THE REAL LOCI OF CALOGERO–MOSER SPACES, REPRESENTATIONS OF RATIONAL CHEREDNIK ALGEBRAS AND THE SHAPIRO CONJECTURE

IAIN GORDON, EMIL HOROZOV, AND MILEN YAKIMOV

ABSTRACT. We prove a criterion for the reality of irreducible representations of the rational Cherednik algebras $H_{0,1}(S_n)$. This is shown to imply a criterion for the real loci of the Calogero–Moser spaces C_n in terms of the Etingof–Ginzburg finite maps $\Upsilon\colon C_n \to \mathbb{C}^n/S_n \times \mathbb{C}^n/S_n$, recovering a result of Mikhin, Tarasov, and Varchenko [MTV2]. As a consequence we obtain a criterion for the real locus of the Wilson's adelic Grassmannian of rank one bispectral solutions of the KP hierarchy. Using Wilson's first parametrisation of the adelic Grassmannian, we give a new proof of a result of [MTV2] on real bases of spaces of quasi polynomials. The Shapiro Conjecture for Grassmannians is equivalent to a special case of our result for Calogero–Moser spaces, namely for the fibres of Υ over $\mathbb{C}^n/S_n \times 0$.

1. Introduction

The n-th Calogero–Moser space C_n is the geometric quotient of

$$\overline{C}_n = \{(X, Z) \in \mathfrak{gl}_n(\mathbb{C})^{\times 2} \mid \operatorname{rank}([X, Z] + I_n) = 1\}$$

by the action of $GL_n(\mathbb{C})$ by simultaneous conjugation. It is a smooth, irreducible, complex, affine variety, [W2]. The space C_n is the phase space of the (complex) Calogero–Moser integrable system [KKS, W2] and parametrizes irreducible representations of the deformed preprojective algebra of a certain quiver [CBH, CB]. We define the real locus RC_n of C_n as the image under π_n of the space of pairs of real matrices inside C_n. It is not hard to see that RC_n is a real algebraic subset of C_n which is isomorphic to the n-th real Calogero–Moser space.

A different interpretation of the spaces C_n was found by Etingof and Ginzburg [EG] in terms of representations of rational Cherednik algebras associated to symmetric groups.

In this paper we show how to use the representation theory of rational Cherednik algebras to obtain results on the real algebraic geometry of C_n. In particular we give new proofs of several theorems of Mukhin, Tarasov and Varchenko, [MTV1, MTV2], including the Shapiro Conjecture for Grassmannians.

The rational Cherednik algebra $H_{0,1}(S_n)$ is a specialisation of a two parameter deformation of the smash product $\mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n] \rtimes \mathbb{C}S_n$. Its irreducible representations all have complex dimension $n!$ and are parametrised by the points of C_n, [EG]. Denote by $\overline{e}$ the symmetrising idempotent of the copy of S_{n-1} inside S_n which permutes only the last $n-1$ indices. Given an irreducible $H_{0,1}(S_n)$ module V, x_1 and y_1 preserve the n-dimensional subspace $\overline{e}V$ and define a point

$\pi_n(x_1|_{\overline{e}V}, y_1|_{\overline{e}V}) \in C_n$. Etingof and Ginzburg proved that this establishes a bijection between the equivalence classes of irreducible representations of $H_{0,1}(S_n)$ and the points of C_n. In order to state the main result of this paper, we note that $H_{0,1}(S_n)$ has a natural real form: the real subalgebra $H^{\mathbb{R}}_{0,1}(S_n)$ generated by the elements of S_n and $x_1, \ldots, x_n, y_1, \ldots, y_n$.

Main Theorem (Theorem 3.2). *If an irreducible representation V of $H_{0,1}$ has the property that $x_1|_{\overline{e}V}$ and $y_1|_{\overline{e}V}$ have only real eigenvalues, then V is the complexification of a real representation of $H^{\mathbb{R}}_{0,1}(S_n)$.*

Now define the real locus RC_n of C_n as the image of the subset of real matrices in $\overline{C}_n$ – this is nothing but the real locus of $\overline{C}_n$ – under the quotient map $\pi_n \colon \overline{C}_n \to C_n$. We show that RC_n is a real algebraic subset of C_n which is isomorphic as a real affine variety to the n-th real Calogero-Moser space.

Set $\mathbb{C}^{(n)} = \mathbb{C}^n/S_n$ and let $\mathrm{Spec}(X)$ stand for the eigenvalues of a square matrix X. In [EG] Etingof and Ginzburg proved that the canonical map

$$\Upsilon \colon C_n \to \mathbb{C}^{(n)} \times \mathbb{C}^{(n)}, \quad \Upsilon(\pi_n(X, Z)) = (\mathrm{Spec}(X), \mathrm{Spec}(Z)), \tag{1.1}$$

is a finite map of degree $n!$. This map, and particularly its fibre over 0×0, was studied in [EG, FG, G]. We obtain from the Main Theorem

Corollary (Theorem 2.3). *Let $\mathbb{R}^{(n)} := \mathbb{R}^n/S_n \subset \mathbb{C}^{(n)}$. We have*

$$\Upsilon^{-1}(\mathbb{R}^{(n)} \times \mathbb{R}^{(n)}) \subset RC_n. \tag{1.2}$$

In elementary terms this claims that if $(X, Z) \in \overline{C}_n$ and both X and Z have real eigenvalues, then X and Z can be simultaneously conjugated (under $GL_n(\mathbb{C})$) to pair of real matrices. This reproves a result of Mukhin, Tarasov and Varchenko.

The Calogero–Moser space C_n parametrises the equivalence classes of representations of a specific dimension vector of the deformed preprojective algebra $\Pi_\nu(Q)$ of a certain quiver, [CBH, CB]. As an immediate consequence of the Main Theorem, we also obtain a criterion for reality of the representations of $\Pi_\nu(Q)$ in this class.

The disjoint union of all Calogero–Moser spaces also parametrizes Wilson's adelic Grassmannian $\mathrm{Gr}^{\mathrm{ad}}$, [W2]. The latter space first arose as the set of all solutions of the KP hierarchy which have bispectral wave functions of rank 1, [W1]. We define and study in detail the real locus of $\mathrm{Gr}^{\mathrm{ad}}$. All possible approaches to the definition of the real locus of $\mathrm{Gr}^{\mathrm{ad}}$ (as the union of the real loci of C_n, or by requiring reality of the associated tau or wave functions) are shown to be equivalent. From the Main Theorem we derive the following criterion:

Corollary (Theorem 5.4). *If $W \in \mathrm{Gr}^{\mathrm{ad}}$ has the property that the specialisations of the tau function $\tau_W(x, 0, \ldots)$ and the bispectral dual tau function $\tau_{bW}(x, 0, \ldots)$ have real roots, then $\tau_W(t_1, t_2, \ldots) \in \mathbb{C}[[t_1, t_2, \ldots]]$ has real coefficients.*

Wilson first defined the adelic Grassmannian (which actually motivated the term) by imposing a set of linear conditions of a special type on the plane of the trivial solution of the KP hierarchy. Translating the criterion for the real locus of $\mathrm{Gr}^{\mathrm{ad}}$ in terms of these conditions leads to another proof of the following result of Mukhin, Tarasov, and Varchenko.

Corollary (Theorem 6.2). *Fix a collection of distinct real numbers* $\mu_1, \dots, \mu_k \in \mathbb{R}$ *and a collection of finite dimensional subspaces* $V_1, \dots, V_k$ *of* $\mathbb{C}[x]$. *If for a given basis* $\{q_1(x), \dots, q_N(x)\}$ *of* $e^{\mu_1 x} V_1 \oplus \cdots \oplus e^{\mu_k x} V_k$ *the polynomial*

$$e^{-(\mu_1+\cdots+\mu_n)x}\,\mathrm{Wr}(q_1(x), \dots, q_N(x))$$

has only real roots, then all vector spaces $V_1, \dots, V_k$ *have real bases.*

This paper was motivated by an attempt to understand the relationship between the Shapiro Conjecture, the Calogero–Moser spaces and the rational Cherednik algebra $H_{0,1}(S_n)$. The Shapiro Conjecture for Grassmannians is a consequence of the Main Theorem.

Corollary. *If* $p_1(x), \dots p_n(x) \in \mathbb{C}[x]$ *are such that the Wronskian* $\mathrm{Wr}(p_1(x), \dots, p_n(x))$ *has only real roots, then* $\mathrm{Span}\{p_1(x), \dots, p_n(x)\}$ *has a real basis.*

This result plays a major role in the real Schubert calculus [S1, S2] and the theory of real algebraic curves [KS]. Considerable numerical evidence to support the conjecture was first obtained in [S3]. The conjecture was proved in the case $n = 2$ by Eremenko and Gabrielov [EG1, EG2]. In the general case, it was proved by Mukhin, Tarasov, and Varchenko [MTV1], who also proved the above generalisation of the conjecture for quasipolynomials [MTV2]. The Shapiro Conjecture is in fact equivalent to the special case of (1.2) $\Upsilon^{-1}(\mathbb{R}^{(n)} \times 0) \subset RC_n$. In fact, our approach to the Main Theorem is to prove the representation theoretic analogue of (1.2) for generic fibres and then to deduce the general case by continuity. In particular this avoids dealing directly with special fibres of Υ, such as those required for the Shapiro Conjecture.

Tracing back the relations between real loci and real representations, we find interesting reformulations of the Shapiro conjecture in different setups. A curious one arises from the setting of the Wilson adelic Grassmannian:

Fix a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_l > 0)$ *and consider the corresponding Schur function* $s_\lambda(p_1, p_2, \dots, p_N)$, $N = \lambda_1 + l - 1$. *If* $c_1, c_2, \dots c_N \in \mathbb{C}$ *are such that* $s_\lambda(x + c_1, c_2, \dots, c_N)$ *has only real roots, then* $c_1, c_2, \dots, c_N \in \mathbb{R}$.

Finally we would like to point out that the real loci of other quiver varieties, and the reality of representations of other deformed preprojective algebras of quivers and Cherednik algebras could be naturally related to other combinatorial problems of the Shapiro–Shapiro type.

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2. Calogero-Moser spaces

First we recall the definition of Calogero–Moser spaces, for details we refer the reader to [W2]. Define the locally closed subset $\overline{C}_n$ of $\mathfrak{gl}_n(\mathbb{C}) \times \mathfrak{gl}_n(\mathbb{C})$, consisting of pairs of matrices (X, Z) such that

$$\operatorname{rank}([X, Z] + I_n) = 1, \tag{2.1}$$

where I_n is the identity matrix of size $n \times n$. The group $GL_n(\mathbb{C})$ acts on $\overline{C}_n$ by simultaneous conjugation

$$g.(X, Z) = (gXg^{-1}, gZg^{-1}), \ g \in GL_n(\mathbb{C}), \tag{2.2}$$

and this action is free and proper, see [W2]. There then exists a geometric quotient

$$\pi_n \colon \overline{C}_n \to C_n = \overline{C}_n / GL_n(\mathbb{C}) \tag{2.3}$$

which is a smooth, irreducible, complex affine variety, [W2]. It is called the n-th Calogero–Moser space. Define the real locus of $\overline{C}_n$ by

$$\overline{R}_n = \{(X, Z) \in \mathfrak{gl}_n(\mathbb{R}) \times \mathfrak{gl}_n(\mathbb{R}) \mid \operatorname{rank}([X, Z] + I_n) = 1\},$$

and define the real locus of C_n as the push-forward of the real locus of $\overline{C}_n$ under π_n, namely

$$RC_n := \pi_n\left(\overline{R}_n\right).$$

The next proposition identifies RC_n with a real Calogero–Moser space and explicitly describes it as a real algebraic subset of C_n. Denote the natural inclusion

$$\overline{i}_n \colon \overline{R}_n \hookrightarrow \overline{C}_n.$$

Here we view $\overline{C}_n$ as a real variety and $\overline{i}_n$ as an embedding of real varieties. We also define

$$\mathcal{O}^{\mathbb{R}}(C_n) = \mathcal{O}(\overline{C}_n)^{GL_n(\mathbb{C})} \cap \mathbb{R}[x_{jl}, z_{jl}]_{j,l=1}^{n} \subset \mathcal{O}(C_n), \tag{2.4}$$

where x_{jl}, z_{jl} are the matrix entries of X, Z considered as regular functions on $\overline{C}_n$.

Proposition 2.1. *Keep the above notation. The set RC_n coincides with the real algebraic subset of C_n*

$$\{c \in C_n \mid f(c) \in \mathbb{R} \text{ for all } f \in \mathcal{O}^{\mathbb{R}}(C_n)\}.$$

There exists a (smooth) geometric quotient for the action of $GL_n(\mathbb{R})$ on $\overline{R}_n$ and thus the natural inclusion $\overline{i}_n \colon \overline{R}_n \hookrightarrow \overline{C}_n$ induces a morphism of real varieties $i_n \colon \overline{R}_n / GL_n(\mathbb{R}) \to C_n$. Furthermore, i_n defines an isomorphism of real varieties $i_n \colon \overline{R}_n / GL_n(\mathbb{R}) \cong RC_n$.

We will call the quotient $R_n := \overline{R}_n / GL_n(\mathbb{R})$ the n-th real Calogero–Moser space.

Proof. Consider the categorical quotient $\nu_n \colon \overline{R}_n \to \overline{R}_n /\!/ GL_n(\mathbb{R})$. By the universal property of categorical quotients $\overline{i}_n$ descends to the map (of real varieties)

$$i_n \colon \overline{R}_n /\!/ GL_n(\mathbb{R}) \to C_n.$$

The fiber of $\pi_n \bar{i}_n \colon \overline{R}_n \to C_n$ through $(X, Z) \in \overline{R}_n$ is

$$\{(X_1, Z_1) \in \overline{R}_n \mid \exists g \in GL_n(\mathbb{C}) \text{ such that } X_1 = gXg^{-1}, Z_1 = gZg^{-1}\}.$$

Lemma 2.2 below implies that the fiber of $i_n \nu_n = \pi_n \bar{i}_n \colon \overline{R}_n \to C_n$ through $(X, Z) \in \overline{R}_n$ is

$$\{(X_1, Z_1) \in \overline{R}_n \mid \exists g \in GL_n(\mathbb{R}) \text{ such that } X_1 = gXg^{-1}, Z_1 = gZg^{-1}\}.$$

This implies that i_n is an injection and that the fibers of ν_n are exactly the $GL_n(\mathbb{R})$ orbits on $\overline{R}_n$. In particular, ν_n is a geometric quotient; hence we will denote its range by $\overline{R}_n / GL_n(\mathbb{R})$. Analogously to [W2, Section 1] the latter is smooth.

It is clear that

$$\operatorname{Im} i_n \subseteq RC_n \subseteq X_n := \{c \in C_n \mid f(c) \in \mathbb{R} \text{ for all } f \in \mathcal{O}^{\mathbb{R}}(C_n)\}.$$

Furthermore, $\mathcal{O}(C_n)$ is the complexification of $\mathcal{O}^{\mathbb{R}}(C_n)$ since as a $GL_n(\mathbb{R})$ module $\mathcal{O}(\overline{C}_n) \cap \mathbb{R}[x_{jl}, z_{jl}]_{j,l=1}^n$ is a direct sum of finite dimensional modules and $\left(\mathcal{O}(\overline{C}_n) \cap \mathbb{R}[x_{jl}, z_{jl}]_{j,l=1}^n\right)_{\mathbb{C}} \cong \mathcal{O}(\overline{C}_n)$. Thus

$$i_n^* \colon \mathcal{O}(X_n) \to \mathcal{O}(R_n)$$

is an isomorphism, where $\mathcal{O}(X_n)$ and $\mathcal{O}(R_n)$ denote the real coordinate rings of X_n and R_n. This implies that $RC_n = X_n$ and that $i_n \colon R_n \to RC_n = X_n$ is an isomorphism of real algebraic varieties. □

Lemma 2.2. *Let $X_1, \ldots, X_k, Y_1, \ldots, Y_k \in \mathfrak{gl}_n(\mathbb{R})$. If there exists $g \in GL_n(\mathbb{C})$ such that*

$$Y_1 = gX_1g^{-1}, \ldots, Y_k = gX_kg^{-1}, \tag{2.5}$$

then there exists $g \in GL_n(\mathbb{R})$ with the same property.

Proof. We can assume that the element $g \in GL_n(\mathbb{C})$ satisfying (2.5) is such that $\operatorname{Re} g$ is nondegenerate. If this is not the case we can substitute g with $a.g$ for an appropriate scalar $a \in \mathbb{C}^*$. Taking real parts in $Y_j g = gX_j$ we find

$$Y_1 = (\operatorname{Re} g)X_1(\operatorname{Re} g)^{-1}, \ldots, Y_k = (\operatorname{Re} g)Y_1(\operatorname{Re} g)^{-1}, \quad \operatorname{Re} g \in GL_n(\mathbb{R})$$

and thus $\operatorname{Re} g \in GL_n(\mathbb{R})$ has the needed property. □

Recall the definition (1.1) of the finite map $\Upsilon \colon C_n \to \mathbb{C}^{(n)} \times \mathbb{C}^{(n)}$. The following Theorem relates the real loci of the Calogero–Moser space C_n and $\mathbb{C}^{(n)} \times \mathbb{C}^{(n)}$ by the map Υ. It was previously proved by Mukhin, Tarasov, and Varchenko [MTV2].

Theorem 2.3. *We have*

$$\Upsilon^{-1}(\mathbb{R}^{(n)} \times \mathbb{R}^{(n)}) \subset RC_n.$$

We postpone the proof of Theorem 2.3 to Sect. 3.

Denote by $\mathbb{C}^n_{\mathsf{reg}}$ the subset of $\mathbb{C}^n$ consisting of $(\lambda_1, \ldots, \lambda_n)$, $\lambda_j \neq \lambda_l$, for $j \neq l$, and by $\mathbb{C}^{(n)}_{\mathsf{reg}}$ its image in $\mathbb{C}^{(n)} = \mathbb{C}^n / S_n$. Wilson proved [W2] that C_n has a

Zariski open subset isomorphic to $T^*\mathbb{C}^{(n)}_{\mathsf{reg}}$. It is the image under π_n of the subset of $\overline{C}_n$ consisting of pairs of matrices
(2.6)

$$X = \operatorname{diag}(\lambda_1, \ldots, \lambda_n), Z = \begin{pmatrix} \alpha_1 & (\lambda_1-\lambda_2)^{-1} & \ldots & (\lambda_1-\lambda_n)^{-1} \\ (\lambda_2-\lambda_1)^{-1} & \alpha_2 & \ldots & (\lambda_2-\lambda_n)^{-1} \\ \ldots & \ldots & \ldots & \ldots \\ (\lambda_n-\lambda_1)^{-1} & (\lambda_n-\lambda_2)^{-1} & \ldots & \alpha_n \end{pmatrix}$$

where $(\lambda_1, \ldots \lambda_n) \in \mathbb{C}^n_{\mathsf{reg}}$, $(\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n$.

The restriction of Theorem 2.3 to this subset leads to the following Corollary for eigenvalues of Calogero–Moser matrices, previously proved by Mukhin, Tarasov, and Varchenko [MTV2].

Corollary 2.4. *Let $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ be an n-tuple of distinct real numbers. If an n-tuple $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{C}^n$ has the property that the Calogero–Moser matrix Z in* (2.6) *has only real eigenvalues, then $\alpha \in \mathbb{R}^n$.*

Proof. Let λ and μ be two real n-tuples as in Corollary 2.4 and X and Z be the matrices (2.6). Then according to Theorem 2.3 there exists $g \in GL_n(\mathbb{C})$, such that $g.(X, Z) \in \mathfrak{gl}_n(\mathbb{R})^{\times 2}$. Since the eigenvalues $\lambda_1, \ldots, \lambda_n$ of the matrix X are distinct, there exists a matrix $g_1 \in GL_n(\mathbb{R})$ for which $g = g_1 A$, where $A = diag(a_1, \ldots, a_n)$ with some complex numbers $a_1, \ldots, a_n$. The fact that the matrix $gZg^{-1} = g_1 AZA^{-1} g_1^{-1}$ is real shows that AZA^{-1} is also real. The off-diagonal entries of this matrix are $(\lambda_i - \lambda_j)^{-1} a_i a_j^{-1}$, i.e. $a_i a_1^{-1} \in \mathbb{R}$. One can write $(a_1, \ldots, a_n) = a_1(b_1, \ldots, b_n)$ for some real n-tuple $(b_1, \ldots, b_n) \in \mathbb{R}^n$. Now put $g_2 = g_1 . \operatorname{diag}(b_1, \ldots, b_n)$. Then $g_2 Z g_2^{-1} = g_1 Z g_1^{-1}$ is real and the matrix g_2 is real. Thus z has real entries and in particular $\alpha \in \mathbb{R}^n$. □

3. Cherednik algebras

Rational Cherednik algebras are two step degenerations of double affine Hecke algebras [Ch]. For details on rational Cherednik algebras and more generally on symplectic reflection algebras we refer the reader to the Etingof–Ginzburg paper [EG]. The rational Cherednik algebra $H_{0,c}(S_n)$ is generated by the polynomial subalgebras $\mathbb{C}[x_1, \ldots, x_n]$ and $\mathbb{C}[y_1, \ldots, y_n]$, and the group algebra $\mathbb{C}S_n$ of the symmetric group, subject to the following deformed crossed product relations

$$s_{ij} x_i = x_j s_{ij}, \quad s_{ij} y_i = y_j s_{ij},$$
$$[x_i, y_j] = c s_{ij} \ (i \neq j), \quad [x_k, y_k] = -c \sum_{i \neq k} s_{ik}.$$

The algebras $H_{0,c}(S_n)$ are isomorphic for different values of $c \neq 0$ and we will mostly restrict our attention to $c = 1$. Denote by $e = (1/n!) \sum_{\sigma \in S_n} \sigma$ the symmetrizing idempotent of $\mathbb{C}S_n \subset H_{0,1}(S_n)$. The spherical subalgebra of $H_{0,1}(S_n)$ is the subalgebra $U = eH_{0,1}(S_n)e$, [EG].

First we recall several results of Etingof and Ginzburg on finite dimensional irreducible $H_{0,1}(S_n)$ representations.

Theorem 3.1. (*Etingof-Ginzburg,* [EG, Theorems 1.23 and 1.24]) *(a) U is a commutative algebra and is isomorphic to the coordinate ring $\mathcal{O}(C_n)$ of the n-th Calogero–Moser space C_n.*

(b) The irreducible $H_{0,1}(S_n)$-representations are parametrized by the points of C_n. Given $p \in C_n$, the corresponding $H_{0,1}(S_n)$ representation is

$$M_p := H_{0,1}(S_n)e \otimes_U \chi_p$$

where $\chi_p \colon U \cong \mathcal{O}(C_n) \to \mathbb{C}$ is the character associated to p.

(c) Each representation M_c has dimension $n!$ and, as an S_n representation, is isomorphic to the regular representation of S_n.

We will also need the following additional fact from [EG] regarding the structure of the representations M_c. First, denote by S_{n-1} the subgroup of S_n permuting the last $n-1$ indices $\{2, \ldots, n\}$ and by $\overline{e} = (1/(n-1)!) \sum_{\sigma \in S_{n-1}} \sigma$ the symmetrizing idempotent of S_{n-1}. The subspace $\overline{e}M_c$ is stable under the action of x_1 and y_1 because x_1 and y_1 commute with $\overline{e}$. In any basis of $\overline{e}M$, x_1 and y_1 act by a pair of matrices $(X_c, Z_c) \in \overline{C}_n$ such that $\pi_n(X_c, Z_c) = c$

Finally, for $c \in \mathbb{R}$ we denote the real subalgebra of $H_{0,c}(S_n)$ generated by $x_1, \ldots, x_n, y_1, \ldots, y_n$ and the elements of S_n by $H^{\mathbb{R}}_{0,c}(S_n)$. It is clear that $H_{0,c}(S_n)$ is the complexification of $H^{\mathbb{R}}_{0,c}(S_n)$.

The following theorem is our main result.

Theorem 3.2. *Fix an irreducible $H_{0,1}(S_n)$ module V. If the restriction of the operators x_1 and y_1 to $\overline{e}V$ have only real eigenvalues, then V is the complexification of a (real) $H^{\mathbb{R}}_{0,1}(S_n)$ module.*

Before we prove Theorem 3.2, we note several lemmas. Let $\mathbb{R}^n_{\mathsf{reg}} = \mathbb{C}^n_{\mathsf{reg}} \cap \mathbb{R}^n$ and $\mathbb{R}^{(n)}_{\mathsf{reg}} = \mathbb{C}^{(n)}_{\mathsf{reg}} \cap \mathbb{R}^{(n)}$.

Lemma 3.3. *(a) The statements of Theorem 3.2 and Theorem 2.3 are equivalent.*

(b) To prove Theorem 2.3, it is sufficient to show that

$$\Upsilon^{-1}(\mathbb{R}^{(n)}_{\mathsf{reg}} \times \mathbb{R}^{(n)}_{\mathsf{reg}}) \subset RC_n. \tag{3.1}$$

To prove Theorem 3.2, it is sufficient to establish the validity of the statement for representations V of $H_{0,1}(S_n)$ for which x_1 and y_1 act on $\overline{e}V$ by regular semisimple operators.

Proof. (a) Assume the validity of the statement of Theorem 3.2 and fix $(X, Z) \in \overline{C}_n$ such that both X and Z have only real eigenvalues. Set $p = \pi_n(X, Z)$. Theorem 3.2 implies that M_p is the complexification of a real $H^{\mathbb{R}}_{0,1}(S_n)$ representation $M^{\mathbb{R}}_p$. Therefore $\overline{e}M^{\mathbb{R}}_p$ is a real vector space which is stable under x_1 and y_1 and such that $(\overline{e}M^{\mathbb{R}}_p)_{\mathbb{C}} = \overline{e}M_p$. If X_1 and Z_1 are the matrix representation of the restriction of the operators x_1 and y_1 to $\overline{e}M^{\mathbb{R}}_p$ in any basis of $\overline{e}M^{\mathbb{R}}_p$, then $(X_1, Z_1) \in \overline{R}_n \cap GL_n(\mathbb{C})(X, Z)$. Thus $\pi_n(X, Z) \in RC_n$.

In the opposite direction, let us assume the validity of the statement of Theorem 2.3. Fix $p \in C_n$ such that the restriction of x_1 and y_1 to $\overline{e}M_c$ have only real eigenvalues. Then p belongs to the real locus RC_n of C_n. One checks by a direct computation that under the Etingof–Ginzburg isomorphism $U \cong \mathcal{O}(C_n)$, $U^{\mathbb{R}} = eH^{\mathbb{R}}_{0,1}(S_n)e$ corresponds to $\mathcal{O}^{\mathbb{R}}(C_n)$, with the notation as in (2.4). Because of Proposition 2.1, $\chi_p \colon U \cong \mathcal{O}(C_n) \to \mathbb{C}$ restricts to a real character

$\chi_p\colon U^{\mathbb{R}} \to \mathbb{R}$. Then M_p is the complexification of the $H^{\mathbb{R}}_{0,1}(S_n)$ representation $M^{\mathbb{R}}_p = H^{\mathbb{R}}_{0,1}(S_n)e \otimes_{U^{\mathbb{R}}} \chi_p$, which establishes Theorem 3.2.

(b) Tracing back the equivalence in part (a), one sees that the second statement is a consequence of the first one. We proceed with the proof of the first statement. Since the map Υ is open in the usual topology,

$$\overline{\Upsilon^{-1}(S)} = \Upsilon^{-1}(\overline{S})$$

for any subset S of $\mathbb{C}^{(n)} \times \mathbb{C}^{(n)}$. Here $\overline{(.)}$ refers to the closure in the usual topology. To obtain the validity of Theorem 2.3 from (3.1), we apply this for the set $S = \mathbb{R}^{(n)}_{\mathsf{reg}} \times \mathbb{R}^{(n)}_{\mathsf{reg}}$ which is dense in

$$\mathbb{R}^{(n)} \times \mathbb{R}^{(n)}.$$

□

We will need a very simple deformation theoretic argument for the proof of Theorem 3.2.

Lemma 3.4. *Suppose that A is a commutative flat finite $\mathbb{R}[t]$-algebra such that for $p \in \mathbb{R}$ each specialisation $A(p) := A/(t-p)A$ is a product of fields. Then $A(p) \cong A(q)$ for all $p, q \in \mathbb{R}$.*

Proof. We are going to prove that for any $p \in \mathbb{R}$ there is an open (analytic) interval I containing p such that $A(v) \cong A(p)$ for any $v \in I$.

Let $f_1(p), \dots, f_m(p)$ be a complete set of primitive idempotents for $A(p)$ that extends to an $\mathbb{R}$-basis of $A(p)$, $f_1(p), \dots, f_m(p), f_{m+1}(p), \dots, f_n(p)$, such that $f_i(p)f_{m+j}(p) = \delta_{ij} f_{m+j}(p)$ for $1 \le i \le m$ and $1 \le j \le n-m$. Since A is a free $\mathbb{R}[t]$-module, this lifts to a basis of A denoted $f_1, \dots, f_n$. For any $u \in \mathbb{R}$ we will denote the induced multiplication in $A(u)$ by $*_u$ so that $f_i(u) *_u f_j(u) = \sum_{k=1}^n \alpha^k_{i,j}(u) f_k(u)$ where $\alpha^k_{i,j} \in \mathbb{R}[t]$. We have

$$\alpha^k_{i,j}(p) = \delta_{i,j}\delta_{i,k} \text{ for } 1 \le i, j \le m \tag{3.2}$$

and

$$\alpha^k_{i,m+j}(p) = \delta_{i,j}\delta_{m+j,k} \text{ for } 1 \le i \le m,\ 1 \le j \le n-m. \tag{3.3}$$

Consider the function $G : \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ that sends $(u; \lambda_1, \dots, \lambda_n)$ to the coefficients of $(\sum \lambda_i f_i(u)) *_u (\sum \lambda_i f_i(u)) - (\sum \lambda_i f_i(u))$ in the basis $(f_1(u), \dots, f_n(u))$ of $A(t)$. In other words $G = (G_1, \dots, G_n)$ where

$$G_k(t; \lambda_1, \dots, \lambda_n) = \sum_{i,j} \lambda_i \lambda_j \alpha^k_{i,j} - \lambda_k.$$

For $1 \le i \le m$ set $p_i = (p; 0, \dots, 0, 1, 0, \dots 0)$, the 1 occurring in the ith place. Then, by (3.2) and (3.3), $(\partial G_k / \partial \lambda_l)_{k,l}(p_i)$ is a diagonal matrix with entries from $\{1, -1\}$. Thus the determinant at p_i is non-zero, so there exists an open interval U_i containing p and continuous mapping $\theta_i : U_i \longrightarrow \mathbb{R}^n$ such that $\tilde{F}_i(u) = \sum_{j=1}^n (\theta_i(u))_j f_j(u)$ is an idempotent for each $u \in U_i$ and $\tilde{F}_i(p) = f_i(p)$. Repeating this argument for each $1 \le i \le m$ produces a continuous family of idempotents $\tilde{F}_1(u), \dots, \tilde{F}_m(u)$ for every $u \in U = \bigcap_i U_i$ and such that $\tilde{F}_i(p) =$

$f_i(p)$. This family can be adjusted inductively to produce orthogonal idempotents $F_1(u), \dots, F_m(u)$. Indeed we set $F_1(u) = \tilde{F}_1(u)$. Then if we have found $\tilde{F}_1(u), \dots, \tilde{F}_{s-1}(u)$ for some $s < m$ we set $F_s(u) = (1 - F_1(u) - \cdots - F_{s-1}(u))\tilde{F}_s(u)$. We finish by setting $F_m(u) = 1 - F_1(u) - \cdots - F_{m-1}(u)$.

Now for any $u \in U$ we have an algebra decomposition

$$A(u) = \bigoplus_{j=1}^{m} F_j(u)A(u)F_j(u) = \bigoplus_{j=1}^{m} A(u)_j.$$

Let $1 \le j \le n-m$ so that $\dim A(p)_j = 2$ with basis $f_j(p), f_{m+j}(p)$. Then on some open set V_j of U we see that $F_{m+j}(u) := F_j(u)f_{m+j}(u)$ is non-vanishing, so that $\dim A(v)_j \ge 2$ for all $v \in V_j$. Hence we produce an open interval $V = U \cap (\bigcap_{j=1}^{n-m} V_j)$ which contains p and on which $\dim A(v)_i \ge \dim A(p)_i$ for $v \in V$ and all $1 \le i \le m$. Since $\dim A(v) = \sum \dim A(v)_i$ is constant we find $\dim A(v)_i = \dim A(p)_i$ for all $1 \le i \le m$ and $v \in V$.

We must show that $A(v)_i \cong A(p)_i$. This is obvious if $\dim_{\mathbb{R}} A(v)_i = 1$, so we assume that $\dim A(v)_i = 2$. By the previous paragraph we have a basis $F_i(v), F_{m+i}(v)$ for $A(v)_i$ where $F_i(v)$ is the identity element of $A(v)_i$. Since $A(p)_i \cong \mathbb{C}$ we may assume without loss of generality that $F_{m+i}(p) *_p F_{m+i}(p) = -F_i(p)$, so if we write $F_{m+i}(v) *_v F_{m+i}(v) = \alpha(v)F_i(v) + \beta(v)F_{m+i}(v)$, then $\alpha(p) = -1$ and $\beta(p) = 0$. Now consider the mapping $H : V \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by sending $(v; \lambda_1, \lambda_2)$ to the coefficients in the basis $F_i(v), F_{m+i}(v)$ of the expression

$$(\lambda_1 F_i(v) + \lambda_2 F_{m+i}(v)) *_v (\lambda_1 F_i(v) + \lambda_2 F_{m+i}(v)) + F_i(v).$$

In other words $H(v; \lambda_1, \lambda_2) = (H_1, H_2) = (\lambda_1^2 + 1 + \lambda_2^2\alpha(v), 2\lambda_1\lambda_2 + \lambda_2^2\beta(v))$. At $(p; 0, 1)$ $\det(\partial H_i/\partial\lambda_j) = 4$, so we can find an open interval $I \subseteq V$ including p and for all $v \in I$ a basis $F_i(v), X_i(v)$ of $A(v)_i$ such that $X_i(v) *_v X_i(v) = -F_1(v)$. Hence $A(v)_i \cong \mathbb{C} \cong A(p)_i$, as required. □

The lemma that follows is well-known, but we have been unable to find a proof in the literature. To ease notation let $\alpha_{ij} = x_i - x_j$ and $\alpha_{ij}^\vee = y_i - y_j$ and let $\delta = \prod_{i<j} \alpha_{ij} \in \mathbb{R}[x_1, \dots, x_n]$ be the discriminant.

Lemma 3.5. *For any $c \in \mathbb{C}$ we have $H_{0,c}[\delta^{-1}] \cong \mathbb{C}[\mathbb{C}^n_{\mathsf{reg}} \times \mathbb{C}^n] * S_n$ and if $c \in \mathbb{R}$ then $H^{\mathbb{R}}_{0,c}[\delta^{-1}] \cong \mathbb{R}[\mathbb{R}^n_{\mathsf{reg}} \times \mathbb{R}^n] * S_n$.*

Proof. Assume that $c \in \mathbb{R}$. We define the isomorphism $\Theta_c^{\mathbb{R}} : H^{\mathbb{R}}_{0,c}[\delta^{-1}] \longrightarrow \mathbb{R}[\mathbb{R}^n_{\mathsf{reg}} \times \mathbb{R}^n] * S_n$ by

$$x \mapsto x, \quad y \mapsto y + c\Delta(y), \quad w \mapsto w,$$

where $\Delta(y) = \sum_{i<j} \frac{\langle y, \alpha_{ij}\rangle}{\alpha_{ij}}((i\,j) - 1)$. Once we show that this mapping is well-defined it is clear that it is an isomorphism since we can remove the $\Delta(y)$-term in $\Theta_c^{\mathbb{R}}(y)$ by subtracting elements from $\mathbb{R}[\mathbb{R}^n_{\mathsf{reg}}] * S_n$.

To prove well-definedness we recall the Dunkl isomorphism constructed in [EG, Proposition 4.5]. For $t \ne 0$ this produces an isomorphism $\Theta_c^t : H_{t,c}[\delta^{-1}] \longrightarrow \mathcal{D}(\mathbb{C}^n_{\mathsf{reg}}) * S_n$ by

$$x \mapsto x, \quad y \mapsto t\partial_y + c\sum_{i<j} \frac{\langle y, \alpha_{ij}\rangle}{\alpha_{ij}}((i\,j) - 1), \quad w \mapsto w.$$

Thus $\Theta_c^{\mathbb{R}}$ is a real form of the semi-classical limit $\lim_{t\to 0}\Theta_c^t$. From the Dunkl isomorphism Θ_1^t and the defining relation for $H_{t,1}$, see [EG, Formula (1.15)], we find

$$[\Delta(y), x] = -\frac{1}{2}\sum_{i<j}\langle y, \alpha_{ij}\rangle\langle\alpha_{ij}^\vee, x\rangle(i\,j) \text{ and } [\Delta(y), \Delta(y')] = 0.$$

From this we deduce that

$$[\Theta_c^{\mathbb{R}}(y), \Theta_c^{\mathbb{R}}(x)] = [y, x] + c[\Delta(y), x] = -c\frac{1}{2}\sum_{i<j}\langle y, \alpha_{ij}\rangle\langle\alpha_{ij}^\vee, x\rangle(i\,j) = \Theta_c^{\mathbb{R}}([y, x]),$$

and

$$\begin{aligned}[\Theta_c^{\mathbb{R}}(y), \Theta_c^{\mathbb{R}}(y')] &= [y, y'] + c[y, \Delta(y')] + c[\Delta(y), y'] + c^2[\Delta(y), \Delta(y')] \\ &= c([y, \Delta(y')] + [\Delta(y), y']).\end{aligned}$$

So it remains to check that $[y, \Delta(y')] + [\Delta(y), y'] = 0$. Well,

$$\begin{aligned}[y, \Delta(y')] + [\Delta(y), y'] &= \sum_{i<j}\left(\frac{\langle y', \alpha_{ij}\rangle}{\alpha_{ij}}[y, (i\,j) - 1] - \frac{\langle y, \alpha_{ij}\rangle}{\alpha_{ij}}[y', (i\,j) - 1]\right) \\ &= \sum_{i<j}\frac{1}{\alpha_{ij}}\left(\langle y', \alpha_{ij}\rangle(y - {}^{(i\,j)}y) - \langle y, \alpha_{ij}\rangle(y' - {}^{(i\,j)}y')\right)(i\,j) \\ &= \sum_{i<j}\frac{1}{\alpha_{ij}}\left(\langle y', \alpha_{ij}\rangle\langle y, \alpha_{ij}\rangle\alpha_{ij}^\vee - \langle y, \alpha_{ij}\rangle\langle y', \alpha_{ij}\rangle\alpha_{ij}^\vee\right)(i\,j) \\ &= 0.\end{aligned}$$

The isomorphism for $H_{0,c}[\delta^{-1}]$ follows by an identical argument. □

Proof of Theorem 3.2. By Lemma 3.3 it is enough to prove this when both x_1 and y_1 have distinct eigenvalues on $\overline{e}V$. Now by [EG, Theorem 11.16] the eigenvalues of x_1 on $\overline{e}V$ coincide with the action of $\mathbb{C}[\mathbb{C}^n]^{S_n} \subset Z(H)$ on V and so we see that $\delta^2 \in \mathbb{C}[\mathbb{C}^n]^{S_n}$ acts by a non-zero scalar on V. In particular V is naturally an irreducible $H[\delta^{-1}]$-representation.

Let $\mathfrak{m}$ be the maximal ideal of $\mathbb{R}[\mathbb{R}^n_{\mathsf{reg}}]^{S_n} \otimes \mathbb{R}[\mathbb{R}^n_{\mathsf{reg}}]^{S_n}$ corresponding to the eigenvalues of x_1 and y_1. For $c \in \mathbb{R}$ set

$$H(c) = H_{0,c}^{\mathbb{R}}/\mathfrak{m}H_{0,c}^{\mathbb{R}} \cong H_{0,c}^{\mathbb{R}}[\delta^{-1}]/\mathfrak{m}H_{0,c}^{\mathbb{R}}[\delta^{-1}],$$

a flat family of algebras over $\mathbb{R}$. By definition V is an irreducible $\mathbb{C}\otimes_{\mathbb{R}} H(1)$-representation, so we must prove that all such representations are extensions of $H(1)$-representations. To do this we will simply prove that $H(c) \cong \mathrm{Mat}_{n!}(\mathbb{R})^{\oplus n!}$.

We translate the problem to $\mathbb{R}[\mathbb{R}^n_{\mathsf{reg}} \times \mathbb{R}^n] * S_n$ by applying $\Theta_c^{\mathbb{R}}$ of Lemma 3.5 to the family $H(c)$. This produces the algebras

$$\tilde{H}(c) = \frac{\mathbb{R}[\mathbb{R}^n_{\mathsf{reg}} \times \mathbb{R}^n] * S_n}{\Theta_c^{\mathbb{R}}(\mathfrak{m})\mathbb{R}[\mathbb{R}^n_{\mathsf{reg}} \times \mathbb{R}^n] * S_n}$$

However, $\Theta_c^{\mathbb{R}}(\mathfrak{m}) \subset Z(\mathbb{R}[\mathbb{R}^n_{\mathsf{reg}} \times \mathbb{R}^n] * S_n) = \mathbb{R}[\mathbb{R}^n_{\mathsf{reg}} \times \mathbb{R}^n]^{S_n}$ so that we actually have

$$\tilde{H}(c) = \left(\frac{\mathbb{R}[\mathbb{R}^n_{\mathsf{reg}} \times \mathbb{R}^n]}{\Theta_c^{\mathbb{R}}(\mathfrak{m})\mathbb{R}[\mathbb{R}^n_{\mathsf{reg}} \times \mathbb{R}^n]}\right) * S_n.$$

Hence we find a flat family of commutative $\mathbb{R}$-algebras of dimension $(n!)^2$

$$A(c) = \frac{\mathbb{R}[\mathbb{R}^n_{\mathsf{reg}} \times \mathbb{R}^n]}{\Theta^{\mathbb{R}}_c(\mathfrak{m})\mathbb{R}[\mathbb{R}^n_{\mathsf{reg}} \times \mathbb{R}^n]}.$$

We consider first $c = 0$. The homomorphism Θ_0 restricts to the inclusion $\mathbb{R}[\mathbb{R}^n]^{S_n} \otimes \mathbb{R}[\mathbb{R}^n]^{S_n} \longrightarrow \mathbb{R}[\mathbb{R}^n_{\mathsf{reg}} \times \mathbb{R}^n]$ and so

$$A(0) = \frac{\mathbb{R}[\mathbb{R}^n_{\mathsf{reg}} \times \mathbb{R}^n]}{\mathfrak{m}\mathbb{R}[\mathbb{R}^n_{\mathsf{reg}} \times \mathbb{R}^n]} \cong \frac{\mathbb{R}[\mathbb{R}^n_{\mathsf{reg}} \times \mathbb{R}^n_{\mathsf{reg}}]}{\mathfrak{m}\mathbb{R}[\mathbb{R}^n_{\mathsf{reg}} \times \mathbb{R}^n_{\mathsf{reg}}]},$$

where the last isomorphism holds since $\mathfrak{m}$ is a maximal ideal of $\mathbb{R}[\mathbb{R}^n_{\mathsf{reg}}]^{S_n} \otimes \mathbb{R}[\mathbb{R}^n_{\mathsf{reg}}]^{S_n}$. Since $S_n \times S_n$ acts freely on $\mathbb{R}^n_{\mathsf{reg}} \times \mathbb{R}^n_{\mathsf{reg}}$ it follows that there are exactly $(n!)^2$ points of $\mathbb{R}^n_{\mathsf{reg}} \times \mathbb{R}^n_{\mathsf{reg}}$ lying above $\mathfrak{m}$ and, in particular, $n!$ distinct free (diagonal) S_n-orbits of points in $\mathbb{R}^n_{\mathsf{reg}} \times \mathbb{R}^n_{\mathsf{reg}}$. Thus we have a direct product expansion

$$A(0) \cong \prod_{i=1}^{n!} \prod_{\sigma \in S_n} \mathbb{R} e_{i,\sigma}$$

where the $e_{i,\sigma}$ are pairwise orthogonal idempotents and $\tau e_{i,\sigma} = e_{i,\tau\sigma}$ for any $\tau \in S_n$.

Note that each $A(c)$ is separable. To see this we may assume that $c \neq 0$ since we dealt with the case $c = 0$ above. Consider the largest nilpotent ideal J of $A(c)$. It must be S_n-stable and so extend to a nilpotent ideal $(J \otimes \mathbb{C}) * S_n$ of $\tilde{H}(c) \otimes \mathbb{C} \cong H(c) \otimes \mathbb{C}$. But this last algebra is semisimple by [EG, Theorem 1.7(i)] and so $J = 0$.

Now Lemma 3.4 shows that all $A(c)$ are isomorphic as $\mathbb{R}$-algebras. Thus since the maximal spectrum of $A(0)$ corresponds to $n!$ distinct free (diagonal) S_n-orbits in $\mathbb{R}^n_{\mathsf{reg}} \times \mathbb{R}^n_{\mathsf{reg}}$, the same is true for all $A(c)$. Therefore $H(c) \cong \prod_{i=1}^{n!} \left(\prod_{\sigma \in S_n} e_{i,\sigma}\right) * S_n$. But $\left(\prod_{\sigma \in S_n} e_{i,\sigma}\right) * S_n \cong \mathrm{Mat}_{n!}(\mathbb{R})$, the isomorphism being given by sending $e_{i,\sigma} \otimes \tau$ to the elementary matrix $E_{\sigma,\tau^{-1}\sigma}$, where we label the rows and columns of $\mathrm{Mat}_{n!}(\mathbb{R})$ by the elements of S_n. Setting $c = 1$ proves the theorem. □

Note that Theorem 2.3 follows from Theorem 3.2 and Lemma 3.3 (a).

4. Deformed preprojective algebras of quivers

Let $Q = (Q_0, Q_1)$ be a finite quiver with vertex set Q_0 and arrow set Q_1. Denote by $\overline{Q}$ its double, obtained by adding a reverse arrow a^* for each arrow a of Q_1. The deformed preprojective algebra of Q of weight $\nu = (\nu_i)_{i \in Q_0} \in \mathbb{C}^{Q_0}$, was defined by Crawley-Boevey and Holland [CBH] as the quotient of the path algebra $\mathbb{C}\overline{Q}$

$$\Pi_\nu(Q) = \mathbb{C}\overline{Q} / \langle \sum_{a \in Q_1} [a, a^*] - \sum_{i \in Q_0} \nu_i e_i \rangle,$$

where the e_i denote the standard idempotents of $\mathbb{C}\overline{Q}$. For a real weight $\nu = (\nu_i)_{i \in I} \in \mathbb{R}^I$, let

$$\Pi^{\mathbb{R}}_\nu(Q) = \mathbb{R}\overline{Q} / \langle \sum_{a \in Q_1} [a, a^*] - \sum_{i \in Q_0} \nu_i e_i \rangle.$$

Thus, for $\nu \in \mathbb{R}^I$, $\Pi_\nu(Q)$ is the complexification of $\Pi^{\mathbb{R}}_\nu(Q)$.

We restrict our attention to the quiver Q with 2 vertices 0 and ∞, and two arrows $v\colon 0 \to \infty$ and $X\colon 0 \to 0$. We set $\nu_0 = -1, \nu_\infty = n$ and denote $w = v^*$, $Z = X^*$. The algebra $\Pi_\nu(Q)$ is then generated by X, Z, v, w and the idempotents e_0, e_∞, and these satisfy the path algebra relations and

$$[X, Z] - wv = -e_0, \quad vw = ne_\infty.$$

A left $\Pi_\nu(Q)$-module is thus a complex vector space $V = V_0 \oplus V_\infty$ with the data of

$$X, Z \in \mathrm{End}(V_0), \quad v \in \mathrm{Hom}(V_0, V_\infty), \quad w \in \mathrm{Hom}(V_\infty, V_0)$$

such that

$$[X, Z] + \mathrm{Id}_{V_0} = wv, \quad vw = n\mathrm{Id}_{V_\infty}.$$

We restrict our attention to representations of $\Pi_\nu(Q)$ of dimension vector $(n, 1)$, i.e. such that $\dim V_0 = n$, $\dim V_\infty = 1$. All such representations are irreducible by [CB, W2].

Theorem 2.3 has the following corollary.

Corollary 4.1. *Consider a representation $V_0 \oplus V_\infty$ of the deformed preprojective algebra $\Pi_\nu(Q)$ of the above quiver for the weight $\nu = (-1, n)$ with dimension vector $(n, 1)$. If the operators $X, Z \in \mathrm{End}(V_0)$ have only real eigenvalues, then this representation is the complexification of a representation of the real algebra $\Pi_\nu^{\mathbb{R}}(Q)$.*

Proof. By Theorem 2.3 we can find a basis of V_0 for which the entries of X and Z are real. Then any non-zero element in the image of this basis under the mapping v provides a basis of V_∞ for which the entries of v and w are real. □

5. The real locus of the Wilson's adelic Grassmannian

In this section we define the real locus of Wilson's adelic Grassmannian. Following Theorem 2.3 we formulate an elementary criterion for a point of Wilson's Grassmannian to belong to its real locus.

We start with a few general facts on the real locus of Wilson's adelic Grassmannian. For details, we refer the reader to Wilson's papers [W1, W2], van Moerbeke's review [vM], and the paper [BHY2].

Sato's Grassmannian is an infinite dimensional Grassmannian of subspaces of $\mathbb{C}[z][[z^{-1}]]$ of a particular type, see [vM]. It is the phase space of the KP hierarchy (an infinite dimensional integrable system). Wilson's adelic Grassmannian is the subset of Sato's Grassmannian which parametrizes all rank 1 bispectral wave functions, [W1]. To a point $W \in \mathrm{Gr}^{\mathrm{ad}}$, in other words a subspace of $\mathbb{C}[z][[z^{-1}]]$ of a particular type, one associates its tau function

$$\tau_W(t_1, t_2, \ldots) \in \mathbb{C}[[t_1, t_2, \ldots]]$$

(defined up to a nonzero factor) which is the image of W under the Plücker embedding of $\mathrm{Gr}^{\mathrm{ad}}$ into the projectivization of $\mathbb{C}[[t_1, t_2, \ldots]]$. To $W \in \mathrm{Gr}^{\mathrm{ad}}$ one also associates its wave function

(5.1)
$$\Psi_W(x, z) = e^{xz}\left(1 + a_1(x)z^{-1} + a_2(x)z^{-2} + \cdots\right), \quad a_1(x), a_2(x), \ldots \in \mathbb{C}(x).$$

The tau and wave functions of W are related by Sato's formula

$$\Psi_W(x,z) = e^{\sum_{k=1}^{\infty} t_k z^k} \frac{\tau\left(t - [z^{-1}]\right)}{\tau(t)}\Big|_{t_1=x, t_2=t_3=\ldots=0}. \tag{5.2}$$

Here and below we abbreviate $t = (t_1, t_2, \ldots)$ and $[z^{-1}] = (z^{-1}, z^{-2}/2, z^{-3}/3, \ldots)$.

We will need two different parametrizations of $\mathrm{Gr}^{\mathrm{ad}}$, both due to Wilson, [W1, W2]. According to [W2, Proposition 2.9] all tau functions $\tau_W(t)$ are polynomials in t_1 with leading coefficient 1 – the other coefficients depend on $t_2, t_3, \ldots$. Denote by $\mathrm{Gr}^{\mathrm{ad}}_n$ the set of tau functions in $\mathrm{Gr}^{\mathrm{ad}}$ which are polynomials of degree n in t_1. The coefficients of τ_W give $\mathrm{Gr}^{\mathrm{ad}}_n$ the structure of a finite-dimensional complex affine variety. Set $C = \sqcup_{n \in \mathbb{Z}_{\geq 0}} C_n$ and define Wilson's map $\beta\colon C \to \mathrm{Gr}^{\mathrm{ad}}$ by

$$\beta(\pi_n(X,Z)) = W \ \text{ where } \ \tau_W := \det(X + \sum_{j=1}^{\infty} j t_j (-Z)^{i-1}), \ \text{for } (X,Z) \in \overline{C}_n. \tag{5.3}$$

Clearly $\beta(C_n) \subset \mathrm{Gr}^{\mathrm{ad}}_n$. The corresponding wave function is given by

$$\Psi_W = e^{xz} \det(I_n - (xI_n + X)^{-1}(zI_n + Z)^{-1}), \tag{5.4}$$

where I_n is the identity matrix of size $n \times n$.

We denote by G the set consisting of a finite collection of distinct complex numbers $\mu_1, \ldots, \mu_k$ ($k \in \mathbb{Z}_{\geq 0}$) and a collection of subspaces $V_1, \ldots, V_k$ of $\mathbb{C}[x]$, associated to each of them. In other words G consists of tuples

$$(\mu_1, \ldots, \mu_k, V_1, \ldots, V_k)$$

where two tuples of this kind are identified if one of them is obtained from the other by a simultaneous permutation of the μ's and the V's. Define Wilson's map [W1] $\gamma\colon G \to \mathrm{Gr}^{\mathrm{ad}}$ by

$$\gamma(\mu_1, \ldots, \mu_k, V_1, \ldots, V_k) = W \ \text{ where } \ \Psi_W(x,z) = \frac{1}{p(z)} P_W(x, \partial_x) e^{xz}. \tag{5.5}$$

Here $p(z) = \prod_j (z - \mu_j)^{\dim V_j}$ and $P_W(x, \partial_x)$ is the monic differential operator of degree $\dim V_1 + \cdots + \dim V_k$ and kernel

$$e^{\mu_1 x} V_1 \oplus \cdots \oplus e^{\mu_k x} V_k.$$

(This is the version of Wilson's map from [BHY2].) Finally, denote by G' the subset of G consisting of $(\mu_1, \ldots, \mu_k, V_1, \ldots, V_k)$ where all subspaces V_j of $\mathbb{C}[x]$ contain no constants except 0.

Theorem 5.1. (*Wilson,* [W1, W2])

(a) The maps $\beta\colon C_n \to \mathrm{Gr}^{\mathrm{ad}}_n$ are bijections.

(b) The map $\gamma\colon G' \to \mathrm{Gr}^{\mathrm{ad}}$ is a bijection. Moreover, for a pair $(y \in G, y' \in G')$, $\gamma(y') = \gamma(y)$ if an only if y is obtained from y' by repeated applications of one of the following two rules:

$$(\mu_1, \ldots, \mu_k, V_1, \ldots, V_k) \mapsto (\mu_1, \ldots, \mu_k, \mu_{k+1}, V_1, \ldots, V_k, \mathbb{C})$$

(for $\mu_{k+1} \neq \mu_j$, $j = 1, \ldots, k$) and

$$(\mu_1, \ldots, \mu_k, V_1, \ldots, V_k) \mapsto (\mu_1, \ldots, \mu_k, V_1, \ldots, V_{k-1}, \widetilde{V}_k)$$

where $\widetilde{V}_k = \{p(x) \mid p'(x) \in V_k\}$.

In the next theorem we describe the set of real points of Wilson's adelic Grassmannian.

Theorem 5.2. *For a point* W *in Wilson's adelic Grassmannian* $\mathrm{Gr}^{\mathrm{ad}}$ *the following conditions are equivalent.*

1) The plane W *has a real basis.*

2) Up to a nonzero scalar the tau function τ_W *has real coefficients.*

3) The wave function $\Psi_W(x,z)$ *has real coefficients, i.e.* $a_1(x), a_2(x), \ldots \in \mathbb{R}(x)$ *in* (5.1).

4) $W = \beta(c)$ *for some* $n \in \mathbb{Z}_{\geq 0}$, $c \in RC_n$, *i.e.* τ_W *is given by* (5.3) *for a pair of real matrices* $(X, Z) \in \overline{R}_n$.

5) $W = \gamma(\mu_1, \ldots, \mu_k, V_1, \ldots, V_k)$ *for some* $(\mu_1, \ldots, \mu_k, V_1, \ldots, V_k) \in G$ *with the following properties:*

(a) each μ_j *appears together with its complex conjugate and*

(b) if $\overline{\mu}_j = \mu_k$, *then* $\overline{V}_j = V_k$.

Here for $p(x) = \sum_{j=1}^m a_j x^j$, we set $\overline{p(x)} = \sum_{j=1}^m \overline{a}_j x^j$. Condition 4 is the same as saying that *one can find bases of* $V_1, \ldots, V_k$ *such that the basis of* V_j *is real if* μ_j *is real and the bases of* V_j *and* V_k *are complex conjugate if* $\overline{\mu}_j = \mu_k$. It can be also restated to: *the space of functions* $e^{\mu_1 x} V_1 \oplus \cdots \oplus e^{\mu_k x} V_k$ *on* $\mathbb{R}$ *has a real basis.*

We define the *real locus of Wilson's adelic Grassmannian* as the set of all $W \in \mathrm{Gr}^{\mathrm{ad}}$ which satisfy any of the five equivalent conditions in Theorem 5.2.

Proof. First we show $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1$. The tau function $\tau_W(t)$ is constructed from W by the so called boson-fermion correspondence [K], from which $1 \Rightarrow 2$ is straightforward. Sato's formula (5.2) shows that $2 \Rightarrow 3$. Finally $3 \Rightarrow 1$ because

$$W = \mathrm{Span}\{\partial_x^j \Psi_W(x,z)|_{x=0}\}_{j=0}^{\infty},$$

cf. [vM]. (If $\Psi_W(x,z)$ is singular at 0, one evaluates its derivatives at any real point r. To get W, one flows back the plane obtained in this fashion, with respect to the first KP flow, see e.g. [BHY2].)

The implications $4 \Rightarrow 2$ and $5 \Rightarrow 3$ are obvious from (5.3) and (5.5).

Next we show $2 \Rightarrow 5$. Assume that up to a nonzero scalar $\tau_W(t)$ has real coefficients and that $W = \gamma(\mu_1, \ldots, \mu_k, V_1, \ldots, V_k)$, for some $(\mu_1, \ldots, \mu_k, V_1, \ldots, V_k) \in G'$. Taking complex conjugates in [BHY1, Theorem 1] (see also [W1, eq. (5.7)]), we find $W = \gamma(\overline{\mu}_1, \ldots, \overline{\mu}_k, \overline{V}_1, \ldots, \overline{V}_k)$. Now 5 follows from the bijectivity of $\gamma \colon G' \to \mathrm{Gr}^{\mathrm{ad}}$, and the fact that $(\overline{\mu}_1, \ldots, \overline{\mu}_k, \overline{V}_1, \ldots, \overline{V}_k) \in G'$. We recall that two tuples of the type $(\mu_1, \ldots, \mu_k, V_1, \ldots, V_k)$ are identified as elements of G', if one of them is obtained from the other by simultaneous permutations of the μ's and the V's.

Finally we prove $2 \Rightarrow 4$. We need to show that for $c \in C_n$, if $\tau_{\beta(c)}(t)$ has real coefficients up to a non-zero factor, then $c \in RC_n$. First we assume that $c = \pi_n(X, Z)$ for some $(X, Z) \in \overline{C}_n$ where X has distinct eigenvalues. Since X has distinct eigenvalues, we can assume that $X = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ and that Z has the form (2.6) for some $\lambda_1, \ldots, \lambda_n, \alpha_1, \ldots, \alpha_n \in \mathbb{C}$. The reality of $\tau_{\beta(c)}(t)$

implies that $\beta(\pi_n(X,Z)) = \beta(\pi_n(\overline{X},\overline{Z}))$. Because $\beta\colon C_n \to \mathrm{Gr}_n^{ad}$ is a bijection by Theorem 5.1 (a), there exists $g \in GL_n(\mathbb{C})$ such that $\overline{X} = gXg^{-1}$, $\overline{Z} = gZg^{-1}$. In particular, possibly after reindexing, we have for some $l \leq n/2$:

$$\lambda_1,\dots,\lambda_{2l} \notin \mathbb{R}, \lambda_2 = \overline{\lambda}_1,\dots,\lambda_{2l} = \overline{\lambda}_{2l-1}, \quad \lambda_{2l+1},\dots,\lambda_n \in \mathbb{R}$$

and $\overline{Z} = gZg^{-1}$ for $g = \mathrm{diag}(B,\dots,B,I_{n-2l})$, where $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. This implies that

$$\alpha_2 = \overline{\alpha}_1,\dots,\alpha_{2l} = \overline{\alpha}_{2l-1}, \quad \alpha_{2l+1},\dots,\alpha_n \in \mathbb{R}.$$

Set

$$B' = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}, \quad S = \mathrm{diag}(B',\dots,B',I_{n-2l})$$

and $X' = SXS^{-1}, Z' = SZS^{-1}$. Then the matrices X' and Z' have real entries and $c = \pi_n(X',Z')$. Thus $c \in RC_n$.

For the general case of $2 \Rightarrow 4$, suppose that, up to a nonzero factor, $\tau_{\beta(c)}(t)$ has real coefficients for some $c \in C_n$. Because of [SW, Proposition 8.6] for almost all $r \in \mathbb{R}$, $\tau_{\beta(c)}(t_1+r,t_2,\dots) = \tau_{\beta(c')}(t_1,t_2,\dots)$ for some $c' \in C_n$ such that $c' = \pi_N(X',Z')$ where X' has distinct eigenvalues. Fix such an r. Since $\tau_{\beta(c')}(t)$ also has real coefficients, the above shows that $c' \in RC_n$. This means that $c' = \pi_n(X'',Z'')$ for some matrices X'', Z'' with real entries. Wilson's result [W2, Lemma 4.1] that β intertwines the Calogero–Moser flows and the KP flows – see Remark 5.3 below – implies that $c = \pi_n(X''+rI_n,Z'') \in RC_n$, where I_n is the identity matrix of size $n \times n$. (This also easily follows from (5.3).) This completes the proof of the theorem. □

Remark 5.3. Wilson's adelic Grassmannian is invariant under the flows of the KP hierarchy, an infinite dimensional hamiltonian system on Sato's Grassmannian. On the level of tau functions the flows of the KP hierarchy account for shifts of the variables $t_1, t_2, \dots$ Similarly the Calogero–Moser spaces are the phase spaces of the Calogero–Moser hamiltonian systems, [W2]. The latter are hamiltonian systems with respect to a natural symplectic form on C_n and hamiltonians $h_k(\pi_n(X,Z)) = (-1)^{k-1}\,\mathrm{tr}\,Z^k$. They are the projections to C_n of the linear flows $(X,Z) \mapsto (X+kt(-Z)^{k-1},Z)$ on $\overline{C}_n$. Wilson proved [W2, Lemma 4.1] that the map (5.3) intertwines the KP and the Calogero-Moser flows. Both the real loci of C_n and the real locus of $\mathrm{Gr}^{\mathrm{ad}}$ are invariant under the Calogero-Moser flows and the KP flows, respectively, for real times.

We finish this section with a criterion for a point of Wilson's adelic Grassmannian to belong to its real locus. Wilson's Grassmannian possesses a remarkable involution called the bispectral involution. It is defined by

$$\Psi_{bW}(x,z) = \Psi_W(z,x), \quad W \in \mathrm{Gr}^{ad}.$$

This means that in (5.1) we expand all coefficients $a_j(x)$ for large x and sum up the powers of z in front of equal powers of x. It is a nontrivial statement proved in [W1] that $\Psi_{bW}(x,z)$ has the form (5.1). On level of the map (5.3), the bispectral involution is given by

$$b(\beta(\pi_n(X,Z))) = \beta(\pi_n(Z^t,X^t)), \quad (X,Z) \in \overline{C}_n,$$

see [W2].

Theorem 5.4. *Assume that $W \in \mathrm{Gr}^{\mathrm{ad}}$. If both $\tau_W(x,0,\dots)$ and $\tau_{bW}(x,0,\dots)$ have only real roots, then W belongs to the real locus of $\mathrm{Gr}^{\mathrm{ad}}$.*

Theorem 5.4 follows from Theorem 2.3 and Theorem 5.2, and the fact that for $W = \beta(\pi_n(X,Z))$

$$\tau_W(x,0,\dots) = (-1)^n \chi_X(-x), \quad \tau_{bW}(x,0,\dots) = (-1)^n \chi_Z(-x),$$

where $\chi_X(x)$ denotes the characteristic polynomial of X.

6. Spaces of quasipolynomials

Fix $\mu = (\mu_1, \dots, \mu_k) \in \mathbb{C}^k_{\mathsf{reg}}$, For a k-tuple of finite dimensional subspaces $V_1, V_2, \dots, V_k$ of $\mathbb{C}[x]$, we define the normalized Wronskian

$$\mathrm{Wr}(e^{\mu_1 x} V_1, \dots, e^{\mu_k x} V_k)$$

as follows: choose a basis $\{q_1(x), \dots, q_N(x)\}$ of $e^{\mu_1 x} V_1 \oplus \cdots \oplus e^{\mu_k x} V_k$ and set

$$\mathrm{Wr}(e^{\mu_1 x} V_1, \dots, e^{\mu_k x} V_k) = a_0^{-1} e^{-(\mu_1 + \cdots + \mu_k)x} \mathrm{Wr}(q_1(x), \dots, q_N(x))$$

where a_0 is the leading coefficient of

$$e^{-(\mu_1 + \cdots + \mu_k)x} \mathrm{Wr}(q_1(x), \dots, q_N(x)).$$

It is easy to check that the latter is a polynomial in x. It is also straightforward to see that this definition does not depend on the choice of the basis $\{q_1(x), \dots, q_N(x)\}$.

Recall the definition of Wilson's map $\gamma\colon G \to \mathrm{Gr}^{\mathrm{ad}}$ from (5.5).

Lemma 6.1. *In the above setting*

$$\tau_{\gamma(\mu_1,\dots,\mu_k,V_1,\dots,V_k)}(x,0,\dots) = s\, \mathrm{Wr}(e^{\mu_1 x} V_1, \dots, e^{\mu_k x} V_k), \quad s \in \mathbb{C}^*.$$

This lemma can be extracted from [SW, Proposition 3.3]. It follows by observing that the coefficient $a_1(x)$ of $\Psi_{\gamma(\mu_1,\dots,\mu_k,V_1,\dots,V_k)}(x,z)$ in (5.1) is given by

$$a_1(x) = -\partial_x \log \tau_{\gamma(\mu_1,\dots,\mu_k,V_1,\dots,V_k)}(x,0,\dots)$$

because of (5.2), and by

$$a_1(x) = -\partial_x \log \mathrm{Wr}(e^{\mu_1 x} V_1, \dots, e^{\mu_k x} V_k)$$

because of (5.5).

Using Wilson's map γ, we rederive the following theorem of Mukhin, Tarasov, and Varchenko [MTV2] from Theorem 2.3.

Theorem 6.2. *Let $(V_1, V_2, \dots, V_k)$ be a k-tuple of subspaces of $\mathbb{C}[x]$. If $\mu_1, \dots, \mu_k$ are real distinct numbers and the polynomial $\mathrm{Wr}(e^{\mu_1 x} V_1, \dots, e^{\mu_k x} V_k)$ has only real roots, then each of the subspaces $V_1, \dots, V_k$ has a basis consisting of polynomials with real coefficients.*

Proof. Fix a collection such that $\mathrm{Wr}(e^{\mu_1 x}V_1, \dots, e^{\mu_k x}V_k)$ has only real roots and $\mu_1, \dots, \mu_k$ are distinct real numbers. We consider $(\mu_1, \dots, \mu_k, V_1, \dots, V_k)$ as an element of G. Let

$$\gamma(\mu_1, \dots, \mu_k, V_1, \dots, V_k) = \beta(\pi_n(X, Z))$$

for some $(X, Z) \in \overline{C}_n$. From (5.4) and (5.5), we obtain that all eigenvalues of Z are among $-\mu_1, \dots, -\mu_k$, and thus are real. Comparing (5.3) and Lemma 6.1, we see that

$$\mathrm{Wr}(e^{\mu_1 x}V_1, \dots, e^{\mu_k x}V_k) = (-1)^n \chi_X(-x). \tag{6.1}$$

Therefore all eigenvalues of X are real as well. Theorem 2.3 then implies that $\pi_n(X, Z) \in RC_n$ and Theorem 5.2 implies that $\beta(\pi_n(X, Z))$ belongs to the real locus of Gr^{ad}. Applying Theorem 5.2 again, we obtain that

$$\gamma(\mu_1, \dots, \mu_k, V_1, \dots, V_k) = \gamma(\nu_1, \dots, \nu_l, U_1, \dots, U_l)$$

for some $(\nu_1, \dots, \nu_l, U_1, \dots, U_k) \in G'$ with the properties of Theorem 5.2(5). Because of part (b) of Theorem 5.1, $\{\nu_1, \dots, \nu_l\}$ is a subset of $\{\mu_1, \dots, \mu_k\}$. So all $\nu_1, \dots, \nu_l$ are real, and therefore each of the spaces $U_1, \dots, U_l$ has a real basis because Condition 5 in Theorem 5.2 is satisfied. Theorem 5.1 now implies that the vector spaces $V_1, \dots, V_k$ have the same properties. This completes the proof of the theorem. □

The Shapiro conjecture is the special case of Theorem 6.2 when $k = 1$ and $\mu_1 = 0$.

Tracing back the relation between Theorem 2.3, Theorem 3.2 and Theorem 5.4, we find three equivalent formulations of the Shapiro in the contexts of Calogero–Moser spaces, representations of $H_{0,1}(S_n)$ and Schur functions.

The tau functions of the KP hierarchy of the type $\tau_{\gamma(0,V)}(t)$ for $V \subset \mathbb{C}[x]$ are exactly the polynomial tau functions, i.e. those which are polynomials and only depend on finitely many of the variables $t_1, t_2, \dots$. Recall that for a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_l > 0)$ the corresponding Schur function is given by

$$s_\lambda(p_1, \dots, p_N) = \det(S_{\lambda_i + j - i}(p))_{i,j=1}^{l},$$
$$s_m(p_1, \dots, p_m) = \sum_{j_1 + 2j_2 + \dots = m} \frac{p_1^{j_1}}{j_1} \frac{p_2^{j_2}}{j_2!} \dots,$$

where $N = \lambda_1 + l - 1$. All polynomial tau functions of the KP hierarchy are of the type

$$s_\lambda(t_1 + c_1, \dots, t_N + c_N)$$

for some complex numbers $c_1, \dots, c_N \in \mathbb{C}$, see [SW, W2]. Finally, $\tau_{\beta(X,Z)}(t)$ is a polynomial tau function if and only if Z is nilpotent, [W2, Proposition 6.1].

We obtain the following corollary for Schur functions.

Corollary 6.3. *Fix a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_l > 0)$ and an N-tuple of complex numbers $c_1, \dots, c_N \in \mathbb{C}$, where $N = \lambda_1 + l - 1$. If the corresponding Schur function $s_\lambda(p_1, \dots, p_N)$ has the property that the polynomial $s_\lambda(x + c_1, c_2, \dots, c_N)$ has only real roots, then $c_1, c_2, \dots, c_N \in \mathbb{R}$.*

On the other hand we obtain that the Shapiro Conjecture is equivalent to the special case of Theorem 3.2 for nilpotent actions of y_1, and to the following special case of Theorem 2.3:

$$\Upsilon^{-1}(\mathbb{R}^{(n)} \times 0) \subset RC_n.$$

The latter equivalence was independently observed in [MTV2].

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I.G.: School of Mathematics and Maxwell Institute for Mathematical Sciences, Edinburgh University, Edinburgh EH9 3JZ, Scotland
E-mail address: igordon@ed.ac.uk

E.H.: Department of Mathematics and Informatics, Sofia University, 5 J. Bourchier Blvd., Sofia 1126, Bulgaria, and

Institute of Mathematics and Informatics, Bulg. Acad. of Sci., Acad. G. Bonchev Str., Block 8, 1113 Sofia, Bulgaria
E-mail address: horozov@fmi.uni-sofia.bg

M.Y.: Department of Mathematics, University of California, Santa Barbara, CA 93106, U.S.A.
E-mail address: yakimov@math.ucsb.edu