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Guofang Wei


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ON THE FUNDAMENTAL GROUPS OF MANIFOLDS WITH ALMOST-NONNEGATIVE RICCI CURVATURE

GUOFANG WEI

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ABSTRACT. We give an upper bound on the growth of \( \pi_1(M) \) for a class of manifolds \( M \) with Ricci curvature \( \text{Ric}_M \geq -\varepsilon \), diameter \( d(M) = 1 \), and volume \( \text{vol}(M) \geq v \).

In [4], Milnor proved that every finitely generated subgroup of the fundamental group of a manifold \( M^n \) with nonnegative Ricci curvature is of polynomial growth with degree \( \leq n \). It is conjectured by Gromov [2] that the fundamental group of a near-elliptic manifold (in the sense of Gromov) is of polynomial growth. The purpose of this note is to present the following theorem.

**Theorem 1.** For any constant \( v > 0 \), there exists \( \varepsilon = \varepsilon(n,v) > 0 \) such that if a complete manifold \( M^n \) admits a metric satisfying the conditions \( \text{Ric}_M \geq -\varepsilon \), \( d(M) = 1 \), and \( \text{vol}(M) \geq v \), then the fundamental group of \( M \) is of polynomial growth with degree \( \leq n \).

Our proof depends essentially on a recent result of M. Anderson [1].

**Theorem 2** (M. Anderson). In the class of compact \( n \)-dimensional Riemannian manifolds \( M \) such that \( \text{Ric}_M \geq (n-1)H \), \( \text{vol}(M) \geq v \), and \( d(M) \leq D \), there are only finitely many isomorphism classes of \( \pi_1(M) \).

**Proof of Theorem 1.** Choose a base point \( \hat{x}_0 \) in the universal covering \( \hat{M} \) of \( M \), and let \( x_0 = p(\hat{x}_0) \) and \( g_1, \ldots, g_r \) be a set of generators of the fundamental group \( \pi_1(M) \) viewed as deck transformations in the isometry group of \( \hat{M} \). Denote \( \Gamma(s) = \{ \text{distinct words in } \pi_1(M) \text{ of length } \leq s \} \), \( \gamma(s) = \# \Gamma(s) \), and \( l = \max_{1 \leq i \leq r} \{ d(x_0, g_i(x_0)) \} \).

Choose a fundamental domain \( F \) of \( \pi_1(M) \) which contains \( \hat{x}_0 \); then

$$
\bigcup_{g \in \Gamma(s)} g(F) \subset B_{sl+d}(\hat{x}_0),
$$

where \( d = d(M) = 1 \). Therefore,

$$
\gamma(s) \cdot \text{vol}(M) \leq \text{vol}(B_{sl+1}(\hat{x}_0)).
$$

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Now suppose, on the contrary, that for any $\varepsilon > 0$, there is a manifold $M^n$ with a metric satisfying $\text{Ric}_M \geq -\varepsilon$, $\text{d}(M) = 1$, $\text{vol}(M) \geq v$, and $\pi_1(M)$ is not of polynomial growth with degree $\leq n$. By the proof of Theorem 2, $\pi_1(M)$ has a presentation which obeys the following:

1. The number of generators $g_1, \ldots, g_N$ is uniformly bounded with $N \leq N(v/D^n, HD^2)$,
2. $d(g_i(\tilde{x}_0), \tilde{x}_0) \leq 3D$,
3. every relation is of the form $g_i g_j = g_k$.

The statements (2) and (3) have already been proved by Gromov [3]. By our assumption, $\pi_1(M)$ is not of polynomial growth with degree $\leq n$. In particular, when taking the above generators, we can find real numbers $s_i$ for all $i$ such that

$$\gamma(s_i) > is_i^n.$$  

It is crucial that this relation is independent of $\varepsilon$, as follows from (1) and (3).

On the other hand, by (1) we have

$$\gamma(s) \leq \frac{1}{v} \int_0^{3s+1} \left( \frac{\sinh \sqrt{\varepsilon t}}{\sqrt{\varepsilon}} \right)^{n-1} dt.$$  

For any fixed, sufficiently large $s_0$, there is $\varepsilon_0 = \varepsilon(s_0)$ such that for all $s \leq s_0$, $\varepsilon \leq \varepsilon_0$,

$$\gamma(s) \leq \frac{6^n}{nv}s^n.$$  

Now take $i_0 > 6^n/nv$. Then for $\varepsilon < \varepsilon(s_0)$, using (2) and (3), we get a contradiction.

We would like to mention that Peter Peterson, working from a different orientation and with different technique, has obtained a slightly weaker result. Instead of a lower volume bound, he imposes a lower bound on the contractibility radius and arrives at the same conclusion.

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Department of Mathematics, State University of New York at Stony Brook, Stony Brook, New York 11794–3651

Current address: Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139