{N+1}) + b_{i-1}((U_1 \cup \cdots \cup U_N) \cap U_{N+1}) \\ &= b_i(U_1 \cup \cdots \cup U_N) + b_i(U_{N+1}) + b_{i-1}(((U_1 \cap U_{N+1}) \cup \cdots \cup (U_N \cap U_{N+1})) \end{aligned}$$

Suppose $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} D$. Then $\operatorname{rank}(wvu)_* \leq \operatorname{rank} v_*$.

Definition 3.3.1 *If $A \subset B$ then $b_i(A, B) = \operatorname{rank} u_*$.*

Let U_i^j, $i = 1, \ldots, N$ be open sets. Index by j with $\bar{U}_i^j \subset U_i^{j+1}$. Set $X^j = \cup U_i^j$. Then $X^0 \subset X^1 \subset \cdots X^{n+1}$.

Prop 3.3.3 $b_i(X^0, X^{n+1}) \leq \sum_{0 \leq j \leq i} b_i(U_{i-j}^i, U_{i-j}^{j+1})$

Lemma 3.3.2 *If $B_r(p_1) \cap \cdots \cap B_r(p_j) \neq \emptyset$ then*

$$\begin{aligned} B_r(p_1) \cap \cdots \cap B_r(p_j) &\subset B_r(p_i) \subset B_{5r}(p_i) \\ &\subset B_{10r}(p_1) \cap \cdots \cap B_{10r}(p_j) \end{aligned}$$

Proof Let $x \in B_r(p_1) \cap \cdots \cap B_r(p_j)$, $q \in B_{5r}(p_i)$. Then

$$\begin{aligned} d(q, p_k) &\leq d(q, p_i) + d(p_i) + d(x, p_k) \\ &\leq 10r \end{aligned}$$

Definition 3.3.2 *Set $b_i(r, p) = b_i(B_r(p), B_{5r}(p))$ and $\operatorname{Cont}(r, p) = \sum_i b_i(r, p)$.*

Corollary $b_i(B_r(p_1) \cap \cdots \cap B_r(p_j), B_{10r}(p_1) \cap \cdots \cap B_{10r}(p_j)) \leq b_i(r, p_i)$, for $1 \leq i \leq j$.

Proof

$$\begin{aligned} \bigcap_i B_r(p_i) &\overset{u}{\hookrightarrow} B_r(p_i) \\ &\overset{v}{\hookrightarrow} B_{5r}(p_i) \overset{w}{\hookrightarrow} \bigcap_i B_{10r}(p_i) \end{aligned}$$

$\operatorname{rank}(wvu)_* \leq \operatorname{rank} v_*$ is equivalent to $b_i(\cap B_r(p_i), \cap B_{10r}(p_i)) \leq b_i(r, p)$.

Cover $B_r(p)$ by N ϵ balls: $B_r(p) \subset \bigcup_{i=1}^{N} B_\epsilon(p_i)$.

Corollary If for all $p' \in B_r(p)$ and $j = 1, \ldots, n+1$, $\operatorname{Cont}(10^{-j} r, p') \leq c$ then $\operatorname{cont}(r, p) \leq (n+1) 2^N c$.

Proof Cover $B_r(p)$ by balls $B_\epsilon(p_i)$ where $\epsilon = 10^{-(n+1)}r$. Let $U_i^j = B_{10^j\epsilon}(p_i)$ and set $X^j = \cup_i U_i^j$. Then

$$b_i(X^0, X^{n+1}) \leq \sum_{j,(i-j)} b_j(U_(i-j)^j, U_(i-j)^{j+1}.$$

Lemma 3.3.3 *Let $B_r(p) \subset M$ with M complete. Assume $5s + d(p,y) \leq 5r$, $d(p,y) \leq 2r$. If $B_r(p)$ does not compress to $B_s(y)$ there is a critical point x of y with*

$$s \leq d(x,y) \leq r + d(y,p)$$

and

$$x \in B_{r+2d(y,p)}(p) \subset B_{5r}(p).$$

Lemma 3.3.4 *Suppose $\text{rank}'(r,p) = j$. Then there is $y \in B_{5r}(p)$ and $x_1, \ldots, x_j \in B_{5r}(p)$ such that for all $i \leq j$, x_j is critical with respect to y and*

$$d(y_i, y) \geq \frac{5d(x_{i-1}, y)}{4}.$$

Proof Assume $B_r(p)$ is incompressible. Observe that if $p' \in B_r(p)$ then $B_{5(r/10)}(p') = B_{r/2}(p') \subset B_{3r/2}(p)$. Put $p_j = p$ and $r_j = r$. Since $\text{rank}(p,r) = j$ there is $p_{j-1} \in B_{r_j}(p_j)$ with $r_{j-1} \leq r_j/10$ such that $\text{rank}(\hat{p}_{j-1}, \hat{r}_{j-1}) = j-1$.

If $B_{\hat{r}_{j-1}}(\hat{p}_{j-1})$ is incompressible set $p_{j-1} = \hat{p}_{j-1}$ and $r_{j-1} = \hat{r}_{j-1}$. Otherwise, compress $B_{\hat{r}_{j-1}}(\hat{p}_{j-1})$ to a ball $B_{r_{j-1}}(p_{j-1})$ which is incompressible and $\text{rank}(p_{j-1}, r_{j-1}) = j-1$.

Since $B_{\hat{r}_{j-1}}(\hat{p}_{j-1}) \mapsto B_{r_{j-1}}(p_{j-1})$,

$$B_{5r_{j-1}}(p_{j-1}) \subset B_{5\hat{r}_{j-1}}(\hat{p}_{j-1}) \subset B_{3r_j/2}(p_j).$$

In either case, $B_{5r_{j-1}}(p_{j-1}) \subset B_{2r_j/2}(p_j)$.

In the first case, $r_{j-1} = \hat{r}_{j-1} \leq r_j/10$.

In the second case, $r_{j-1} \leq \hat{r}_{j-1} \leq r_j/10$.

Repeat to get $B_{r_i}(p_i)$ for $i = 0, \ldots, j$ such that $B_{r_i}(p_i)$ is incompressible, $r_{i-1} \leq r_i/10$, $B_{5r_{i-1}}(p_{i-1}) \leq B_{3r_i/2}(p_i)$.

Put $y = p_0$. Then $y \in B_{3r_i/2}(p_i)$ for all i. Since the balls $B_{r_i}(p_i)$ are incompressible, $B_{r_i}(p_i)$ does not compress to $B_{r_i/2}(y)$. Thus there is a critical point x_i with

$$\begin{aligned} r_i/2 \leq d(x_i, y) &\leq r_i + d(y, p_i) \\ &\leq r_i + 3r_i/2 \leq 4r_i. \end{aligned}$$

Hence

$$\begin{aligned} d(x_i, y) &\geq r_i/2 \\ &\geq 5r_{i-1} \qquad (r_{i-1} \leq r_i/10) \\ &= (5/4)4r_{i-1} \\ &\geq (5/4)d(x_{i-1}, y). \end{aligned}$$

Corollary $\operatorname{rank}(r,p) \leq \begin{cases} N(n) & \text{if } H = 0 \\ N(n, HD^2) & \text{if } H < 0 \end{cases}$

Proof Since $d(x_i, y) \geq 5d(x_i, y)/4$, there is a bound on the number of critical points.

Two operators δ, d; d derivative.

$$\begin{aligned} H_i(U_1) \oplus H_i(U_2) &\rightarrow H_i(U_1 \cup U_2) \\ (\alpha, \beta) &\mapsto (\alpha - \beta) \end{aligned}$$

We also have

$$(\delta\omega)_{\alpha_0 \cdots \alpha_{i+1}} = \sum_{k=1}^{i+1} (-1)^k \omega_{\alpha_0 \cdots \hat{\alpha}_k \cdots \alpha_{i+1}}.$$

Theorem 3.3.7 (Cheeger's Finiteness Theorem (1970)) *There are only finitely many diffeomorphism types of Riemannian manifolds M^n with $|K_M| \leq H$, $\mathrm{vol}_M \geq V$ and $\mathrm{diam}(M) \leq D$. This number is explicit, and depends on n, HD^2 and VD^{-n}. The main lemma in the proof is the injectivity radius estimate.*

Lemma 3.3.5 *For manifolds M^n as above, $\mathrm{inj}_M \geq i_0(n, D, F, V)$. In particular, the small balls are contractible.*

Note that all conditions are necessary for finiteness. For example, every surface of genus ≥ 2 can be given a metric that has $K \equiv -1$ and vol $\geq V$. Also $S^3/\mathbb{Z}_p$ has $K \equiv 1$, diam $\leq D$.

Theorem 3.3.8 (Grove-Petersen 1988) *For the class of manifolds M^n satisfying $K_M \geq H$, $\mathrm{vol}_M \geq V$ and $\mathrm{diam}(M) \leq D$, there are only finitely many homotopy types. (For future reference, such manifolds are said to satisfy $(*)$).*

In this case there is no uniform bound on inj_M. For example, cones have arbitrarily small closed geodesics. The idea is that small balls are contractible in bigger balls. For Ricci curvature, the theorem is true for $n = 3$ (Zhu) and false for $n \geq 4$ (Perelman).

Lemma 3.3.6 (Main Lemma) *For M satisfying $(*)$ there is $\delta = \delta(n, D^n/V, HD^2) > 0$ and a deformation of the $\delta D/2$-tubular neighborhood of the diagonal Δ in $M \times M$ retracting it to the diagonal*

$$H_t : T_{\delta D/2}(\Delta) \to M \times M.$$

Moreover, the curve $t \mapsto H_t(p, q)$ has length

$$L(H_t(p, q)) \leq R(n, D^n/V, HD^2)\bar{pq}.$$

Corollary If $f_1 : N \to M^n$, $f_2 : N \to M^n$, N arbitrary and M satisfies $(*)$ such that $d(f_1(x), f_2(x)) \leq \delta D/2$ for all x, then $f_1 \simeq f_2$.

Idea Proof of Main Lemma $\leftrightarrow$ Uniform Estimate on the Distance of Pairs of Mutually Critical Points

Given $(p, q) \in T_{\delta D/2}(\Delta)$ if q is not critical to p we deform q and get closer to p. Continue until both are critical to each other.

Definition 3.3.3 *q is ϵ-almost critical to p if for all $v \in T_qM$ there is a minimal geodesic γ from q to p such that $\measuredangle(v, \gamma'(0)) \leq \pi/2 + \epsilon$.*

Main Lemma follows from the following estimate for mutually critical points.

Theorem 3.3.9 *There exist $\epsilon(n, D/V, HD^2), \delta(n, D/V, HD^2) > 0$ such that if $p, q \in M$, where M satisfies $(*)$, and $d(p, q) < \delta D$ then at least one of p, q is not p, q ϵ-almost critical to each other.*

Proof Suppose that for all $\epsilon, \delta > 0$ there are p, q with $d(p, q) < \delta D$ and p, q ϵ-almost critical to each other. Scale M so that $D = 1$, $d(p, q) < \delta$. Let

$$M_p = \{x \in M : d(x, p) \leq d(x, q)\}.$$

We will show that $\mathrm{vol}(M_p) \leq V/3$.

Let $\dot{\Gamma}_{pq} = \{\gamma'(0) : \gamma'(0) \in S^{n-1} \subset T_pM, \gamma \text{ minimal connecting } p \text{ to } q\}$, and define $\dot{\Gamma}_{qp} \subset T_qM$. Since p, q are ϵ-critical to each other, $\dot{\Gamma}_{pq}$ and $\dot{\Gamma}_{qp}$ are $\pi/2 + \epsilon$ dense in $S^{n-1} \subset T_pM$. To compute the volume of M_p, we consider

$$\ell + p = \{x : x = \gamma(\ell), \gamma'(0) \notin T_{\pi/2-\epsilon}\dot{\Gamma}_{pq}\}.$$

Choose ϵ so that

$$\left(\frac{v_{n-1,1}(\pi/2+\epsilon) - v_{n-1,1}(\pi/2-\epsilon)}{v_{n-1,1}(\pi/2+\epsilon)}\right) v_{n,H}(1) = V/6.$$

$$\begin{aligned}
\mathrm{vol}(\ell+p) &\leq \mathrm{vol}(\text{corresponding part in } M_H^n) \\
&= v_{n,H}(1) - \frac{\mathrm{vol}(T_{\pi/2}\dot{\Gamma}_{pq})}{\mathrm{vol}(S_1^{n-1})} v_{n,H}(1) \\
&= v_{n,H}(1)\left(1 - \frac{\mathrm{vol}(T_{\pi/2-\epsilon}\dot{\Gamma}_{pq})}{\mathrm{vol}(T_{\pi/2+\epsilon}\dot{\Gamma}_{pq})}\right) \\
&\leq v_{n,H}(1)\left(1 - \frac{\mathrm{vol}(\pi/2-\epsilon)}{\mathrm{vol}(\pi/2+\epsilon)}\right) = V/6
\end{aligned}$$

Pick r such that $v_{n,H}(r) = V/6$. If $y = \sigma(t), \sigma'(0) \in T_{\pi/2-\epsilon}\dot{\Gamma}_{pq}$ and $t > r$ then $y \notin M_p$. To see this use Toponogov and compare with M_H^n.

$$d(y,q) < d(\bar{y},\bar{q}) < d(\bar{y},\bar{p}) = d(y,p)$$

Thus $\mathrm{vol}(M_p) \leq \mathrm{vol}(n,H,r) = V/6$.

This theorem implies the following: If $d(p,q) < \delta$ then there is $H(t,p,q)$ that deforms q to p and $L(H_t(p,q)) \leq Rd(p,q)$, where $R = 1/\cos(\pi/2-\epsilon)$ (c.f. proof of isotopy lemma).

If $(\sigma_1(t), \sigma_2(t)) : [a,b] \to M \times M$ realizes the distance from (p,q) to Δ then $\sigma'(b) = -\sigma_2'(b)$. Thus $\sigma_2 \cdot \sigma_1$ is a minimal geodesic connecting p and q. Thus minimal geodesics $(p,q) \to \Delta$ are in 1-1 correspondence with minimal geodesics $p \to q$.

We proved that if $f_1 : X \to M$, $f_2 : X \to M$ where M satisfies $(*)$ then $d(f_1(x), f_2(x)) < \delta D/2$ then f_1, f_2 are homotopic to each other. Also, if $d(p,q) < \delta$ then there is H_t from q to p such that $L(H_t) \leq Rd(p,q)$.

Lemma 3.3.7 (Center of Mass) *Suppose* $\{p_0, \ldots, p_k\} \subset B(p,r) \subset M$, *where* M *satisfies* $(*)$. *Assume*

$$2r\frac{R^k-1}{R-1} < \delta.$$

Then for a simplex

$$\Delta^k = \{(x_0,\ldots,x_k) \in \mathbb{R}^{k+1} : \sum x_i = 1, x_i \in [0,1]\}$$

there is a continuous map $f : \Delta^k \to B(p, r + 2rR\frac{R^k-1}{R-1}) \subset M$ *such that* $f(e_i) = p_i, 1$ *where* e_i *is the* i*th vertex of* Δ^k.

Proof Induction: If $k = 1$ we use the previous lemma. Assume true for k. We have $p_0, \ldots, p_{k+1} \in B(p,r)$ and $2r\frac{R^{k+1}-1}{R-1} < \delta$. Find $f : \Delta^k \to B(p, r + 2rR\frac{R^k-1}{R-1})$ such that $f(e_i) = p_i$, $0 \leq i \leq k$. Define

$$\bar{f}(x_0,\ldots,x_{k+1}) = H(x_{k+1}, f(\frac{x_0}{\sum x_i},\ldots,\frac{x_k}{\sum x_i}), p_{k+1}).$$

This is well defined and continuous provided

$$d(f(\frac{x_0}{\sum x_i},\ldots,\frac{x_k}{\sum x_i}, p_{k+1}) < \delta.$$

But

$$\begin{aligned} d(f(\frac{x_0}{\sum x_i},\ldots,\frac{x_k}{\sum x_i}, p_{k+1}) &\leq d(f(\cdots),p) + d(p,p_{k+1}) \\ &\leq 2rR\frac{R^k-1}{R-1} + r + r \\ &= 2r(1 + R\frac{R^k-1}{R-1} \\ &= 2r\frac{R^{k+1}-1}{R-1} < \delta \end{aligned}$$

Also

$$\begin{aligned} d(\bar{f}(x_0,\ldots,x_{k+1}, p_{k+1}) &\leq Rd(f(\cdots),p_{k+1}) \\ &\leq R \cdot 2R\frac{R^{k+1}-1}{R-1} \end{aligned}$$

The next tool we will use will be the Gromov-Hausdorff distance, with which we will measure the closeness of M^n satisfying $(*)$.

(Insert definitions/basic results regarding Gromov-Hausdorff distance)

Lemma 3.3.8 *For any two manifolds M and N satisfying $(*)$ there is $\epsilon(n, H, V, D) > 0$ such that if $d_{GH}(M, N) < \epsilon$ M, N are homotopy equivalent.*

Proof Triangulate M, N so that any simplex lies in a ball of radius ϵ. Use this triangulation of M to construct a continuous map $f : M \to N$. Let $\{p_\alpha\}$ denote the vertices of the triangulation M. Since $d_{GH}(M, N) < \epsilon$ there are $q_\alpha \in N$ such that $d(p_\alpha, q_\alpha) < \epsilon$.

Let $\{p_{\alpha_0}, \dots, p_{\alpha_n}\}$ be a simplex in M. Then $\{p_{\alpha_0}, \dots, p_{\alpha_n}\} \subset B(x, \epsilon)$ for some $x \in M$ and $\{q_{\alpha_0}, \dots, q_{\alpha_n}\} \subset B(q_{\alpha_0}, 4\epsilon)$.

If $8\epsilon R\frac{R^n-1}{D-1} < \delta$ there is a continuous map $f : \{p_{\alpha_0}, \dots, p_{\alpha_n}\} \to N$, which gives a map $f : M \to N$. For $x \in M$, assume x lies in the simplex spanned by $\{p_{\alpha_0}, \dots, p_{\alpha_n}\}$. Then

$$\begin{aligned} d(x, f(x)) &\leq d(x, p_{\alpha_0}) + d(p_{\alpha_0}, f(x)) \\ &\leq 2\epsilon + d(p_{\alpha_0}, q_{\alpha_0}) + d(q_{\alpha_0}, f(x)) \\ &\leq 3\epsilon + d(q_{\alpha_0}, f(x)) \\ &\leq 3\epsilon + 4\epsilon + 8\epsilon R\frac{R^n - 1}{R - 1} \\ &\leq 7\epsilon + 8\epsilon R\frac{R^n - 1}{R - 1} \end{aligned}$$

Similarly we have $g : N \to M$ with $d(y, g(y)) \leq 7\epsilon + 8\epsilon R\frac{R^n-1}{R-1}$ for $y \in N$. Consider $f \circ g$ and $g \circ f$:

$$\begin{aligned} d(g \circ f(x), x) &\leq d(g \circ f(x), f(x)) + d(x, f(x)) \\ &\leq 14\epsilon + 16\epsilon R\frac{R^n - 1}{R - 1}. \end{aligned}$$

Similarly for $f \circ g$. Pick ϵ so $14\epsilon + 16\epsilon R\frac{R^n-1}{R-1} < \delta$. Then $g \circ f \simeq \mathrm{id}_M$ and $f \circ g \simeq \mathrm{id}_N$.

Theorem 3.3.10 (Grove-Petersen 1988) - *The class of Riemannian manifolds M^n with $K_M \geq H$, $\mathrm{vol}_M \geq V$ and $\mathrm{diam}(M) \leq D$ has only finitely many homotopy types.*

Proof If M_1, M_2 satisfy the above conditions, $(*)$, then there is $\epsilon(n, H, V, D)$ such that $D_{GH}(M_1, M_2) < \epsilon$ implies M_1, M_2 homotopy equivalent. But the class $\mathcal{C}$ of manifolds satisfying $(*)$ is precompact, so $\mathcal{C}$ can be covered by finitely many ϵ-balls. Thus there are only finitely many homotopy types.

Theorem 3.3.11 (Cheeger's Finiteness Theorem) *For the class of manifolds M^n with $|K_M| \leq H$, $\operatorname{diam}(M) \leq D$ and $\operatorname{vol}_M \geq V$, there is $\epsilon = \epsilon(n, H, D, V) > 0$ such that if $d_{GH}(M_1, M_2) < \epsilon$ then $M_1 \overset{diffeo}{\simeq} M_2$.*

Convergence of maps Let $\phi_i : \Omega \subset \mathbb{R}^n \to \mathbb{R}^k$ be a sequence of maps. $\phi_i \to \phi$ in the $\mathcal{C}^{m,\alpha}$ topology if $\|\phi_i - \phi\|_{\mathcal{C}^{m,\alpha}} \to 0$ as $i \to \infty$. Here

$$\|U\|_{\mathcal{C}^{m,\alpha}} = \|U\|_{\mathcal{C}^m} + \sum_{|i|=m} \|\partial^i U\|_\alpha,$$

where $i = (i_1, \dots, i_n)$, $|i| = i_1 + \cdots + i_n$ and for $U(x_1, \dots, x_n)$ we have

$$\partial^i U = \frac{\partial^{|i|} U}{\partial^{i_1} x_1 \cdots \partial^{i_n} x_n}.$$

Also,

$$\|U\|_{\mathcal{C}^m} = \sup_{x \in \Omega} |U(x)| + \sum_{|i| \leq m} \sup_{x \in \Omega} |\partial^i U(x)|$$

and

$$\|U\|_\alpha = \sup_{x, y \in \Omega} \frac{|U(x) - U(y)|}{|x - y|^\alpha}.$$

A sequence of manifolds (M_i^n, g_i) converges to (M, g) in the $\mathcal{C}^{m,\alpha}$ topology if there are diffeomorphism $\phi_i : M \to M_i$ for i large such that $\phi_i^* g_i \to g$ in the $\mathcal{C}^{m,\alpha}$ topology.

Theorem 3.3.12 (Cheeger-Gromov Compactness) *The class of M^n with $|K_M| \leq H$, $\operatorname{diam}(M) \leq D$ and $\operatorname{vol}_M \geq V$ $(*)$ is precompact in the $C^{1,\alpha}$ topology for $\alpha \in (0,1)$. We also have $\operatorname{inj}_M \geq i(n, H, V, D) > 0$.*

Theorem 3.3.13 (Anderson 1990) *The class of M^n, $|\operatorname{Ric}_M| \leq (n-1)H$, $\operatorname{inj}_M \geq i_0$ and $\operatorname{diam}(M) \leq D$ is precompact in the $\mathcal{C}^{1,\alpha}$ topology. If we only have $\operatorname{Ric}_M \geq (n-1)H$ then the class is precompact in the $\mathcal{C}^{0,\alpha}$ topology.*

Idea of Proof Use harmonic coordinates.

Theorem 3.3.14 (Cheeger-Colding) *If M^n is a closed manifolds there exists and $\epsilon(M) > 0$ such that if N^n is a manifold with $\operatorname{Ric} \geq -(n-1)$ and $d_{GH}(M, N) < \epsilon$ then $M \overset{diffeo}{\simeq} N$.*

This result is true even though M^n with $\mathrm{Ric} \le (n-1)H$, $\mathrm{vol} \ge V$ and $\mathrm{diam} \le D$ has infinitely many homotopy types.

Theorem 3.3.15 (Colding) *For $r > 0$ consider all metric balls of radius r in all complete Riemannian manifolds M^n with $\mathrm{Ric} \ge -(n-1)$. The volume function is continuous with respect to the Gromov-Hausdorff topology.*