Curvature formulas in §6.1 of [P]

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This little note presents explicit curvature formulas in §6.1 of Perelman's paper [P]. In particular it verifies (mod N^{-1}) the geometric interpretation of Hamilton's matrix (trace) Harnarck quadratic and that the Ricci tensor of the warped metric are equal to zero. The mod N^{-1} computation of the curvatures is also done in [STW] using Christoffel symbols. Here we do the computation using Gauss equation and Koszul's formula.

Recall that $\tilde{M} = M \times \mathbb{S}^N \times \mathbb{R}^+$ with the metric:

$$\tilde{g}_{ij} = g_{ij},\ \tilde{g}_{\alpha\beta} = \tau g_{\alpha\beta},\ \tilde{g}_{00} = \frac{N}{2\tau} + R,\ \tilde{g}_{i\alpha} = \tilde{g}_{i0} = \tilde{g}_{\alpha 0} = 0, \tag{1}$$

where i, j denote coordinate indices on the M factor, α, β denote those on the $\mathbb{S}^N$ factor, and the coordinate τ on $\mathbb{R}^+$ had index 0; g_{ij} evolves with τ by the backward Ricci flow $(g_{ij})_\tau = 2R_{ij}, g_{\alpha\beta}$ is the metric on $\mathbb{S}^N$ of constant curvature $\frac{1}{2N}$.

We first compute the curvatures of $\tilde{M}$ without τ-direction using Gauss equation by viewing $M \times \mathbb{S}^N$ as isometrically embedded submanifold of $\tilde{M}$. Let $n = (\frac{N}{2\tau} + R)^{-1/2}\frac{\partial}{\partial\tau}$ be its unit normal vector. Denote $\{X_i\}$ local coordinate fields of M and $\{U_\alpha\}$ local coordinate fields of $\mathbb{S}^N$. Then the second fundamental form

$$\begin{aligned}\langle B(X_i, X_j), n\rangle &= \langle \tilde{\nabla}_{X_i} X_j, n\rangle = -\frac{1}{2} n \langle X_i, X_j\rangle \\ &= -\frac{1}{2}(\frac{N}{2\tau} + R)^{-1/2}(g_{ij})_\tau = -(\frac{N}{2\tau} + R)^{-1/2} R_{ij}.\end{aligned}$$

So

$$B(X_i, X_j) = -(\frac{N}{2\tau} + R)^{-1/2} R_{ij} n. \tag{2}$$

Similarly

$$B(U_\alpha, U_\beta) = -\frac{1}{2}(\frac{N}{2\tau} + R)^{-1/2} g_{\alpha\beta} n,\ B(U_\alpha, X_i) = 0. \tag{3}$$

By Gauss equation

$$\begin{aligned}\langle \tilde{R}(X_i, X_j)X_k, X_l\rangle &= \langle R(X_i, X_j)X_k, X_l\rangle - \langle B(X_i, X_l), B(X_j, X_k)\rangle + \langle B(X_j, X_l), B(X_i, X_k)\rangle \\ &= \langle R(X_i, X_j)X_k, X_l\rangle - (\frac{N}{2\tau} + R)^{-1}\left[R_{il}R_{jk} - R_{jl}R_{ik}\right], && (4)\\ \langle \tilde{R}(X_i, U_\alpha)U_\beta, X_j\rangle &= 0 - \frac{1}{2}(\frac{N}{2\tau} + R)^{-1} R_{ij} g_{\alpha\beta}, && (5)\\ \langle \tilde{R}(U_\alpha, U_\beta)U_\gamma, U_\theta\rangle &= \langle R(U_\alpha, U_\beta)U_\gamma, U_\theta\rangle - \frac{1}{4}(\frac{N}{2\tau} + R)^{-1}\left[g_{\alpha\theta}g_{\beta\gamma} - g_{\alpha\gamma}g_{\beta\theta}\right] && (6)\\ &= \frac{1}{2N\tau}\left[\tilde{g}_{\alpha\theta}\tilde{g}_{\beta\gamma} - \tilde{g}_{\alpha\gamma}\tilde{g}_{\beta\theta}\right] - \frac{1}{4}(\frac{N}{2\tau} + R)^{-1}\left[g_{\alpha\theta}g_{\beta\gamma} - g_{\alpha\gamma}g_{\beta\theta}\right] \\ &= (\frac{N}{2\tau} + R)^{-1}\frac{\tau R}{2N}\left[g_{\alpha\theta}g_{\beta\gamma} - g_{\alpha\gamma}g_{\beta\theta}\right] && (7)\\ \langle \tilde{R}(X_i, X_j)X_k, U_\alpha\rangle &= 0 && (8)\\ \langle \tilde{R}(X_i, U_\beta)U_\gamma, U_\theta\rangle &= 0. && (9)\end{aligned}$$

For curvatures involve normal direction, note that

$$[U_\alpha, n] = 0, \quad [n, X_i] = \frac{1}{2}(\frac{N}{2\tau} + R)^{-1}(X_i R)n.$$

By Koszul's formula we have

$$\begin{aligned}
\tilde{\nabla}_{U_\alpha} X_i &= 0, \\
\tilde{\nabla}_n n &= -\frac{1}{2}(\frac{N}{2\tau} + R)^{-1} \sum_{l,k} (X_l R) g^{lk} X_k, \\
\tilde{\nabla}_{U_\alpha} n &= \frac{1}{2\tau}(\frac{N}{2\tau} + R)^{-1/2} U_\alpha, \\
\tilde{\nabla}_{X_i} n &= (\frac{N}{2\tau} + R)^{-1/2} \sum_{l,k} R_{il} g^{lk} X_k.
\end{aligned}$$

Therefore

$$\begin{aligned}
\langle \tilde{R}(U_\alpha, n)n, X_i \rangle &= 0, & (10) \\
\langle \tilde{R}(U_\alpha, n)n, U_\beta \rangle &= -\langle \tilde{\nabla}_n \tilde{\nabla}_{U_\alpha} n, U_\beta \rangle \\
&= -(\frac{N}{2\tau} + R)^{-1/2} \frac{d}{d\tau}\left[(\frac{N}{2\tau} + R)^{-1/2} \frac{1}{2\tau}\right] \tau g_{\alpha\beta} - \frac{1}{4\tau^2}(\frac{N}{2\tau} + R)^{-1} \tau g_{\alpha\beta} \\
&= \frac{1}{4}(\frac{N}{2\tau} + R)^{-2}(R_\tau + \frac{R}{\tau}) g_{\alpha\beta}. & (11)
\end{aligned}$$

By (2)

$$\tilde{\nabla}_{X_i} X_j = -(\frac{N}{2\tau} + R)^{-1/2} R_{ij} n + \nabla_{X_i} X_j. \quad (12)$$

So

$$\begin{aligned}
\langle \tilde{\nabla}_{X_i} \tilde{\nabla}_n n, X_j \rangle &= -\frac{1}{2} X_i [(\frac{N}{2\tau} + R)^{-1} \sum_{l,k} (X_l R) g^{lk}] g_{kj} - \frac{1}{2}(\frac{N}{2\tau} + R)^{-1} \sum_{l,k} (X_l R) g^{lk} \langle \nabla_{X_i} X_k, X_j \rangle \\
&= \frac{1}{2}(\frac{N}{2\tau} + R)^{-2} (X_i R)(X_j R) - \frac{1}{2}(\frac{N}{2\tau} + R)^{-1} Hess(R)(X_i, X_j), \\
-\langle \tilde{\nabla}_n \tilde{\nabla}_{X_i} n, X_j \rangle &= -(\frac{N}{2\tau} + R)^{-1/2} \sum_{l,k} \frac{d}{d\tau}\left[(\frac{N}{2\tau} + R)^{-1/2} R_{il} g^{lk}\right] g_{kj} - (\frac{N}{2\tau} + R)^{-1} \sum_{l,k} R_{il} g^{lk} R_{kj} \\
&= \frac{1}{2}(\frac{N}{2\tau} + R)^{-2} \left(-\frac{N}{2\tau^2} + R_\tau\right) R_{ij} + (\frac{N}{2\tau} + R)^{-1} \left[\sum_{l,k} R_{il} g^{lk} R_{kj} - (R_{ij})_\tau\right], \\
-\langle \tilde{\nabla}_{[X_i, n]} n, X_j \rangle &= -\frac{1}{4}(\frac{N}{2\tau} + R)^{-2} (X_i R)(X_j R).
\end{aligned}$$

Thus

$$\begin{aligned}
\langle \tilde{R}(X_i, n)n, X_j \rangle &= \frac{1}{2}(\frac{N}{2\tau} + R)^{-2} \left[\left(-\frac{N}{2\tau^2} + R_\tau\right) R_{ij} + \frac{1}{2}(X_i R)(X_j R)\right] \\
&\quad + (\frac{N}{2\tau} + R)^{-1} \left[\sum_{l,k} R_{il} g^{lk} R_{kj} - \frac{1}{2} Hess(R)(X_i, X_j) - (R_{ij})_\tau\right]. & (13)
\end{aligned}$$

Last we need to look at the normal component of the curvature tensor. By (3) we have

$$\tilde{\nabla}_{U_\alpha} U_\beta = -\frac{1}{2}(\frac{N}{2\tau} + R)^{-1/2} g_{\alpha\beta} n + \nabla_{U_\alpha} U_\beta. \quad (14)$$

Therefore

$$
\begin{aligned}
\langle \tilde{R}(X_i,X_j)X_k,n\rangle &= \langle \tilde{\nabla}_{X_i}\tilde{\nabla}_{X_j}X_k-\tilde{\nabla}_{X_j}\tilde{\nabla}_{X_i}X_k,n\rangle \\
&= \frac{1}{2}(\frac{N}{2\tau}+R)^{-\frac{3}{2}}[(X_iR)R_{jk}-(X_jR)R_{ik}] \\
&\quad -(\frac{N}{2\tau}+R)^{-\frac{1}{2}}[(\nabla_{X_i}Ric)(X_j,X_k)-(\nabla_{X_j}Ric)(X_i,X_k)] \\
\langle \tilde{R}(U_\alpha,U_\beta)U_\gamma,n\rangle &= 0 \\
\langle \tilde{R}(X_i,U_\alpha)U_\beta,n\rangle &= \frac{1}{4}(\frac{N}{2\tau}+R)^{-\frac{3}{2}}(X_iR)g_{\alpha\beta} \\
\langle \tilde{R}(U_\alpha,X_i)X_j,n\rangle &= 0.
\end{aligned}
$$

From above, mod N^{-1}, all curvature tensors of $\tilde{M}$ are zero except

$$
\begin{aligned}
\langle \tilde{R}(X_i,X_j)X_k,X_l\rangle &= \langle R(X_i,X_j)X_k,X_l\rangle \\
\langle \tilde{R}(X_i,\frac{\partial}{\partial\tau})\frac{\partial}{\partial\tau},X_j\rangle &= -\frac{1}{2\tau}R_{ij}+\sum_{l,k}R_{il}g^{lk}R_{kj}-\frac{1}{2}Hess(R)(X_i,X_j)-(R_{ij})_\tau \\
&= -\frac{1}{2\tau}R_{ij}+\Delta R_{ij}+2R_{ikjl}R_{kl}-\sum_{l,k}R_{il}g^{lk}R_{kj}-\frac{1}{2}Hess(R)(X_i,X_j) \\
\langle \tilde{R}(X_i,X_j)X_k,\frac{\partial}{\partial\tau}\rangle &= -(\nabla_{X_i}Ric)(X_j,X_k)+(\nabla_{X_j}Ric)(X_i,X_k).
\end{aligned}
$$

These are exactly the coefficients $R_{ijkl}, M_{ij}, P_{ijk}$ of Hamilton's Harnack quadratic.

Any two form ω on $\tilde{M}$ can be written as

$$\omega = U_{ij}X_i^*\wedge X_j^*+W_id\tau\wedge X_i^*+\text{two form with sphere components.}$$

Then mod N^{-1}, the curvature operator $\tilde{\mathcal{R}}$ acts on ω is

$$\langle \tilde{\mathcal{R}}(\omega),\omega\rangle = \langle \tilde{R}(X_i,X_j)X_k,X_l\rangle U_{ij}U_{kl}+2\langle \tilde{R}(X_i,X_j)X_k,\frac{\partial}{\partial\tau}\rangle U_{ij}W_k+\langle \tilde{R}(X_i,\frac{\partial}{\partial\tau})\frac{\partial}{\partial\tau},X_j\rangle W_iW_j,$$

which is exactly Hamilton's matrix Harnack quadratic. Therefore Hamilton's matrix Harnack inequality can be interpreted as the curvature operator $\tilde{\mathcal{R}}$ is nonnegative (mod N^{-1}). This is suggested to me by John Lott.

By taking trace in the manifold directions we get the trace Harnack quadratic. Namely let

$$\omega_k = e_k^*\wedge(d\tau+X^*),$$

where $\{e_k\}$ is an orthonormal basis of TM and X is a vector field on M. Then

$$
\begin{aligned}
\sum_k\langle \tilde{\mathcal{R}}(\omega_k),\omega_k\rangle &= \sum_k\langle \tilde{R}(X,e_k)e_k,X\rangle+2\sum_k\langle \tilde{R}(X,e_k)e_k,\frac{\partial}{\partial\tau}\rangle+\sum_k\langle \tilde{R}(e_k,\frac{\partial}{\partial\tau})\frac{\partial}{\partial\tau},e_k\rangle \\
&= Ric(X,X)-\langle\nabla R,X\rangle-\frac{1}{2\tau}R-\frac{1}{2}R_\tau \quad \text{mod } N^{-1},
\end{aligned}
$$

which is exactly Hamilton's trace Harnack quadratic. Note that this is not $\tilde{Ric}(\frac{\partial}{\partial\tau}+X,\frac{\partial}{\partial\tau}+X)$, since we need to take trace in all directions for $\tilde{Ric}$.

For Ricci curvatures we take trace of the curvature tensors and get

$$\tilde{Ric}(X_i,X_j) = \langle \tilde{R}(X_i,n)n,X_j\rangle+R_{ij}-(\frac{N}{2\tau}+R)^{-1}\sum_{k,l}[R_{ij}R_{kl}-R_{kj}R_{il}]\,g^{kl}-\frac{N}{2\tau}(\frac{N}{2\tau}+R)^{-1}R_{ij}$$

$$
\begin{aligned}
&= \frac{1}{2}(\frac{N}{2\tau}+R)^{-2}\left[\left(-\frac{N}{2\tau^2}+R_\tau\right)R_{ij}+\frac{1}{2}(X_iR)(X_jR)\right] \\
&\quad +(\frac{N}{2\tau}+R)^{-1}\left[2\sum_{l,k}R_{il}g^{lk}R_{kj}-(R_{ij})_\tau-\frac{1}{2}Hess(R)(X_i,X_j)\right] \qquad (15)
\end{aligned}
$$

$$\tilde{Ric}(X_i,U_\alpha) = 0 \qquad (16)$$

$$
\begin{aligned}
\tilde{Ric}(U_\alpha,U_\beta) &= \langle\tilde{R}(U_\alpha,n)n,U_\beta\rangle-\frac{1}{2}(\frac{N}{2\tau}+R)^{-1}\left(\sum_{ij}R_{ij}g^{ij}-\frac{N-1}{N}R\right)g_{\alpha\beta} \\
&= \frac{1}{4}(\frac{N}{2\tau}+R)^{-2}(R_\tau-\frac{2R^2}{N})g_{\alpha\beta} \qquad (17)
\end{aligned}
$$

$$\tilde{Ric}(n,n) = \frac{1}{4}(\frac{N}{2\tau}+R)^{-2}\|\nabla R\|^2 \qquad (18)$$

$$\tilde{Ric}(X_i,n) = -\frac{1}{2}(\frac{N}{2\tau}+R)^{-3/2}[\sum_{j,k}(X_jR)R_{ik}g^{jk}] \qquad (19)$$

$$\tilde{Ric}(U_\alpha,n) = 0. \qquad (20)$$

So $\tilde{Ric}=0 \bmod N^{-1}$.

The scalar curvature of $\tilde{M}$ is

$$
\begin{aligned}
\tilde{R} &= \frac{1}{2}(\frac{N}{2\tau}+R)^{-2}\|\nabla R\|^2-\frac{1}{2}(\frac{N}{2\tau}+R)^{-1}(\Delta R+\frac{R}{\tau}+R_\tau) \\
&= \frac{1}{2}(\frac{N}{2\tau}+R)^{-2}\|\nabla R\|^2+(\frac{N}{2\tau}+R)^{-1}(\|\mathrm{Ric}\|^2-\frac{R}{2\tau}) \\
&= \frac{1}{2}(\frac{N}{2\tau}+R)^{-2}\|\nabla R\|^2+(\frac{N}{2\tau}+R)^{-1}[\|\mathrm{Ric}_o\|^2+\frac{R}{n}(R-\frac{n}{2\tau})].
\end{aligned}
$$

So the scalar curvature is positive when $R<0$ or $R>\frac{n}{2\tau}$.

References

[P] G. Perelman *The entropy formula for the Ricci flow and its geometric applications*, math.DG/0211159.

[STW] N. Sesum, G. Tian, X. Wang, *Notes on Perelman's paper on the entropy formula for the Ricci flow and its geometric applications.*