A Comparison-Estimate of Toponogov Type for Ricci Curvature*

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Abstract

A comparison estimate is given for small hinges in a complete manifold with a lower Ricci bound. The estimate is then used to show that a complete nonnegatively Ricci curved manifold with diameter growth $= o(r^{1/n})$ $(n = \dim)$ and a positive conjugate radius is of finite topological type.

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1 Introduction

It seems a natural question to ask to what extent the results and tools for sectional curvature remain valid for Ricci curvature. There is rapid progress in both positive and negative directions. Toponogov Comparison Theorem has been the most powerful tool in the study of sectional curvature, underlying the proof of the Soul Theorem, the diameter sphere theorem, the uniform estimate of betti numbers and the finiteness theorems. Toponogov Comparison Theorem is also the characterizing property of lower (or upper) sectional curvature bounds, which led to generalizations of the concept of (sectional) curvature bounds to non-smooth space. Thus it could not possibly hold for Ricci curvature, which makes problems a lot harder for Ricci curvature (from a geometric point of view). It also makes the Ricci curvature very different from the sectional curvature, as one gradually comes to realize. It is interesting then that we find a comparison *estimate* of Toponogov type for Ricci curvature.

A hinge in a complete Riemannian manifold consists of two geodesic segments γ_1, γ_2 such that $\gamma_1(l_1) = \gamma_2(0)$. We denote it by $(\gamma_1, \gamma_2, \alpha)$, where $\alpha = \angle(-\gamma_1'(l_1), \gamma_2'(0))$. We will be using the following modified version of the conjugate radius function $\rho_c(p)$:

$$\rho_c(p) = \sup\{\rho > 0 | \mathrm{conj}(q) \geq \rho, \ \forall q \in B_\rho(p)\}.$$

(The reason being, unlike the case of sectional curvature bound, we need the control of the conjugate radius at nearby points as well.)

Theorem 1.1 (Toponogov type comparison-estimate) *Let M^n be a complete manifold with Ric $\geq (n-1)\lambda$ and $(\gamma_1, \gamma_2, \alpha)$ a hinge such that*

$$L[\gamma_1], L[\gamma_2] \leq r_0,$$

where $4r_0 = \rho_c(\gamma_2(0))$. (We call such a hinge small.*) Let $(\bar{\gamma}_1, \bar{\gamma}_2, \alpha)$ be a hinge in the Euclidean plane with $L[\bar{\gamma}_i] = L[\gamma_i], i = 1, 2$. Then*

$$d(\gamma_1(0), \gamma_2(l_2)) \leq e^{C(n,\lambda,r_0)l^{1/2}} d(\bar{\gamma}_1(0), \bar{\gamma}_2(l_2)),$$

where C is an explicit uniform constant depending only on n, λ and r_0, and $l = \max(L[\gamma_1], L[\gamma_2])$.

So, within the conjugate radius, the distance can be bounded by the Euclidean one up to a uniform constant. In other words the exponential map can be bounded in terms of Ricci curvature. Thus stated, it is very closely related to [A-C].

We refer to [W] for an angle version of Toponogov comparison estimate and its applications.

Theorem 1.1 follows from the following Rauch type comparison estimate.

Theorem 1.2 (Rauch type comparison-estimate) *Let M^n be a complete manifold with $Ric \geq (n-1)\lambda$ and M^n_λ the model space of constant sectional curvature λ, $p_0 \in M$ and $\bar{p}_0 \in M^n_\lambda$. Let $4r_0 = \rho_c(p_0)$ and*

$$I \ : \ T_{p_0}M \to T_{\bar{p}_0}M^n_\lambda$$

be a linear isometry. Then for any curve

$$c \ : \ [0, \ l] \to B_{r_0}(p_0),$$

we have

$$L(c) \leq e^{C(n,\lambda,r_0)l^{1/2}} L(\exp_{\bar{p}_0} \circ I \circ \exp^{-1}_{p_0}(c)).$$

Remark 1. Note that $\exp_{p_0} : B(r_0) \subset T_{p_0}M \to B_{r_0}(p_0)$ is a local diffeomorphism. Thus the right hand side of the estimate above should be interpreted as the infimum of the lengths of the images of all the lifts of c. Thus stated, Theorem 1.2 implies that for any curve $\bar{c} : [0, l] \to B_{r_0}(\bar{p}_0)$ and any linear isometry $I : T_{\bar{p}_0}M^n_\lambda \to T_{p_0}M$,

$$L(\bar{c}) \geq e^{-C(n,\lambda,r_0)l^{1/2}} L(\exp_{p_0} \circ I \circ \exp^{-1}_{\bar{p}_0}(\bar{c})).$$

This is the direction that is used in the proof of Theorem 1.1.

Remark 2. One can actually take R^n as the comparison space.

Theorem 1.2 again follows from a Rauch type comparison estimate for Jacobi field. Namely it follows from the following.

Theorem 1.3 *Let M^n be a complete manifold with $Ric \geq (n-1)\lambda$ and M^n_λ the model space of constant sectional curvature λ. Let $\gamma, \ \gamma_0 \ : [0, l] \to M, \ M_\lambda$ be normal geodesics, and set $T = \gamma', \ T_0 = \gamma_0'$. Assume γ has no conjugate* *point in $[-l, l]$ (with respect to $\gamma(-l)$). Let $J(t)$, $J_0(t)$ be Jacobi fields along γ, γ_0 such that $J(0)$, $J_0(0)$ are tangent to γ, γ_0 and*

$$\|J(0)\| = \|J_0(0)\|, \quad \langle T, J'(0)\rangle = \langle T_0, J_0'(0)\rangle, \quad \|J'(0)\| = \|J_0'(0)\|.$$

Then for all $t \in [0, \frac{1}{2}l]$,

$$\|J(t)\| \leq e^{C(n,\lambda,l)t^{1/2}} \|J_0(t)\|.$$

The passage from Theorem 1.3 to Theorem 1.2 and then to Theorem 1.1 is standard. It is the same as in the sectional curvature case (see [C-E] for detail). The proof of Theorem 1.3 is based on the recent work of R. Brocks [B1] and will be given in §3.

As an application of Theorem 1.1 we show that the sectional curvature lower bound in Abresch-Gromoll's theorem [A-G] can be replaced by a lower bound for the conjugate radius, i.e.

Theorem 1.4 *Let M^n be a complete manifold with Ric ≥ 0 outside a compact set and its conjugate radius $r_0 > 0$. If further the diameter growth $= o(r^{1/n})$ then M is of finite topological type.*

This technique applies to other forms of finite topological type results [A-G, S, S-W]. In particular when we apply it to a result in [S-W, Corollary 1.2] we have the following result.

Theorem 1.5 *Let M^n be a complete manifold with Ric ≥ 0 outside a compact set and inj $\geq i_0 > 0$. If further vol$(B(p, r)) = o(r^{1+1/n})$ for some $p \in M$ then M is of finite topological type.*

Remark 1. There are examples of complete manifolds with Ric > 0 and injectivity radius bounded from below uniformly but having infinite topological type (see [S-Y] for example). So the growth condition is necessary here. But the question of finding the optimal growth condition remains.

Remark 2. Recently Perelman has announced an example of a complete manifold with positive Ricci curvature and constant diameter growth, but of infinite topological type. This showes that one can not simply do away with the sectional curvature lower bound in Abresch-Gromoll's theorem.

The main point in the proof of Theorem 1.4 or Theorem 1.5 is a uniform positive lower bound for the excess function at critical points, see §3.

We refer to [B2] for similar estimate for Jacobi fields.

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2 Jacobi fields and geodesic spheres

Let $p_0 \in M$ be a fixed point, and $r(p) = d(p_0, p)$ the distance function from p_0. Away from the cut locus the Hessian of r is also the second fundamental form of its level surfaces, i.e. the geodesic spheres (with respect to the inward normal). We denote $A = \operatorname{Hess} r$. Now let $c(t)$ $(0 \leq t \leq l)$ be a *minimal* geodesic starting at p_0 and $T = c'(t)$. If $J(t)$ is a Jacobi field along c such that $J(0) = 0$ and $J'(0) \perp T$, then

$$J(t) = \frac{\partial}{\partial s}|_{s=0} \exp_p t\alpha(s), \tag{2.1}$$

where $\alpha(s)$ is a smooth curve in T_pM such that $\alpha(0) = T,\ \alpha'(0) = J'(0)$ and $\|\alpha(s)\| = 1$. From here one derives

$$J'(t) = A(t)J(t), \tag{2.2}$$

where $A(t) = A(c(t))$, i.e. $AJ = \nabla_J T$.

The Jacobi equation for J translates into a Riccati equation for A:

$$A' + A^2 + R = 0, \tag{2.3}$$

where $R(\cdot) = R(\cdot, T)T$. It should also be pointed out that $A(t)$ has a singularity at $t = 0$. In fact a straightforward computation in normal coordinates shows

$$A(t) = \frac{1}{t} I + B(t) \tag{2.4}$$

with $B(t)$ smooth at $t = 0$.

The above discussion can actually be extended to the case when c is only assumed to have no conjugate point in $[0, l]$ (i.e. c is only locally minimizing).

In this case $A(t)$ has to be interpreted as the second fundamental form of the *immersed* hypersurface $S_t = \exp_p(B_t(0))$. But the relationship of $A(t)$ with the distance function is no longer valid. The trick here is to pull everything back to the tangent space.

Thus, consider the exponential map $\exp_p : T_pM \to M$. The collection of line segments tT $(0 \leq t \leq l)$ in T_pM such that $\exp_p tT$ has no conjugate points in $[0, l]$ form an open starshaped set in T_pM. We denote it by $\tilde{U} \subset T_pM$. Then

$$\exp_p : \tilde{U} \to M$$

is a local diffeomorphism onto its image.

Consider $\tilde{M}_c = \tilde{U}$ with the pull-back metric $\tilde{g} = \exp_p^* g$. The geodesic $c(t)$ lifts to a geodesic $\tilde{c}(t) = tT$ $(0 \leq t \leq l)$ in $\tilde{M}_c$. Now the main observation is that, on $\tilde{M}_c$ the geodesic $\tilde{c}(t)$ is indeed minimal. Thus our previous discussion on $J(t)$, $A(t)$ and the geodesic spheres continues to hold provided we do everything on $(\tilde{M}_c, \tilde{g})$.

3 A Rauch type comparison estimate for Jacobi fields

Our starting point is a very sharp lower bound estimate of the Laplacian of the distance function, due to R. Brocks.

As in the previous section we let $p_0 \in M$ be a fixed point, and $r(p) = d(p_0, p)$ the distance function from p_0. Also let $c(t)$ $(0 \leq t \leq l)$ be a minimal geodesic starting at p_0 and $A(t) = A(c(t))$. Define functions $\mathrm{ct}_\lambda(t)$, $\mathrm{sn}_\lambda(t)$ by

$$\mathrm{ct}_\lambda(t) = \begin{cases} \frac{\sqrt{\lambda}\cos\sqrt{\lambda}t}{\sin\sqrt{\lambda}t} & \lambda > 0 \\ \frac{1}{t} & \lambda = 0 \\ \frac{\sqrt{|\lambda|}\cosh\sqrt{|\lambda|}t}{\sinh\sqrt{|\lambda|}t} & \lambda < 0 \end{cases},$$

$$\mathrm{sn}_\lambda(t) = \begin{cases} \frac{1}{\sqrt{\lambda}}\sin\sqrt{\lambda}t & \lambda > 0 \\ t & \lambda = 0 \\ \frac{1}{\sqrt{|\lambda|}}\sinh\sqrt{|\lambda|}t & \lambda < 0 \end{cases}.$$

The following is a recent result of R. Brocks [B1].

Theorem 3.1 (R. Brocks) *Let M^n be a complete manifold with $Ric \geq (n-1)\lambda$. Assume that c is actually minimal over $-l \leq t \leq l$. Then there exists a constant $C_0(n, \lambda, l)$ depending continuously on l and decreasing with l such that*

$$0 \leq ct_\lambda(t) - \frac{trA(t)}{n-1} \leq C_0(n, \lambda, l) \quad \textit{for} \quad 0 \leq t \leq \frac{l}{2}, \tag{3.1}$$

where $ct_\lambda(t)$ is defined above.

Remark . The left hand side of (3.1) is the well-known upper bound estimate for the Laplacian of the distance function [C-G]. It holds without any restriction. For previous lower bound estimate we refer to Anderson-Cheeger [A-C].

Proof. We refer to [B1, Satz 5.6] for detail. For our purpose we would like to mention the main idea. The minimality of the geodesic c gives rise to the convexity of the associated excess functions at c. This enables one to convert upper bounds for the Laplacian of distance function to lower bounds for the Laplacian of distance function (with different base point). However, to obtain the optimal lower bound, one has to play with several excess functions and, is much more subtle. ■

From (3.1) we derive the following estimate for Jacobi fields (see [D-S-W, Lemma 5.2] for a similar kind of estimate).

Proposition 3.2 *Let M^n be a complete manifold with $Ric \geq (n-1)\lambda$ and $c: \ [-l, l] \to M$ a geodesic without conjugate point (with respect to $c(-l)$). Let $J(t)$ be a Jacobi field along c such that $J(0) = 0$, $\langle J'(0), T\rangle = 0$, where $T = c'(t)$. Then we have*

$$\|J(t)\| \leq e^{C(n,\lambda,l)t^{1/2}} sn_\lambda(t) \|J'(0)\|, \quad \textit{for } 0 \leq t \leq \frac{l}{2}, \tag{3.2}$$

where $C(n, \lambda, l)$ is a constant depending only on n, λ, l.

Remark . Note that it follows from (the proof of) Myers' Theorem that $2l \leq \pi/\sqrt{\lambda}$ if $\lambda > 0$.

Proof. First we assume c has no cut point in $[-l, l]$. For this type of Jacobi fields we have equation (2.2). Write $J(t) = \mathrm{sn}_\lambda(t)U(t)$ and $A(t) = B_\lambda(t) + \mathrm{ct}_\lambda(t)I$. From (2.2) (and (2.4)) we obtain

$$\begin{aligned} U'(t) &= B_\lambda(t)U(t) \\ U(0) &= J'(0). \end{aligned}$$

Therefore
$$\|U(t)\|' \leq \|U'(t)\| \leq \|B_\lambda(t)\|\|U(t)\|,$$
and so
$$\|J(t)\| = \mathrm{sn}_\lambda(t)\|U(t)\| \leq e^{\int_0^t \|B_\lambda(s)\|ds}\mathrm{sn}_\lambda(t)\|J'(0)\|.$$
Thus estimate (3.2) follows if we show that
$$\int_0^t \|B_\lambda(s)\|ds \leq C(n,\lambda,l)t^{1/2}. \tag{3.3}$$

The Riccati equation (2.3) for A gives
$$B_\lambda'(s) + B_\lambda^2(s) + 2\mathrm{ct}_\lambda(s)B_\lambda(s) + R - \lambda I = 0,$$
which, after taking trace, yields
$$\mathrm{tr}B_\lambda' + \|B_\lambda\|^2 + 2\mathrm{ct}_\lambda(s)\mathrm{tr}B_\lambda + (\mathrm{Ric}(T) - (n-1)\lambda) = 0. \tag{3.4}$$
Multiply (3.4) by $s^{1/2}$ and integrate along $c(t)$:
$$\begin{aligned}\int_0^t s^{1/2}\|B_\lambda(s)\|^2 ds &= -t^{1/2}\mathrm{tr}B_\lambda(t) + \int_0^t (\frac{1}{2}s^{-1/2} - 2s^{1/2}\mathrm{ct}_\lambda(s))\mathrm{tr}B_\lambda(s)ds \\ &\quad - \int_0^t s^{1/2}(\mathrm{Ric}(T) - (n-1)\lambda)ds \\ &\leq C(n,\lambda,l),\end{aligned}$$
where the last inequality comes from using Theorem 3.1, i.e. $-C_0(n,\lambda,l) \leq \mathrm{tr}B_\lambda(t) \leq 0$.

Hence
$$\begin{aligned}\int_0^t \|B_\lambda(s)\|ds &\leq (\int_0^t s^{-1/2}ds)^{1/2}(\int_0^t s^{1/2}\|B(s)\|^2 ds)^{1/2} \\ &\leq C(n,\lambda,l)t^{1/2},\end{aligned}$$
which is (3.3).

In the general case, we consider the Riemannian manifold $(\tilde{M}_c, \tilde{g})$ introduced in the previous section, with $p = c(-l)$. Let $\tilde{J}(t)$ be the Jacobi field along $\tilde{c}(t)$ such that
$$(\exp_p)_*\tilde{J}(t) = J(t).$$

Then $\tilde{J}(0) = 0$, and $\tilde{J} \perp \tilde{c}'(t)$. More importantly, $\tilde{c}(t)$ is now a minimal geodesic. Therefore (3.2) holds for $\tilde{J}(t)$ since $\exp_p$ is a local isometry. Consequently (3.2) holds for $J(t)$ as well.

■

It should be pointed out that there is some subtlety in the last part of the proof since our manifold $\tilde{M}_c$ is not complete. However, in applying Brock's estimate, all we need is that any two points can be connected by a smooth geodesic, and we need this only for points in a small neighborhood of $\tilde{c}(t)$. It is not hard to see that such a neighborhood can be chosen so that any pair of points in it can be connected by a smooth geodesic lying in $\tilde{M}_c$.

Now we are ready to proof Theorem 1.3.

Proof of Theorem 1.3. We first assume that J, J_0 are perpendicular to T, T_0 and $J(0) = J_0(0) = 0$. Since $R_0 = \lambda I$ for M^n_λ, it is easy to see that

$$J_0(t) = \mathrm{sn}_\lambda(t) J_0'(0).$$

Therefore, by Proposition 3.2

$$\begin{aligned} \|J(t)\| &\le e^{C(n,\lambda,l)t^{1/2}} \mathrm{sn}_\lambda(t) \|J'(0)\| \\ &= e^{C(n,\lambda,l)t^{1/2}} \|J_0(t)\|. \end{aligned}$$

In the general case let

$$J(t) = J^T(t) + J^\perp(t),$$

where $\langle J^\perp(t), T\rangle = 0$ and $J^\perp(0) = 0$, $J^T(t) = (\langle J(0), T\rangle + \langle J'(0), T\rangle t)T$. Decompose $J_0(t)$ similarly. Then

$$\|J^\perp(t)\| \le e^{C(n,\lambda,l)t^{1/2}} \|J_0^\perp(t)\|$$

as above and $\|J^T(t)\| = \|J_0^T(t)\|$. Hence

$$\|J(t)\| \le e^{C(n,\lambda,l)t^{1/2}} \|J_0(t)\|.$$

■

4 A uniform lower bound for the excess function at critical points

For $p_0,\ p_1 \in M$, the excess function $e_{p_0,p_1} : M^n \to R$ is defined by

$$e_{p_0,p_1}(p) = d(p_0, p) + d(p_1, p) - d(p_0, p_1).$$

It measures the "excess" in the triangle inequality. In [A-G] Abresch-Gromoll proved a beautiful and very important inequality for this function.

Theorem 4.1 (Abresch-Gromoll) *Let M^n be a complete manifold with $Ric \geq 0$. Then*

$$e_{p_0,p_1}(p) \leq 4(\frac{h^n}{s})^{1/(n-1)},$$

where $s = \min(d(p_0,p), d(p_1,p))$ and $h = \min_{\gamma,t} d(p, \gamma(t))$, where γ is a minimal segment from p_0 to p_1.

Remark . The more general version of Abresch-Gromoll's inequality assumes only a Ricci lower bound.

Corollary 4.2 *Assume further that the diameter growth of $M = o(r^{1/n})$. Then*

$$e_{p_0,\infty}(p) \to 0 \quad \text{as } p \to \infty.$$

Note that the excess function is "monotonic" in the following sense. If p_0', p_1' are any two points lying on a minimal geodesic connecting p_0, p and p_1, p respectively, then

$$e_{p_0,p_1}(p) \geq e_{p_0',p_1'}(p). \tag{4.1}$$

This is an easy consequence of the triangle inequality.

We now have

Lemma 4.3 *Let M^n be a complete manifold with $Ric \geq (n-1)\lambda$. For any $p_0 \in M$ if p is a critical point for the distance function $d(p_0,p)$ and $4r_0 = \rho_c(p)$, then for any $p_1 \in M$ and $0 < \rho \leq \min\{d(p_0,p), d(p_1,p), r_0\}$,*

$$e_{p_0,p_1}(p) \geq [2 - \sqrt{2}e^{C(n,\lambda,r_0)\rho^{1/2}}]\rho.$$

In particular, if the conjugate radius of $M \geq 2r_0 > 0$, then there exists $e_0 = e_0(n,\lambda,r_0) > 0$ such that

$$e_{p_0,p_1}(p) \geq e_0 \tag{4.2}$$

whenever p is a critical point of p_0 and $d(p_1,p) \geq r_0$.

Proof. Let γ_1 be a minimal geodesic from p to p_1. Since p is a critical point of p_0, there exists a minimal geodesic γ_2 from p to p_0 such that $\angle(\gamma_1'(0), \gamma_2'(0)) \leq \frac{\pi}{2}$. Take ρ as in the Lemma and let $p_0' = \gamma_2(\rho),\ p_1' = \gamma_1(\rho)$. Then by the monotonicity (4.1)

$$e_{p_0,p_1}(p) \geq e_{p_0',p_1'}(p) = 2\rho - d(p_0', p_1'). \tag{4.3}$$

On the other hand by Theorem 1.1,

$$\begin{aligned} d(p_0', p_1') &\leq e^{C(n,\lambda,r_0)\rho^{1/2}} d(\bar{p}_0', \bar{p}_1') \\ &= e^{C(n,\lambda,r_0)\rho^{1/2}} \sqrt{2}\rho. \end{aligned} \tag{4.4}$$

Combining (4.3) and (4.4) we have

$$e_{p_0,p_1}(p) \geq [2 - \sqrt{2} e^{C(n,\lambda,r_0)\rho^{1/2}}]\rho. \tag{4.5}$$

In particular, if we take ρ such that $e^{C(n,\lambda,r_0)\rho^{1/2}} = \sqrt{\frac{3}{2}}$ (i.e. $\rho = (\frac{1}{2}\ln\frac{3}{2})^2 \frac{1}{C(n,\lambda,r_0)^2}$) in (4.5), equation (4.5) gives (4.2). ■

Theorem 1.4 now follow from Corollary 4.2 and Lemma 4.3 (Cf. [A-G] for assuming Ric ≥ 0 only outside a compact set).

After hearing our result, S. Zhu has found a different proof of Lemma 4.3 using the rescaling argument in [A-C] under a lower injective radius bound.

References

[A-G] Abresch, U., Gromoll, D.: *On complete manifolds with nonnegative Ricci curvature.* Journal of A. M. S. 3, 355-374 (1990)

[A-C] Anderson, M., Cheeger, J.: *C^α-compactness for manifolds with Ricci curvature and injectivity radius bounded below.* J. Diff. Geom. 35, 265-281 (1992)

[B1] Brocks, R.: *Abstandsfunktion, Riccikrümmung und injektivitätsradius.* Diplomarbeit, University of Münster, 1993

[B2] Brocks, R.: *Convexity and Ricci curvature.* preprint, 1994

[C-E] Cheeger, J., Ebin, D.: *Comparison theorems in Riemannian geometry.* North-Holland, Amsterdam, 1975

[C-G] Cheeger, J., Gromoll, D.: *The splitting theorem for manifolds of nonnegative Ricci curvature*. J. Differential Geom. 6, 119-129 (1971)

[D-S-W] Dai, X., Shen, Z., Wei, G.: *Negative Ricci curvature and isometry group*. to appear in Duke Math J. (1994)

[S-Y] Sha, J.P., Yang, D.Y.: *Examples of metrics of positive Ricci curvature*. J. Differential Geom. 29, 95-103 (1989)

[S] Shen, Z.: *Finiteness and vanishing theorems for complete open riemannian manifolds*. Bull. Amer. Math. Soci. 21, 241-244 (1989)

[S-W] Shen, Z., Wei, G.: *Volume growth and finite topological type*. Proc. Symposia in Pure Math. 54, 539-549 (1993)

[W] Wei, G.: *Ricci curvature and betti numbers*. preprint, 1994