Describing the Universal Cover of a Compact Limit *

John Ennis Guofang Wei

Abstract

If X is the Gromov-Hausdorff limit of a sequence of Riemannian manifolds M_i^n with a uniform lower bound on Ricci curvature, Sormani and Wei have shown that the universal cover $\tilde{X}$ of X exists [13, 14]. For the case where X is compact, we provide a description of $\tilde{X}$ in terms of the universal covers $\tilde{M}_i$ of the manifolds. More specifically we show that if $\bar{X}$ is the pointed Gromov-Hausdorff limit of the universal covers $\tilde{M}_i$ then there is a subgroup H of $Iso(\bar{X})$ such that $\tilde{X} = \bar{X}/H$.

1 Introduction

In 1981 Gromov proved that any finitely generated group has polynomial growth if and only if it is almost nilpotent [7]. In his proof, Gromov introduced the Gromov-Hausdorff distance between metric spaces [7, 8, 9]. This distance has proven to be especially useful in the study of n-dimensional manifolds with Ricci curvature uniformly bounded below since any sequence of such manifolds has a convergent subsequence [10]. Hence we can follow an approach familiar to analysts, and consider the closure of the class of all such manifolds. The limit spaces of this class have path metrics, and one can study these limit spaces from a geometric or topological perspective.

Much is known about the limit spaces of n-dimensional Riemannian manifolds with a uniform lower bound on sectional curvature. These limit spaces are Alexandrov spaces with the same curvature bound [1], and at all points have metric tangent cones which are metric cones. Since Perelman has shown that Alexandrov spaces are locally homeomorphic to their tangent cones [12], these limit spaces are locally contractible. In this case, an argument of Tuschmann's shows that there is eventually a surjective map from the fundamental groups of the manifolds in the sequence onto the fundamental group of the limit space [16].

We seek similar results when Ricci curvature is uniformly bounded from below. Cheeger and Colding have made considerable progress studying the geometric and

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regularity properties of the limit spaces of this class [2, 3, 4], but the local topology of the limit spaces could be very complicated. For instance, Menguy has shown that the limit spaces can have infinite topology on arbitrarily small balls [11], even when the sequence has nonnegative Ricci curvature. In addition, it is not known whether the limit space is locally or even semilocally simply connected.

Sormani and Wei have shown that the limit space X of a sequence of manifolds M_i^n with uniform Ricci curvature lower bound has a cover $\tilde{X}$ with a universal mapping property [13, 14]. This cover is called the universal cover, and it is not assumed to be simply connected.

In this note we use the notation $M_i \xrightarrow{GH} X$ to mean that the manifolds M_i converge to the space X in the Gromov-Hausdorff sense. For a compact limit space we describe the universal cover $\tilde{X}$ in terms of the universal covers of the manifolds.

Theorem 1.1. *Suppose M_i^n have $\mathrm{Ric}_{M_i} \geq (n-1)H$ and $\mathrm{diam}_{M_i} \leq D$. Assume*

$$M_i \xrightarrow{GH} X$$

and that

$$(\tilde{M}_i, \tilde{p}_i) \xrightarrow{GH} (\bar{X}, \bar{x}).$$

Then there is a closed subgroup $H \leq Iso(\bar{X})$ such that $\bar{X}/H$ is the universal cover of X.

We start by reviewing some results, then give an example to show that the universal cover of the limit space may not be the limit of the universal covers of the manifolds. This example leads us to consider equivariant Hausdorff convergence, due to Fukaya [5, 6], which extends Gromov-Hausdorff convergence to include group actions. We then combine results of Fukaya and Yamaguchi with results of Sormani and Wei [13] to prove Theorem 1.1.

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2 Background

In this paper, a manifold is a complete Riemannian manifold without boundary. An essential result in the study of Gromov-Hausdorff limits of manifolds with uniform lower bound on curvature is the Gromov Precompactness Theorem.

Theorem 2.1 (Gromov Precompactness Theorem). *Let $\mathcal{R}$ be the set of all closed, connected, Riemannian n-manifolds with*

$$\mathrm{diam} \leq D \text{ and } \mathrm{Ric} \geq (n-1)H,$$

and let $\mathcal{M}$ be the set of all isometry classes of compact metric spaces. Let d_{GH} denote the Gromov-Hausdorff distance, which is a metric on $\mathcal{M}$. Then $\mathcal{R} \subset (\mathcal{M}, d_{GH})$ is precompact.

The Gromov-Hausdorff limit of length spaces is a length space. An effective way to study the coverings of these spaces is using δ-covers, which were introduced by Sormani and Wei in [13]. Suppose X is a complete length space. For $x \in X$ and $\delta > 0$, let $\pi_1(X, x, \delta)$ be the subgroup of $\pi_1(X, x)$ generated by elements of the form $= [\alpha * \beta * \alpha^{-1}]$, where α is a path from x to some $y \in X$ and β is a loop contained in some open δ-ball in X.

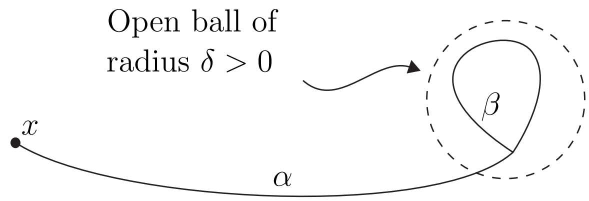


Figure 1: A typical generator for $\pi_1(X, x, \delta)$

Definition 2.2 (δ-cover). *The δ-cover of a metric space X is the covering space*

$$\pi^\delta : \tilde{X}^\delta \to X$$

with

$$(\pi^\delta)_*(\pi_1(\tilde{X}^\delta, \tilde{x})) = \pi_1(X, x, \delta).$$

Note that

$$(\pi^\delta)_* : \pi_1(\tilde{X}^\delta, \tilde{x}) \to \pi_1(X, x)$$

is the map $[\gamma] \to [\pi^\delta(\gamma)]$.

Intuitively, a δ-cover is the result of unwrapping all but the loops generated by small loops in X.

Remarks 2.3.

1. $\tilde{X}^{\delta'}$ covers $\tilde{X}^\delta$ for $\delta' \leq \delta$.

2. δ-covers exist for connected, locally path connected spaces. See [15] for more details.

Sormani and Wei have used δ-covers to show that when manifolds with a uniform Ricci curvature lower bound and a uniform diameter upper bound converge in the Gromov-Hausdorff sense to a limit space X, the universal cover of X exists [13].

Theorem 2.4 (Sormani-Wei). *Suppose M_i^n have $\mathrm{Ric}_{M_i} \geq (n-1)H$, $\mathrm{diam}_{M_i} \leq D$, and*

$$M_i \xrightarrow{GH} X.$$

Then the universal cover $\tilde{X}$ exists, and there is $\delta = \delta(X) > 0$ such that

$$(\tilde{X}, \tilde{x}) = GH \lim_{i\to\infty} (\tilde{M}_i^\delta, \tilde{p}_i).$$

Note that $\tilde{X}$ may not be the limit of the universal covers of the manifolds in the sequence. The following example shows this need not be the case.

Example 2.5. *Consider* $S^3/\mathbb{Z}_p \xrightarrow{GH} S^2$*. Then*

$$\begin{array}{ccc} S^3 & \xrightarrow{GH} & S^3 \\ \downarrow & & \vdots \\ S^3/\mathbb{Z}_p & \xrightarrow{GH} & S^2 \end{array}$$

Here the loops in the lens spaces $S^3/\mathbb{Z}_p$ shrink to points as p goes to infinity, so $S^3/\mathbb{Z}_p$ collapses to S^2. In this case the fundamental group $\mathbb{Z}_p$ of $S^3/\mathbb{Z}_p$ fills up S^1 as p grows and we have

$$S^3/\mathbb{Z}_p \xrightarrow{GH} S^3/S^1 = S^2.$$

Example 2.5 indicates the need for considering group actions as well as convergence of spaces. For this reason we use equivariant Hausdorff convergence, introduced by Fukaya [5, 6].

Consider pointed group metric spaces (X, G, x), where X is a complete metric space, G is a group of isometries of X and $x \in X$. Set

$$G(R) = \{g \in G |\, d(g(x), x) < R\}$$

for each $R > 0$.

Definition 2.6 (Equivariant ϵ-Hausdorff Approximation). *Suppose* (X_1, G_1, x_1) *and* (X_2, G_2, x_2) *are pointed group metric spaces. Let* d *be a metric on*

$$B_{1/\epsilon}(x_1, X_1) \cup B_{1/\epsilon}(x_2, X_2),$$

and let $\phi : G_1(1/\epsilon) \to G_2(1/\epsilon)$ *and* $\psi : G_2(1/\epsilon) \to G_1(1/\epsilon)$ *be maps. The triple* (d, ϕ, ψ) *is said to be an equivariant* ϵ*-Hausdorff approximation if*

1. d *extends the original metrics on* $B_{1/\epsilon}(x_i, X_i)$ *for* $i = 1, 2$.

2. *For each* $y_1 \in B_{1/\epsilon}(x_1, X_1)$ *there is* $y_2 \in B_{1/\epsilon}(x_2, X_2)$ *such that*

$$d(y_1, y_2) < \epsilon,$$

and for each $y_2' \in B_{1/\epsilon}(x_2, X_2)$ *there is* $y_1' \in B_{1/\epsilon}(x_1, X_1)$ *with*

$$d(y_1', y_2') < \epsilon.$$

3. $d(x_1, x_2) < \epsilon$.

4. For each $y_i \in B_{1/3\epsilon}(x_i, X_i)$ with $d(y_1, y_2) \le \epsilon$ and $g_i \in G_i(1/3\epsilon)$ we have

$$|d(y_1, g_1 y_1) - d(y_2, \phi(g_1)(y_2))| < \epsilon, \quad |d(y_2, g_2 y_2) - d(y_1, \psi(g_2)(y_1))| < \epsilon.$$

Definition 2.7 (Equivariant Hausdorff Convergence). *The sequence (X_i, G_i, x_i) of pointed group metric spaces converges to the pointed group metric space (X, G, x) in the equivariant Hausdorff sense if there are equivariant ϵ_i-Hausdorff approximations between (X_i, G_i, x_i) and (X, G, x), where $\epsilon_i \to 0$ as $i \to \infty$. We write*

$$(X_i, G_i, x_i) \xrightarrow{eH} (X, G, x).$$

Note that equivariant Hausdorff convergence implies Gromov-Hausdorff convergence.

Fukaya and Yamaguchi have given a constructive proof of the following important theorem [6].

Theorem 2.8 (Fukaya-Yamaguchi). *If*

$$(X_i, p_i) \xrightarrow{GH} (Y, q)$$

in the Gromov-Hausdorff sense, and $G_i \le Iso(X_i)$ are closed subgroups then there is $G \le Iso(Y)$ such that, after passing to a subsequence,

$$(X_i, G_i, p_i) \xrightarrow{eH} (Y, G, q).$$

In addition, Fukaya has shown a natural relationship between equivariant convergence and Gromov-Hausdorff convergence [5].

Theorem 2.9 (Fukaya). *If*

$$(X_i, G_i, p_i) \xrightarrow{eH} (Y, G, q)$$

then

$$(X_i/G_i, [p_i]) \xrightarrow{GH} (Y/G, [q]).$$

Remark 2.10. If

$$G_i = \pi_1(M_i),$$

$$(M_i, p_i) \xrightarrow{GH} (X, x)$$

and

$$(\tilde{M}_i, \pi_1(M_i), \tilde{p}_i) \xrightarrow{eH} (\bar{X}, G, \bar{x}),$$

then Theorem 2.9 implies $X = \bar{X}/G$.

3 Description of Universal Cover

Suppose M_i^n is a sequence of manifolds with $\text{Ric}_{M_i} \geq (n-1)H$ and $\text{diam}_{M_i} \leq D$. By Theorem 2.1, there is a length space X such that, after passing to a subsequence we have

$$M_i \xrightarrow{GH} X.$$

If we pick a sequence of points $\tilde{p}_i \in \tilde{M}_i$, where $\tilde{M}_i$ is the universal cover of M_i, a further subsequence of $(\tilde{M}_i, \tilde{p}_i)$ converges to a length space $(\bar{X}, \bar{x})$ in the pointed Gromov-Hausdorff sense.

Since $\pi_1(M_i)$ is a discrete subgroup of $Iso(\tilde{M}_i)$, $\pi_1(M_i)$ is closed. Thus Theorem 2.8 implies that there is $G \leq Iso(\bar{X})$ such that, after passing to a subsequence,

$$(\tilde{M}_i, \pi_1(M_i), \tilde{p}_i) \xrightarrow{eH} (\bar{X}, G, \bar{x}).$$

Set

$$G_i = \pi_1(M_i, p_i)$$

for each i. Then set

$$G_i^\epsilon = < g \in G_i |\, d(g\tilde{q}, \tilde{q}) \leq \epsilon \text{ for some } \tilde{q} \in \tilde{M}_i >$$

for each $\epsilon > 0$. Note that G_i^ϵ is closed, since G_i is a discrete group, and is a normal subgroup of G_i.

Thus we may consider the quotient G_i/G_i^ϵ, and its isometric action on $\tilde{M}_i/G_i^\epsilon$ by $[g][\tilde{q}] = [g\tilde{q}]$.

Lemma 3.1. *G_i/G_i^ϵ is a discrete group that acts freely on $\tilde{M}_i/G_i^\epsilon$.*

Proof. If $[g] \in G_i/G_i^\epsilon$ is not trivial, then $d([g][\tilde{q}], [\tilde{q}]) > \epsilon$ for all $[\tilde{q}] \in \tilde{M}_i/G_i^\epsilon$. In particular, G_i/G_i^ϵ acts freely on $\tilde{M}_i/G_i^\epsilon$. □

Remark 3.2. Lemma 3.1 implies that $\tilde{M}_i/G_i^\epsilon$ covers

$$(\tilde{M}_i/G_i^\epsilon)/(G_i/G_i^\epsilon) = M_i.$$

Next we prove two lemmas relating the covering spaces $\tilde{M}_i/G_i^\epsilon$ to the δ-covers $\tilde{M}_i^\delta$.

Lemma 3.3. *For $0 < \epsilon/2 < \delta$, $\tilde{M}_i/G_i^\epsilon$ covers $\tilde{M}_i^\delta$.*

Proof. We show that $G_i^\epsilon \leq \pi_1(M_i, \delta, p_i)$. Suppose g is a generator for G_i^ϵ. There is $\tilde{q}_i \in \tilde{M}_i$ with

$$d(\tilde{q}_i, g\tilde{q}_i) \leq \epsilon.$$

Connect $\tilde{q}_i$ to $g\tilde{q}_i$ by a distance minimizing path $\tilde{\beta}$, and connect $\tilde{p}_i$ to $\tilde{q}_i$ by a path $\tilde{\alpha}$. Note that the length of $\tilde{\beta}$, $\ell(\tilde{\beta})$, is at most ϵ.

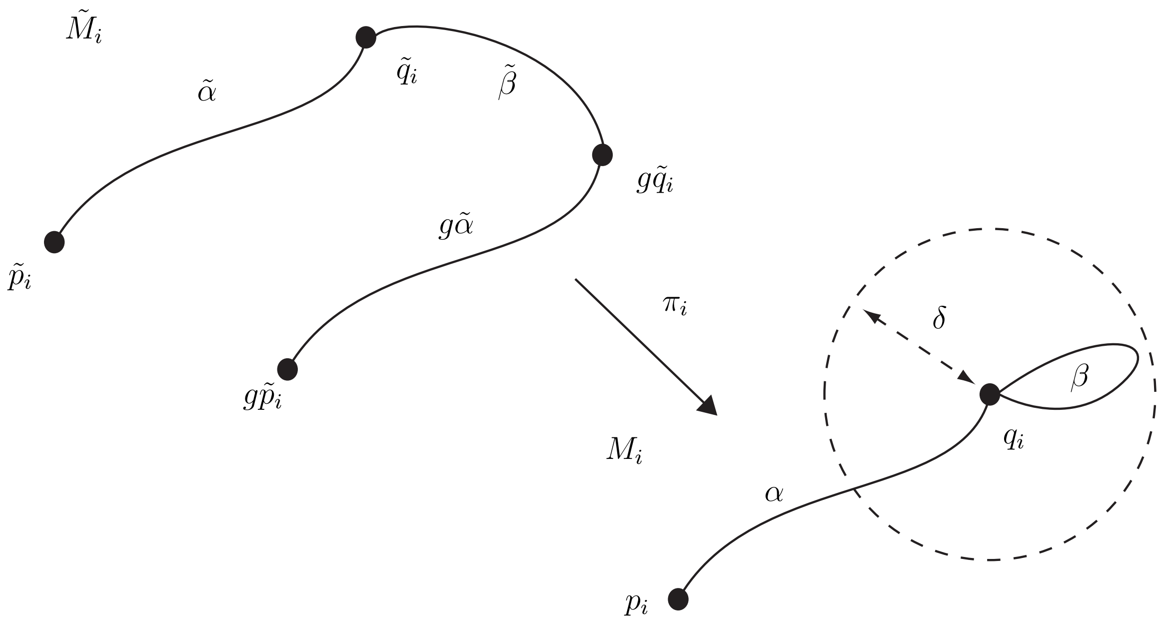


Figure 2: $\alpha * \beta * \alpha^{-1}$ lifts to $\tilde{\alpha} * \tilde{\beta} * (g\tilde{\alpha})^{-1}$

Set $\alpha = \pi_i(\tilde{\alpha})$ and $\beta = \pi_i(\tilde{\beta})$. By uniqueness of path lifting, the lift of $\alpha * \beta * \alpha^{-1}$ beginning at $\tilde{p}_i$ is $\tilde{\alpha} * \tilde{\beta} * (g\tilde{\alpha})^{-1}$.

Thus

$$[\alpha * \beta * \alpha^{-1}]\tilde{p}_i = g\tilde{p}_i,$$

so $g = [\alpha * \beta * \alpha^{-1}]$. Moreover, $\ell(\tilde{\beta}) \leq \epsilon$ implies that β is contained in $B(\beta(0), \epsilon/2)$, which lies in the open δ-ball centered at $\beta(0)$. Thus $g \in \pi_1(M_i, \delta, p_i)$, whence

$$G_i^\epsilon \leq \pi_1(M_i, \delta, p_1).$$

□

Lemma 3.4. *For each $0 < \delta < \epsilon/5$, $\tilde{M}_i^\delta$ covers $\tilde{M}_i/G_i^\epsilon$.*

Proof. Here we show that $\pi_1(M_i, \delta, p_i) \leq G_i^\epsilon$. Suppose g is a generator for $\pi_1(M_i, \delta, p_i)$. Then $g = [\alpha * \beta * \alpha^{-1}]$, where α is a path in M_i from p_i to some q_i and β_δ is a loop in $B(q_i, 2\delta)$.

Let $\tilde{\alpha}$ be the lift of α to $\tilde{M}_i$ beginning at $\tilde{p}_i$, set $\tilde{q}_i = \tilde{\alpha}(1)$ and let $\tilde{\beta}$ be the lift of β to $\tilde{M}_i$ beginning at $\tilde{q}_i$.

Observe that if $\ell(\beta) < \epsilon$,

$$d(\tilde{q}_i, g\tilde{q}_i) \leq \ell(\tilde{\beta}) = \ell(\beta) < \epsilon.$$

In this case, $g = [\alpha * \beta * \alpha^{-1}] \in G^\epsilon$.

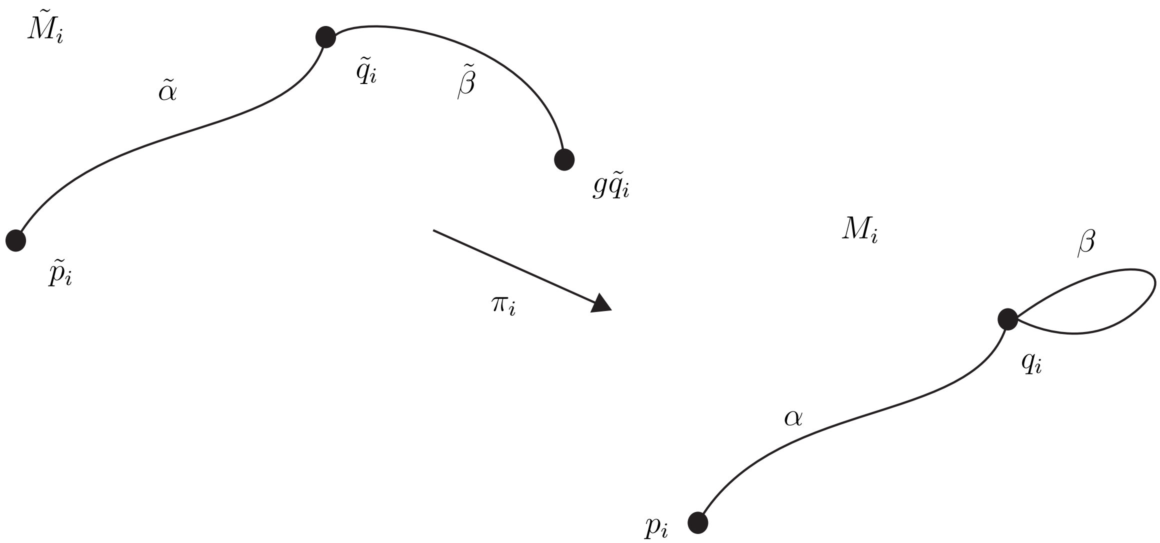


Figure 3: Lemma 3.4

In general, if $\beta \subset B(q_i, 2\delta)$ is a loop based at q_i, we can find loops $\beta_1, \dots, \beta_k$ based at q_i with $\ell(\beta_j) < 5\delta$ and

$$[\beta] = [\beta_1][\beta_2] \cdots [\beta_k].$$

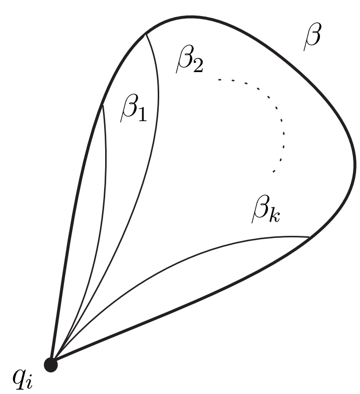


Figure 4: Dividing β

For each j, $[\alpha * \beta_j * \alpha^{-1}] \in G^{5\delta} \leq G^{\epsilon}$. Then

$$g = [\alpha * \beta * \alpha^{-1}] = [\alpha * \beta_1 * \alpha^{-1}] \cdots [\alpha * \beta_k * \alpha^{-1}] \in G^{\epsilon}.$$

□

Now we show a key relationship between δ-covers and the group actions coming from the sequence of manifolds.

Lemma 3.5. *Suppose*

$$(\tilde{M}_i, G_i, \tilde{p}_i) \xrightarrow{eH} (\bar{X}, G, \bar{x})$$

and that for $\delta > 0$,

$$(X^\delta, x^\delta) = GH \lim_{i\to\infty} (\tilde{M}_i^\delta, \tilde{p}_i^\delta).$$

Then there is $\epsilon > 0$ *such that* $\bar{X}/H^\epsilon$ *is a covering space of* X *that also covers* X^δ.

Proof. By Lemma 3.3 we may pick $\epsilon > 0$ so that

$$\phi_i : \tilde{M}_i/G_i^\epsilon \to \tilde{M}_i^\delta$$

are covering maps. In particular, each ϕ_i is distance nonincreasing. By Lemma 2.8 we may pass to a subsequence and obtain a closed subgroup H^ϵ of $G \le Iso(\bar{X})$ such that

$$(\tilde{M}_i, G_i^\epsilon, \tilde{p}_i) \xrightarrow{eH} (\bar{X}, H^\epsilon, \bar{x}) \ .$$

Note that by Theorem 2.9,

$$\tilde{M}_i/G_i^\epsilon \xrightarrow{GH} \bar{X}/H^\epsilon.$$

Thus the Arzela-Ascoli lemma implies that some subsequence of $\{\phi_i\}$ converges to a distance nonincreasing map $\phi : \bar{X}/H^\epsilon \to X^\delta$.

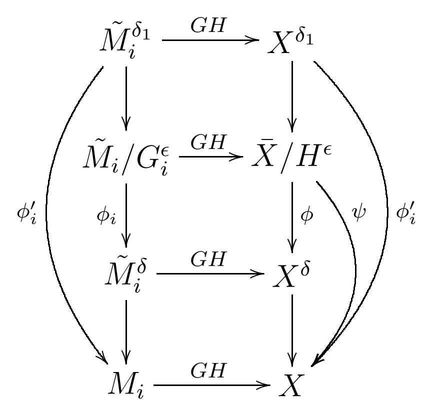
Set $\delta_1 = \epsilon/5$. By Lemma 3.4, $\tilde{M}_i^{\delta_1}$ covers $\tilde{M}_i/G_i^\epsilon$. As above, if

$$\phi_i' : \tilde{M}_i^{\delta_1} \to M_i$$

is a covering map, we may pass to a subsequence and obtain a distance nonincreasing map

$$\phi' : X^{\delta_1} \to X.$$

We have



where we have chosen basepoints so the downward pointing arrows commute.

Now each ϕ_i is an isometry on balls of radius less than δ_1, so $\phi = \lim_{i\to\infty} \phi_i$ is an isometry on balls of radius less than δ_1. In particular, ϕ is a covering map. Thus

$\bar{X}/H^\epsilon$ covers X^δ. To see that $\bar{X}/H^\epsilon$ covers X, observe that a similar argument as above shows that ϕ' is an isometry on balls of radius less than δ_1. Since this map factors through $\psi : \bar{X}/H^\epsilon \to X$, ψ is also an isometry on balls of radius less than δ_1. Thus ψ is a covering and the proof is complete. $\square$

Combining this result with Theorem 2.4, we obtain Theorem 1.1.

Proof of Theorem 1.1. By Theorem 2.4, there is $\delta > 0$ such that the universal cover of X is $\tilde{X} = X^\delta = GH\lim_{i\to\infty}(\tilde{M}_i^\delta, \tilde{p}_i^\delta)$. By Lemma 3.5 we may choose $H = H^\epsilon$ so that $\bar{X}/H$ covers both X and X^δ. Thus $\bar{X}/H = \tilde{X}$. $\square$

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