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A DEODHAR-TYPE STRATIFICATION ON THE DOUBLE FLAG VARIETY

BEN WEBSTER*

Department of Mathematics
University of California
Berkeley, CA 94720

MILEN YAKIMOV**

Department of Mathematics
University of California
Santa Barbara, CA 93106

Abstract. We describe a partition of the double flag variety $G/B^+ \times G/B^-$ of a complex semisimple algebraic group G analogous to the Deodhar partition on the flag variety G/B^+. This partition is a refinement of the stratification into orbits both for $B^+ \times B^-$ and for the diagonal action of G, just as Deodhar's partition refines the orbits of B^+ and B^-.

We give a coordinate system on each stratum, and show that all strata are coisotropic subvarieties. Also, we discuss possible connections to the positive and cluster geometry of $G/B^+ \times G/B^-$, which would generalize results of Fomin and Zelevinsky on double Bruhat cells and Marsh and Rietsch on double Schubert cells.

1. Introduction

Let G be a complex semisimple algebraic group with a fixed pair of dual Borel subgroups $B^{\pm}$ and let $T = B^+ \cap B^-$ be the corresponding maximal torus of G. The Weyl group of the pair (G, T) will denoted by W and its identity element by e. The flag variety, that is the set of Borels of G, is naturally identified with G/B for any Borel B by the map $gB \mapsto gBg^{-1}$ and naturally has the structure of a projective variety.

In [Deo85], Deodhar described a remarkable partition of the flag variety $G/B^+ = \mathcal{B}$. It is a refinement of the stratification of $\mathcal{B}$ both into B^+ and into B^- orbits, and thus of the stratification of $\mathcal{B}$ into intersections of dual Schubert cells $\mathcal{R}_{v,w} = B^+w \cdot B^+ \cap B^-v \cdot B^+$ which we call *double Schubert cells*. In particular, it provides a stratification of each double Schubert cell. Furthermore, he showed that each stratum is isomorphic to $\mathbb{C}^k \times (\mathbb{C}^*)^m$, where $m + 2k = \dim \mathcal{R}_{v,w} = \ell(w) - \ell(v)$, and the strata are indexed by certain subexpressions of a reduced expression for w.

Throughout the paper by a *stratification* of a quasiprojective variety X we mean a partition of X

$$X = \bigsqcup_{\alpha \in A} X_\alpha$$

into smooth, locally closed, irreducible subsets X_α, the closure of each of which is a union of strata X_β, $\beta \in A(\alpha) \subset A$.

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In this paper, we will describe an analogue of Deodhar's partition in the double flag variety $\mathcal{B} \times \mathcal{B}$.

Playing the role of the subgroups B^+ and B^- are the actions of G_Δ (the diagonal subgroup of $G \times G$) and $B^+ \times B^-$ as subgroups of $G \times G$. These actions are in a certain sense (Poisson) dual, much as B^+ and B^- are. To be more precise, recall that $G \times G$ is a double Poisson algebraic group for the standard Poisson structure on G. The image of G inside $G \times G$ is G_Δ and the normalizer of its dual is $B^+ \times B^-$. Similarly a double of B^+ is closely related to G and its dual is B^-.

As is well known, the orbits of G_Δ on $\mathcal{B} \times \mathcal{B}$ are indexed by the Weyl group W of G, and $B^+ \times B^-$-orbits are indexed by $W \times W$, both with the partial order induced by closure being the standard strong Bruhat order (or its opposite, depending on indexing conventions). Thus, to each ordered triplet $(u, v, w) \in W^3$, we associate the intersection of the G_Δ orbit attached to u and $B^+ \times B^-$-orbit associated to (v, w).

$$\mathcal{P}^u_{v,w} = \left(G_\Delta \cdot (B^+, u \cdot B^-)\right) \bigcap \left((B^+ \times B^-) \cdot (v \cdot B^+, w \cdot B^-)\right). \tag{1}$$

We will prove a number of results about these remarkable varieties. Here, we collect the principal results of Theorem 5.2 and Proposition 8.4:

Theorem. *For all $u, v, w \in W^3$, the variety $\mathcal{P}^u_{v,w}$ has a stratification indexed by combinatorial objects related to u, v, w called double distinguished subexpressions, such that each stratum has explicit coordinates, which realize an isomorphism with $(\mathbb{C}^*)^k \times \mathbb{C}^\ell$ for some integers k, ℓ. Furthermore, each stratum is coisotropic in the double Poisson structure on $\mathcal{B} \times \mathcal{B}$.*

Interest in these varieties has arisen from different directions in mathematics. In [EL04, Example 4.9], they were identified with the torus orbits of symplectic leaves of $\mathcal{B} \times \mathcal{B}$ considered as a Poisson subvariety of variety of Lagrangian subalgebras associated to the standard Poisson structure on G.

On the other hand, similar varieties appear in the study of positivity in algebraic groups and cluster algebras. The orbit intersection $\mathcal{P}^e_{v,w}$ is just a reduced double Bruhat cell as defined by Fomin and Zelevinsky in [FZ99], and thus has a natural cluster algebra structure. Furthermore, the intersections with double Bruhat cells give a cell decomposition of the non-negative part of G. The stratum $\mathcal{P}^{w_0}_{v,w}$ is naturally a double Schubert cell, and the intersections of the non-negative part of $\mathcal{B}$ with double Schubert cells gives a similar cell decomposition.

When G is of adjoint type, the double flag variety $\mathcal{B} \times \mathcal{B}$ sits inside the wonderful compactification $\overline{G}$ of G as the so called lowest stratum (the unique closed $G \times G$ orbit). For the purposes of understanding total positivity in $\overline{G}$ better, Lusztig [Lus98b] defined a partition of $\overline{G}$ which is a refinement of of the partition by $G \times G$ orbits. The partition (1) differs from the restriction of Lusztig's partition to $\mathcal{B} \times \mathcal{B}$. The latter consists of products of Schubert cells.

In Section 2, we will define the varieties in question, and discuss their relationship to double Schubert cells and double Bruhat cells. In Section 3, we will recall the background on relative positions and Bruhat decomposition that we will need later, and in Section 4, we cover the analogue, for our varieties, of Deodhar's theory of distinguished subexpressions, which provide the necessary combinatorial setup. In Section 5, we will define our stratification and define a parameterization of each stratum of it, as well as characterization of each stratum in terms of generalized minors in Section 6. In Sections 7 and 8, we will consider connections to positivity and Poisson geometry respectively.

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2. The varieties $\mathcal{P}^u_{v,w}$

The varieties $\mathcal{P}^u_{v,w}$ can be defined in terms of the more classical terminology of relative position. For any two flags $B', B'' \in \mathcal{B}$, we can define a "distance" (in a very crude sense) between them as follows: The orbits of B' on $\mathcal{B}$ are naturally indexed by W, and relative position $r(B', B'') \in W$ indexes the orbit in which B'' lies. Following [MR04], we use the notation $B \xrightarrow{w} B'$ to express that B and B' are in relative position w. Alternatively, relative positions can be characterized as the G_Δ-orbits on $\mathcal{B} \times \mathcal{B}$. That is, $B \xrightarrow{w} B'$ if and only if $(g \cdot B, g \cdot B') = (B^+, w \cdot B^+)$ for some $g \in G$. Relative positions can be used to define several interesting subvarieties of $\mathcal{B}$ and $\mathcal{B} \times \mathcal{B}$.

For ease of notation let $v^* = w_0 v w_0$, where $w_0 \in W$ is the longest element of W.

- The Schubert cells of $\mathcal{B}$ with respect to B^+ and B^- can be defined by:

$$\mathcal{B}_w = B^+ w \cdot B^+ = \{B \in \mathcal{B} | B^+ \xrightarrow{w} B\},$$
$$\mathcal{B}^v = B^- v \cdot B^- = \{B \in \mathcal{B} | B^- \xrightarrow{v^*} B\}$$

- The double Schubert cells can be defined by

$$\mathcal{R}_{w,v} = \mathcal{B}_v \cap \mathcal{B}^{w w_0} = \{B \in \mathcal{B} | B^+ \xrightarrow{v} B \xleftarrow{w_0 w} B^-\}. \tag{2}$$

- In $\mathcal{B} \times \mathcal{B}$, as we described above

$$G_\Delta \cdot (B^+, u \cdot B^-) = \{(B_1, B_2) \in \mathcal{B} \times \mathcal{B} | B_1 \xrightarrow{u w_0} B_2\}$$
$$(B^+ \times B^-) \cdot (v \cdot B^+, w \cdot B^-) = \{(B_1, B_2) \in \mathcal{B} \times \mathcal{B} | B^+ \xrightarrow{v} B_1, B^- \xrightarrow{w^*} B_2\}$$

- Thus, the intersection of these orbits is the variety

$$\mathcal{P}^u_{v,w} = \{(B_1, B_2) \in \mathcal{B} \times \mathcal{B} | B^+ \xrightarrow{v} B_1 \xrightarrow{u w_0} B_2 \xleftarrow{w^*} B^-\} \tag{3}$$

While this choice of indexing of relative positions may seem a little strange, we will see in Sections 4 and 5 that this will simplify the combinatorics necessary to describe these varieties.

We consider some already known special cases: For all $v, w \in W$, we let

$$G_{v,w} = B^+ w B^+ \cap B^- v B^- \subset G$$

denote a double Bruhat cell of G, and let $L_{v,w} = G_{v,w}/T$ be the corresponding reduced double Bruhat cell.

Proposition 2.1. *If $u = e$, then $\mathcal{P}^e_{v,w}$ is isomorphic to $L_{v,w}$. Similarly, the varieties $\mathcal{P}^{w_0}_{v,w} \cong \mathcal{P}^{w_0 w^{-1}}_{v,e} \cong \mathcal{P}^{v w_0}_{e,w}$ are isomorphic to $\mathcal{R}_{ww_0,v}$.*

Proof. Rewriting the definition of a double Bruhat cell, any $g \in G$ is in $G_{u,v}$ if and only if $B^+ \xrightarrow{w} g \cdot B^+$ and $B^- \xrightarrow{v^*} g \cdot B^-$.

Since $g \cdot B^+ \xrightarrow{w_0} g \cdot B^-$, the image of the action map sending $g \in G_{v,w}$ to $(g \cdot B^+, g \cdot B^-)$ is contained in $\mathcal{P}^e_{v,w}$, and this map is surjective (since G acts transitively on pairs of flags in fixed relative position). Since T is the stabilizer of (B^+, B^-), this map is an isomorphism $L_{v,w} = G_{v,w}/T \cong \mathcal{P}^e_{v,w}$.

On the other hand, if either of v, w is e, or $u = w_0$ then the definition of equation (3) collapses to the definition in (2), for the appropriate choice of new v, w. □

3. Relative position and Bruhat decomposition

In this section, we will restate many well-known properties of relative position and the Bruhat stratification of $\mathcal{B}$, which will be used later.

Proposition 3.1. *Relative position is*

(1) *invariant under the diagonal action of G, i.e.*
$$r(B', B'') = r(g \cdot B', g \cdot B''),$$
(2) *anti-symmetric, i.e.*
$$r(B', B'') = \left(r(B'', B')\right)^{-1},$$
(3) *sub-multiplicative with respect to Bruhat order, i.e. there exist $u, v \in W$ such that $u \leq r(B', B''), v \leq r(B'', B''')$ and*
$$uv = r(B', B'''),$$
(4) *multiplicative if*
$$\ell\left(r(B', B'')\right) + \ell\left(r(B'', B''')\right) = \ell\left(r(B', B'') \cdot r(B'', B''')\right).$$

Complementary to property 4 is the following essential result:

Proposition 3.2. *If $B \xrightarrow{w} B'$, and $w = w'w''$ for $\ell(w') + \ell(w'') = \ell(w)$, then there is a unique flag $\tilde{B}$ such that $B \xrightarrow{w'} \tilde{B} \xrightarrow{w''} B'$.*

In the case where $B = B^+$, we can use this to define a surjective projection map $\pi^w_{w'} : \mathcal{B}_w \to \mathcal{B}_{w'}$. We will also use this notation for analogous projections between $B^+ \times B^-$-orbits on $\mathcal{B} \times \mathcal{B}$. This makes $\mathcal{B}_w$ a trivial fiber bundle over $\mathcal{B}_{w'}$. In the case where $w's_i = w$, we can explicitly trivialize this bundle. This will be done in Proposition 3.3. The general case can be treated by iteration.

As usual, we let $x_i(t) = \exp(te_i)$ and $x_{-i}(t) = y_i(t) = \exp(tf_i)$, where e_i, f_i are the standard Chevalley generators for the simple root spaces, and s_i are the corresponding simple reflections in the Weyl group.

Following Fomin and Zelevinsky, we pick a lift of W to G as follows: for simple reflections, we let $\bar{s}_i = x_i(-1)y_i(1)x_i(-1)$, and $\bar{w} = \bar{w}_1\bar{w}_2$ if $w = w_1w_2$ and $\ell(w) = \ell(w_1) + \ell(w_2)$.

Note that this is **not** a homomorphism, since $\bar{s}_i^2 \neq 1$. However, it is independent of the reduced word we choose.

We will use the notation $x_{w(\alpha_i)}(t) = \bar{w}x_i(t)\bar{w}^{-1}$.

Proposition 3.3. *Fix two flags B_1 and B_2 of relative position $B_1 \xrightarrow{u} B_2$; that is $(B_1, B_2) = g \cdot (u_1 \cdot B^+, u_2 \cdot B^+)$ for some $u_1, u_2 \in W$ such that $u = u_1^{-1}u_2$. Let $X \subset \mathcal{B}$ be the subvariety of flags B' such that $B_1 \xrightarrow{s_i} B'$. Then $X \cong \mathbb{C}$, and the following hold.*

(1) If $s_i u > u$, then $B' \xrightarrow{s_i u} B_2$ for all $B' \in X$ and the map

$$t \mapsto g x_{u_1(\alpha_i)}(t) u_1 s_i \cdot B^+$$

defines an isomorphism $\mathbb{C} \to X$. Furthermore, if $u_1 s_i > u_1$ then

$$\left(\pi^{(u_1,u_2)}_{(u_1 s_i, u_2)}\right)^{-1}(B_1, B_2) = g x_{u_1(\alpha_i)}(t) \cdot \left(u_1 s_i \cdot B^+, u_2 \cdot B^+\right) \qquad (t \in \mathbb{C}).$$

(2) If $s_i u < u$, then there is a unique flag $B_0 \in X$ such that $B_0 \xrightarrow{s_i u} B_2$ and for all other $B' \in X$, $B' \xrightarrow{u} B_2$. The map

$$t \mapsto g y_{u_1(\alpha_i)}(t) u_1 \cdot B^+$$

is an isomorphism $\mathbb{C}^* \to X - B_0$, and $B_0 = g u_1 s_i \cdot B^+$. Furthermore, if $u_1 s_i > u_1$ then

$$\left(\pi^{(u_1,u_2)}_{(u_1 s_i, u_2)}\right)^{-1}(B_1, B_2) - B_0 = g y_{u_1(\alpha_i)}(t) \cdot \left(u_1 \cdot B^+, u_2 \cdot B^+\right) \qquad (t \in \mathbb{C}^*).$$

In Proposition 3.3, the term u_1 can be incorporated into g, so that

$$g \cdot (u_1 B^+, u_2 B^+) = g u_1 \cdot (B^+, u B^+)$$

which somewhat simplifies the statement. We state the proposition as above since this is the form in which it will be used later. The proof is straightforward and is left to the reader.

4. Double distinguished subexpressions

In this section, we will discuss the combinatorics of the "double Weyl group" $W \times W$. As with "double flag variety," this terminology can be justified by the fact that this is the Weyl group of the double $G \times G$. In particular, we will generalize Deodhar's notion of a *distinguished subexpression* to the double case. In Section 5, we will see that these double distinguished subexpressions play the same role for the varieties $\mathcal{P}^u_{v,w}$ that distinguished subexpressions do for $\mathcal{R}_{v,w}$.

First let us describe briefly the "single Weyl group" case: Consider a reduced expression $w = s_{i_1} \cdots s_{i_n}$ in W, and let $w_{(k)} = s_{i_1} \cdots s_{i_k}$, and $\mathbf{w} = \left(w_{(0)}, \ldots, w_{(n)}\right)$. A sequence $\mathbf{v} = \left(v_{(0)}, \ldots, v_{(n)}\right)$ is called a **subexpression** of $\mathbf{w}$ if $v_{(0)} = e$ and

$$v_{(k)} \in \{v_{(k-1)}, v_{(k-1)} s_{i_k}\}$$

for all $1 \leq k \leq n$. Informally, $\mathbf{v}$ has been obtained by throwing some of the simple reflections out of the word $\mathbf{w}$.

Such an expression is called **distinguished** if whenever $v_{(k-1)}s_{i_k} < v_{(k-1)}$, we have $v_{(k)} = v_{(k-1)}s_{i_k}$, that is, $\mathbf{v}$ decreases in length whenever possible.

Following the convention of Fomin and Zelevinsky, we consider $W \times W$ as a Coxeter group with simple reflections s_{-i}, s_i for $i \in \Pi$, the simple roots of G.

Pick a reduced word $s_{i_1} \cdots s_{i_n}$ for $(v, w) \in W \times W$. As before, we let

$$\big(v_{(k)}, w_{(k)}\big) = s_{i_1} \cdots s_{i_k}.$$

We let $\epsilon(k) = 1$ if $i_k > 0$ and -1 if $i_k < 0$.

Now, fix a sequence $\mathbf{u} = \big(u_{(0)}, \dots, u_{(n)}\big)$ with $u_{(i)} \in W$. We call $\mathbf{u}$ a **double subexpression of** $(\mathbf{v}, \mathbf{w})$ if $u_{(0)} = e$ and

(1) if $i_k > 0$, then $u_{(k)} \in \{u_{(k-1)}, u_{(k-1)}s_{|i_k|}\}$,
(2) if $i_k < 0$, then $u_{(k)} \in \{u_{(k-1)}, s_{|i_k|}u_{(k-1)}\}$.

We can write this more compactly as

$(*)$ $u_{(k)}^{\epsilon(k)} \in \Big\{u_{(k-1)}^{\epsilon(k)}, u_{(k-1)}^{\epsilon(k)}s_{|i_k|}\Big\}$.

We call **u double distinguished** if $u_{(k)}^{\epsilon(k)} = u_{(k-1)}^{\epsilon(k)}s_{|i_k|}$ for all k such that $u_{(k-1)}^{\epsilon(k)}s_{|i_k|} < u_{(k-1)}^{\epsilon(k)}$.

For each expression $\mathbf{u}$, we let

- $J_{\mathbf{u}}^{\circ}$ be the set of indices $k \in [1, n]$ such that $u_{(k-1)} = u_{(k)}$,
- $J_{\mathbf{u}}^{+}$ be the set of indices $k \in [1, n]$ such that $u_{(k-1)} < u_{(k)}$,
- $J_{\mathbf{u}}^{-}$ be the set of indices $k \in [1, n]$ such that $u_{(k-1)} > u_{(k)}$.

Obviously, a subexpression $\mathbf{u}$ of $(\mathbf{v}, \mathbf{w})$ is uniquely determined by these subsets.

Note that each double distinguished subexpression writes $u_{(n)}$ as a product of $u_{(n)} = (u^{\mathbf{w}})^{-1} u^{\mathbf{v}}$ with $u^{\mathbf{w}} \leq w, u^{\mathbf{v}} \leq v$.

We call a double distinguished subexpression **positive** if $J_{\mathbf{u}}^{-} = \emptyset$. As is the case for usual (single) subexpressions, a positive subexpression with $u_{(n)} = u$ exists if and only if any does.

Proposition 4.1. *The following are equivalent:*

(a) There is a unique positive double distinguished subexpression of any reduced expression for (v, w) with $u_{(n)} = u$.
(b) There is a double distinguished subexpression of some reduced expression for (v, w) with $u_{(n)} = u$.
(c) $u = v'w'$ for $v' \leq v^{-1}$ and $w' \leq w$.

Proof. The implications $(a) \Rightarrow (b) \Rightarrow (c)$ are obvious.

Now, we consider the implication $(c) \Rightarrow (a)$.

Since $v' < v^{-1}$ and $w' < w$, there are positive subexpressions for both these elements. We can combine these into a double expression $v'w'$, which is a subexpression $\bar{u}$ for u. We can assume by the deletion property of Coxeter groups that $\bar{u}$ is positive (i.e. reduced). The only question is whether it is distinguished.

Let k be the largest index for which $u_{(k-1)}^{\epsilon(k-1)} > u_{(k-1)}^{\epsilon(k-1)}s_{|i_k|}$. Using the deletion property again, there is an index $k' < k$ for which $u_{(k')} > u_{(k'-1)}$, such that if we delete k' from our original subexpression and add k, we have a positive double subexpression which agrees with our original for all indices outside $k' \leq i < k$ (in particular n), and for which the largest index where it is not distinguished is strictly smaller than k. Using our cancellation argument inductively, we obtain a positive double distinguished subexpression for u.

Uniqueness follows from the fact that if a positive double distinguished subexpression exists, it is forced by the distinguishment and positivity conditions to be the following: we set $u_{(n)} = u$ and let $\tilde{u}^{\epsilon(k)}_{(k-1)} = \min\left(\tilde{u}^{\epsilon(k)}_{(k)}, \tilde{u}^{\epsilon(k)}_{(k)} s_{|i_k|}\right)$. ☐

It is worth noting that when either of $v = e$ (resp. $w = e$), condition (c) of Proposition 4.1 simply reduces to requiring $u < w$ (resp. $u < v^{-1}$).

We can obtain a similar reduction when $u = w_0$.

Proposition 4.2. *There exists a double distinguished subexpression of* (v, w) *with* w_0 *if and only if* $vw_0 \leq w$.

Proof. By Proposition 4.1, if such a double distinguished subexpression exists, then $w_0 = v'w'$ with $v' \leq v, w' \leq w$. Thus $vw_0 \leq (v')^{-1}w_0 = w' \leq w$.

On the other hand, if $vw_0 \leq w$, then $v' = v, w' = vw_0$ realizes condition (c) of Proposition 4.1. ☐

5. The stratification of $\mathcal{P}^u_{v,w}$

For each reduced expression $(\mathbf{v}, \mathbf{w})$ for (v, w), we have the standard projection maps

$$\pi_k : \mathcal{B}_v \times \mathcal{B}^w \to \mathcal{B}_{v_{(k)}} \times \mathcal{B}^{w_{(k)}}, \quad \pi_k = \pi^v_{v_{(k)}} \times \pi^w_{w_{(k)}}.$$

Fix such an expression, and for a pair $(B_1, B_2) \in \mathcal{P}^u_{v,w} \subset \mathcal{B}_v \times \mathcal{B}^w$, consider the sequence $u_{(k)} = r(\pi_k(B_1, B_2))w_0$. That is, $u_{(k)}$ is the unique element such that $\pi_k(B_1, B_2) \in \mathcal{P}^{u_{(k)}}_{v_{(k)}, w_{(k)}}$.

Theorem 5.1. *The sequence* $\mathbf{u} = (u_{(0)}, \ldots, u_{(n)})$ *is a double distinguished subexpression of* $(\mathbf{v}, \mathbf{w})$.

Proof. To simplify notation, we let

$$\pi^1_k = \pi^v_{v_{(k)}} : \mathcal{B}_v \to \mathcal{B}_{v_{(k)}}, \quad \pi^2_k = \pi^w_{w_{(k)}} : \mathcal{B}_w \to \mathcal{B}_{w_{(k)}}.$$

Assume for simplicity that $i_k < 0$ (the proof for $i_k > 0$ is the same). Then $\pi^2_k(B_2) = \pi^2_{k-1}(B_2)$, and $r(\pi^1_k(B_1), \pi^1_{k-1}(B_1)) = s_{|i_k|}$.

By Proposition 3.3, $u_{(k)} \in \{u_{(k-1)}, s_{|i_k|}u_{(k-1)}\}$, so $\mathbf{u}$ is a double subexpression of $(\mathbf{v}, \mathbf{w})$. If $u_{(k-1)} > s_{|i_k|}u_{(k-1)}$, since left multiplication by w_0 is order-reversing for Bruhat order,

$$r\left(\pi^1_k(B_1), \pi^1_{k-1}(B_1)\right) \cdot r\left(\pi_{k-1}(B_1, B_2)\right) = r\left(\pi_k(B_1, B_2)\right)$$

by property (4) of Proposition 3.1. That is, $u_{(k)} = s_{|i_k|}u_{(k-1)}$. Thus, $\mathbf{u}$ is also distinguished. ☐

Let $\mathcal{P}^{\mathbf{u}}_{\mathbf{v},\mathbf{w}}$ to be the subvariety of $\mathcal{P}^u_{v,w}$ defined by

$$\mathcal{P}^{\mathbf{u}}_{\mathbf{v},\mathbf{w}} = \{(B_1, B_2) | \pi_k(B_1, B_2) \in \mathcal{P}^{u_{(k)}}_{v_{(k)},w_{(k)}}\}$$

The theorem above shows that $\mathcal{P}^u_{v,w}$ has a partition,

$$\mathcal{P}^u_{v,w} = \bigsqcup_{\mathbf{u}} \mathcal{P}^{\mathbf{u}}_{\mathbf{v},\mathbf{w}} \tag{4}$$

which we will show in Section 8 is a stratification. First, let us understand the topology of the strata a bit better.

We let

$$g_k = \begin{cases} y_{u^{\mathbf{v}}_{(k)}\alpha_{i_k}}(t_k) & i_k > 0, k \notin J^+_{\mathbf{u}} \\ x_{u^{\mathbf{w}}_{(k)}\alpha_{i_k}}(t_k) & i_k < 0, k \notin J^+_{\mathbf{u}} \\ 1 & k \in J^+_{\mathbf{u}} \end{cases}$$

and let $g_{\mathbf{u}}(t_1, \ldots, t_n) = g_1(t_1) \cdots g_n(t_n)$ where $t_i = 1$ if $i \in J^+_{\mathbf{u}}$, $t_i \in \mathbb{C}^*$ if $i \in J^\circ_{\mathbf{u}}$ and $t_i \in \mathbb{C}$ if $i \in J^-_{\mathbf{u}}$.

Theorem 5.2. *For any double distinguished subexpression* $\mathbf{u}$*,*

$$\mathcal{P}^{\mathbf{u}}_{\mathbf{v},\mathbf{w}} \cong (\mathbb{C}^*)^{|J^\circ_{\mathbf{u}}|} \times \mathbb{C}^{|J^-_{\mathbf{u}}|}.$$

In particular, $\dim \mathcal{P}^{\mathbf{u}}_{\mathbf{v},\mathbf{w}} = n - |J^+_{\mathbf{u}}| \leq \ell(w) + \ell(v) - \ell(u)$*.*

Furthermore, the map

$$\varphi_{\mathbf{u}} : (t_1, \ldots, t_n) \mapsto g_u(t_1, \ldots, t_n) \cdot (u^{\mathbf{w}} \cdot B^+, u^{\mathbf{v}} \cdot B^-)$$

is an isomorphism between $(\mathbb{C}^*)^{|J^\circ_{\mathbf{u}}|} \times \mathbb{C}^{|J^-_{\mathbf{u}}|}$ *and* $\mathcal{P}^{\mathbf{u}}_{\mathbf{v},\mathbf{w}}$*.*

Corollary 5.3. *The set* $\mathcal{P}^u_{v,w}$ *is a smooth, locally closed, and irreducible subvariety of* $\mathcal{B} \times \mathcal{B}$*. It is nonempty if and only if* $u = w'v'$ *for some* $w' \leq w^{-1}$ *and* $v' \leq v$*.*

The first part of this corollary is a special case of [EL04, Proposition 4.2]. That part and Proposition 4.2 in [EL04] also follow directly from Richardson's result [R92, Corollary 1.5].

Proof. The intersection between $B^+ \times B^-$ and G_Δ-orbits is transverse, since $\mathfrak{g} \oplus \mathfrak{g}$ is spanned by $\mathfrak{b}_+ \oplus \mathfrak{b}_-$ and $\mathfrak{g}_\Delta$. Thus, $\mathcal{P}^u_{v,w}$ is a locally closed, smooth subvariety of $\mathcal{B} \times \mathcal{B}$. of dimension $\ell(v) + \ell(w) - \ell(u)$. In particular, in each irreducible component, there is a stratum of dimension $\ell(v) + \ell(w) - \ell(u)$. Since

$$\dim \mathcal{P}^{\mathbf{u}}_{\mathbf{v},\mathbf{w}} = \ell(v, w) - |J^+_{\mathbf{u}}| = \ell(v, w) - \ell(u) - |J^-_{\mathbf{u}}|$$

a stratum is of maximal dimension if $\mathbf{u}$ is positive, and there is a unique such double distinguished subexpression. Thus, $\mathcal{P}^u_{v,w}$ only has one component, and thus is irreducible.

The second part follows from Proposition 4.1 and Theorem 5.2. □

Applying Proposition 2.1, we see that Corollary 5.3 also gives a characterization of when the varieties $\mathcal{R}_{v,w}$ are nonempty (in fact, it provides 3 different descriptions). Proposition 4.2 and the discussion preceding it show that these characterizations all reduce to a previously known characterization.

Corollary 5.4. ([Deo85, Corollary 1.2]) *The variety $\mathcal{R}_{v,w}$ is nonempty if and only if $v \leq w$.*

Proof of Theorem 5.2. By induction, assume the result is true for $u_{(n-1)}, v_{(n-1)}, w_{(n-1)}$, and for simplicity, assume $i_n > 0$, and let

$$X = \{(B_1, B_2) \in \mathcal{B} \times \mathcal{B} | B_1 \xrightarrow{s_{i_n}} B_1', (B_1', B_2) \in \mathcal{P}^{\mathbf{u_{(n-1)}}}_{\mathbf{v_{(n-1)}}, \mathbf{w_{(n-1)}}}\}.$$

Then, if $s_{i_n} u_{(n-1)} < u_{(n-1)}$, then $X = \mathcal{P}^{\mathbf{u}}_{\mathbf{v},\mathbf{w}}$, and by Proposition 3.3, part (1),

$$\begin{aligned} &g_u(t_1, \dots, t_n) \cdot (u^{\mathbf{w}} \cdot B^+, u^{\mathbf{v}} \cdot B^-) \\ &\qquad = g_{u_{(n-1)}}(t_1, \dots, t_{n-1}) \cdot y_{u^{\mathbf{w}}(\alpha_{i_n})}(t_n) \cdot (s_i u^{\mathbf{w}}_{(n-1)} \cdot B^+, u^{\mathbf{v}} \cdot B^-) \end{aligned}$$

is a parameterization of $\mathcal{P}^{\mathbf{u}}_{\mathbf{v},\mathbf{w}}$.

On the other hand, if $s_{i_n} u_{(n-1)} > u_{(n-1)}$, then $X = \mathcal{P}^{\mathbf{u_{(n-1)}}}_{\mathbf{v},\mathbf{w}} \cup \mathcal{P}^{\mathbf{s}_{i_n}\mathbf{u_{(n-1)}}}_{\mathbf{v},\mathbf{w}}$, and by a similar calculation, part (2) of Proposition 3.3 confirms that g_u provides a parameterization. □

If $u = e$, then the unique non-decreasing sequence is obviously given by $e^+_{(k)} = e$ for all k.

In [FZ99], Fomin and Zelevinsky construct a dense subset of $G_{v,w}$ for each reduced word for (v, w) by the factorization map

$$\begin{aligned} x_{\mathbf{v},\mathbf{w}} : H \times (\mathbb{C}^*)^n &\to G_{v,w} \\ (h, t_1, \dots t_n) &\mapsto x_{i_1}(t_1) \cdots x_{i_n}(t_n) h. \end{aligned}$$

Clearly, the composition $\varphi \circ x_{\mathbf{v},\mathbf{w}}$ does not depend on h, and so gives an injection $(\mathbb{C}^*)^n \to \mathcal{P}^e_{v,w}$.

Corollary 5.5. $\mathcal{P}^{\mathbf{e}^+}_{\mathbf{v},\mathbf{w}}$ *is exactly the set of elements of the form $(g \cdot B^+, g \cdot B^-)$ where $g = hx_{i_1}(t_1) \cdots x_{i_n}(t_n)$ with $t_i \in \mathbb{C}^*$.*

Thus the stratification of $\mathcal{P}^e_{v,w}$ induces a stratification of the double Bruhat cell $B^+vB^+ \cap B^-wB^-$ with maximal stratum that coincide with the open subsets of Fomin-Zelevinsky [FZ99]. In addition, all strata have dimension greater than more half the dimension of $G^{u,v}$; in fact, they are coisotropic with respect to the standard Poisson structure on G, as we will show in the last section.

Furthermore, if $w = 1$, then we recover a rewriting of the parameterizations of [MR04] for double Schubert cells: to produce our parameterization from theirs, simply commute all Weyl group elements past the unipotent elements until they are collected at the end of the word.

On the other hand, if $v = 1$, then we recover the double Schubert cell $\mathcal{R}_{uw_0,vw_0}$, with the opposite parameterization.

Remark 1. These results all have a straight-forward generalization to the intersection of a G_Δ-orbit with an orbit of $B \times B'$ with B and B' two Borels, which may not be opposite. Unlike the varieties $\mathcal{P}^u_{v,w}$, we do not expect these more general orbit intersections to have good properties with respect to Poisson or positive structures. However, in the case where $B = B'$, we will recover results of Curtis [Cur88], which are relevant to the study of Hecke algebras.

The varieties of Curtis appear in a second way in our theory. If one chooses a reduced word for (v, w) which puts all reflections in the first copy of W before those appear in the second copy, i.e. an "unmixed" word, then $\pi_{\ell(v)}$ is simply the projection onto the first factor $\mathcal{P}^u_{v,w} \subset \mathcal{B} \times \mathcal{B} \to \mathcal{B}$. Each fiber of this map will be isomorphic to one of Curtis's varieties, though which variety it is will depend on which double Bruhat cell we are taking the fiber over. In this case, each of our strata projects to a Deodhar stratum in $\mathcal{B}$, with fiber given by one of Curtis's strata. In particular, considering the intersection of our stratification with each fiber will recover Curtis's stratification.

6. Chamber minors

Just as in [BZ97, MR04], we can define $\mathcal{P}^{\mathbf{u}}_{\mathbf{v},\mathbf{w}}$ in terms of certain generalized minors, which one can call "generalized chamber minors." Although Marsh and Rietsch have already claimed this name for the generalized minors which appear in their "generalized Chamber Ansatz," since our situation subsumes theirs, it should cause no confusion.

For a fixed highest weight vector h_λ in the representation V_λ, let h^*_λ denote the unique dual highest weight vector in V^*_λ, and $\langle -, - \rangle$ denote the standard pairing between these spaces. The generalized minors $\Delta^{w\lambda}_{v\lambda}$ are defined by

$$\Delta^{w\lambda}_{v\lambda}(g) = \langle g\bar{v} \cdot h_\lambda, \bar{w} \cdot h^*_\lambda \rangle.$$

These are precisely the matrix coefficients of extremal weight vectors in representations of G. They play a central role in the papers [BFZ05, BZ97, FZ99, KZ02, Zel00].

Let U^+ and U^- be the unipotent radicals of B^+ and B^-, and for all $w \in W$, let $U^{\pm}_w = U^{\pm} \cap wU^{\mp}w^{-1}$. As usual, the map $\alpha_{v,w} : U^+_v \times U^-_w \to \mathcal{B}_v \times \mathcal{B}^w$ given by action on $(v \cdot B^+, w \cdot B^-)$ is an isomorphism. Furthermore,

Proposition 6.1. *For all $B_1 \in \mathcal{B}_w, B_2 \in \mathcal{B}^v$,*

$$\pi_k(B_1, B_2) = \alpha^{-1}_{v,w}(B_1, B_2) \cdot (v_{(k)} \cdot B^+, w_{(k)} \cdot B^-).$$

This allows us to identify the varieties $\mathcal{P}^{\mathbf{u}}_{\mathbf{v},\mathbf{w}}$ using chamber minors:

Theorem 6.2. *Let $(z_1, z_2) = \alpha^{-1}_{v,w}(B_1, B_2)$. Then $(B_1, B_2) \in \mathcal{P}^{\mathbf{u}}_{\mathbf{v},\mathbf{w}}$ if and only if*

$$\begin{aligned}
\Delta^{v_{(k)}u_{(k)}w_0\omega_{i_k}}_{w_{(k)}w_0\omega_{i_k}}(z_1^{-1}z_2) &\neq 0 \text{ for all } k \in J^\circ_{\mathbf{u}}, i_k < 0,\\
\Delta^{v_{(k)}u_{(k-1)}w_0\omega_{i_k}}_{w_{(k)}w_0\omega_{i_k}}(z_1^{-1}z_2) &= 0 \text{ for all } k \in J^+_{\mathbf{u}}, i_k < 0,\\
\Delta^{w_{(k)}u^{-1}_{(k)}\omega_{i_k}}_{v_{(k)}\omega_{i_k}}(z_2^{-1}z_1) &\neq 0 \text{ for all } k \in J^\circ_{\mathbf{u}}, i_k > 0,\\
\Delta^{w_{(k)}u^{-1}_{(k-1)}\omega_{i_k}}_{v_{(k)}\omega_{i_k}}(z_2^{-1}z_1) &= 0 \text{ for all } k \in J^+_{\mathbf{u}}, i_k > 0.
\end{aligned}$$

Proof. By definition $(B_1, B_2) = (z_1 v \cdot B^+, z_2 w \cdot B^-)$ and

$$\pi_k(B_1, B_2) = (z_1 v_{(k)} \cdot B^+, z_2 w_{(k)} \cdot B^-),$$

so for all k, we calculate

$$r(\pi_k(B_1,B_2)) = r(z_1 v_{(k)}\cdot B^+, z_2 w_{(k)}\cdot B^-) = r(B^+, v_{(k)}^{-1} z_1^{-1} z_2 w_{(k)}\cdot B^-)$$

is $u_{(k)}w_0$ if and only if $v_{(k)}^{-1}z_1^{-1}z_2 w_{(k)} \in B^+ u_{(k)} B^-$.

By basic representation theory, $g \in B^+ u B^-$ if and only if

$$\begin{aligned} &\Delta_{w_0\omega_i}^{uw_0\omega_i}(g) \neq 0 \\ &\Delta_{w_0\omega_i}^{u'w_0\omega_i}(g) = 0 \qquad u'\omega_i < u\omega_i \end{aligned}$$

or equivalently

$$\begin{aligned} &\Delta_{\omega_i}^{u^{-1}\omega_i}(g^{-1}) \neq 0 \\ &\Delta_{\omega_i}^{u'\omega_i}(g^{-1}) = 0 \qquad u'\omega_i < u^{-1}\omega_i. \end{aligned}$$

Applying these to $g = v_{(k)}^{-1}z_1^{-1}z_2 w_{(k)}$, we find that the relations are necessary.

Now assume that they are sufficient for all (v',w') with $\ell(v',w') < \ell(v,w)$. Thus, the first $n-1$ relations are sufficient to assure that $\pi_{n-1}(B_1,B_2) \in \mathcal{P}_{\mathbf{v_{(n-1)}},\mathbf{w_{(n-1)}}}^{\mathbf{u_{(n-1)}}}$. For ease, again assume that $i_n > 0$.

If $n \in J_{\mathbf{u}}^-$, we are done, so assume $n \in J_{\mathbf{u}}^\circ \cup J_{\mathbf{u}}^+$, that is $s_{|i_n|}u_{(n-1)} > u_{(n-1)}$. Then (B_1,B_2) must be in $\mathcal{P}_{v,w}^{u_{(n-1)}}$ or $\mathcal{P}_{v,w}^{s_{|i_n|}u_{(n-1)}}$, and this determines which Deodhar stratum it lives in. Since $u_{(n-1)}^{-1}s_{|i_n|}\omega_i > u_{(n-1)}^{-1}\omega_i$, if $u_{(n)} = s_{|i_n|}u_{(n-1)}$, then by the formulae above, $\Delta_{v\omega_i}^{wu_{(n-1)}^{-1}\omega_i}(z_2^{-1}z_1) = 0$, whereas if $u_{(n)} = u_{(n-1)}$ then

$$\Delta_{v\omega_i}^{wu_{(n-1)}^{-1}\omega_i}(z_2^{-1}z_1) = \Delta_{v\omega_i}^{wu^{-1}\omega_i}(z_2^{-1}z_1) \neq 0.$$

Thus, the hypotheses of the theorem are sufficient as well as necessary. □

In the case where $w = 1$, there is an explicit change of coordinates between our parameterization of Theorem 5.2, and a coordinate system given by chamber minors, known as the Generalized Chamber Ansatz, described in [MR04].

Question 1. *Is there a generalization of the Chamber Ansatz to the coordinate systems in Theorem 5.2?*

7. Positivity

Previous work along these lines has been closely related to the theory of total positivity. While this the concept of totally positive matrices has existed for decades, it was developed in its modern form by Lusztig, followed by Berenstein, Fomin and Zelevinsky (see the papers [BZ97, BZ01, FZ99, Lus94, Lus98a]).

Let $G^{>0}$, the strictly totally positive part of G, be the subset of G on which all generalized minors are positive. By [FZ99], this is the same as the set of G of the form

$$g = x_{i_1}(t_1)\cdots x_{i_n}(t_n)$$

for $t_i \in \mathbb{R}_{>0}$ and $s_{i_1} \cdots s_{i_n}$ a reduced word for (w_0, w_0).

The set of non-negative elements of G is simply the closure $G^{\geq 0} = \overline{G^{>0}}$. Similarly, in any based G-space (X, x), we can define a positive subset relative to x as the subset $X^{\geq 0} = \overline{G^{>0} \cdot x}$.

In [FZ99], Fomin and Zelevinsky show that $G^{\geq 0}$ has a natural (real) cell decomposition, in which each cell being the intersection of $G^{\geq 0}$ with a double Bruhat cell.

In [MR04], Marsh and Rietsch show that $\mathcal{B}^{\geq 0}$ has a similar cell decomposition in terms of double Schubert cells, as was conjectured by Lusztig.

Now, we define *the positive part of* $\mathcal{B}\times\mathcal{B}$ to be $(\mathcal{B}\times\mathcal{B})^{\geq 0} = \overline{G^+_\Delta \cdot (B^+, B^-)}$. We prove below that this is not the product of the positive flag varieties $\mathcal{B}^{\geq 0} \times \mathcal{B}^{\geq 0} \subset \mathcal{B}\times\mathcal{B}$. In particular, it differs from the restriction of the nonnegative part of the wonderful compactification $\overline{G}$, defined by Lusztig [Lus98b], intersected with the lowest stratum $\mathcal{B}\times\mathcal{B}$. In the terminology of [FG03], these different "positive parts" correspond to different positive structures on $\mathcal{B}\times\mathcal{B}$, one the product the standard positive structures on the flag variety, and one descending from the standard positive structure on G.

For example, if $G = \mathrm{SL}_2\mathbb{C}$, one can identify $\mathcal{B}\times\mathcal{B}$ with $\mathbb{CP}^1 \times \mathbb{CP}^1$. Then $\mathcal{B}^{\geq 0} \times \mathcal{B}^{\geq 0}$ is the subset

$$\{(a,b) | a, b \in [0, \infty]\}$$

whereas $(\mathcal{B}\times\mathcal{B})^{\geq 0}$ is

$$\{(a,b) | a \in [0,\infty], b \in [1/a, \infty]\}.$$

While this definition may look asymmetric in a and b, in fact, it is invariant under the switch map.

Theorem 7.1. *For any simple algebraic group G,*

$$(\mathcal{B}\times\mathcal{B})^{\geq 0} \subsetneq \mathcal{B}^{\geq 0} \times \mathcal{B}^{\geq 0}.$$

Proof. Since $G^{>0} = {U^+}^{\geq 0} H^{\geq 0} {U^-}^{\geq 0} = {U^-}^{\geq 0} H^{\geq 0} {U^+}^{\geq 0}$ by [FZ99],

$$G^{>0} \cdot (B^+, B^-) \subset {U^-}^{\geq 0} \cdot B^+ \times {U^+}^{\geq 0} \cdot B^-.$$

Taking closure of both sides yields the desired inclusion.

Now, consider the Schubert cell $\overline{\mathcal{B}_{s_i} \times \mathcal{B}^{s_i}}$ for some simple root s_i. This naturally identified with the flag variety of the corresponding root subgroup, isomorphic to $\mathrm{SL}_2\mathbb{C}$. The intersection $\mathcal{B}^{\geq 0} \times \mathcal{B}^{\geq 0} \cap \overline{\mathcal{B}_{s_i} \times \mathcal{B}^{s_i}}$ is precisely the same as the corresponding product in the double flag of $\mathrm{SL}_2\mathbb{C}$ (this is clear from the parameterization of $\mathcal{B}$ given in [MR04]). On the other hand $(\mathcal{B}\times\mathcal{B})^{\geq 0} \cap \mathcal{P}^e_{s_i,s_i}$ is precisely $(G^{s_i,s_i})^{\geq 0} \cdot (B^+, B^-)$. By our calculation for $\mathrm{SL}_2\mathbb{C}$, these sets do not coincide. □

By analogy with previous positivity results [FZ99, He04, MR04], we conjecture that the varieties $\mathcal{P}^u_{v,w}$ and the parameterizations of them we have described are closely connected to the positive part of $\mathcal{B}\times\mathcal{B}$.

Conjecture 1. $(\mathcal{B}\times\mathcal{B})^{\geq 0}$ *has a real cell decomposition, given by its intersections with the subvarieties $\mathcal{P}^u_{v,w}$ where u, v, w range over W. Each such intersection ${\mathcal{P}^u_{v,w}}^{\geq 0} =$*

$\mathcal{P}^u_{v,w} \cap (\mathcal{B}\times\mathcal{B})^{\geq 0}$ *is contained in* $\mathcal{P}^{\mathbf{u}^+}_{\mathbf{v},\mathbf{w}}$*, and* $\varphi_{\mathbf{u}^+}$ *gives an isomorphism between* $\mathbb{R}^{\ell(w)+\ell(v)-\ell(w)}_{>0}$ *and* ${\mathcal{P}^u_{v,w}}^{\geq 0}$.

The positive structure on G can be strengthened to a cluster algebra structure (for background on cluster algebras, see [BFZ05, FG03, FZ02]), so the varieties $\mathcal{P}^e_{v,w} = L^{v,w}$ have a natural cluster algebra structure.

Question 2. *Does* $\mathcal{P}^{\mathbf{u}}_{\mathbf{v},\mathbf{w}}$ *have a cluster algebra structure for* $u \neq e$*?*

While quite natural from the perspective that the varieties $\mathcal{P}^{\mathbf{u}}_{\mathbf{v},\mathbf{w}}$ are a generalization of double Bruhat cells, this question appears to be quite difficult, and is beyond the scope of this paper.

8. The Poisson geometry of strata

In this section, we show how our partition (and that of Deodhar) are compatible with the natural Poisson structure induced by the standard Poisson structure on G and the double structure on $G \times G$. We assume from now on that G and $G \times G$ are endowed with these Poisson structures, unless stated otherwise.

There seems to be a clear, if somewhat imprecise, connection between positive and Poisson structures on $\mathcal{B}\times\mathcal{B}$. Just as $\mathcal{B}\times\mathcal{B}$ has two positive structures,

- one coming from the product of standard positive structures on $\mathcal{B}$
- one from the standard positive structure on G_Δ,

it also has two Poisson structures,

- one from the product of the Poisson structure induced on $\mathcal{B}$ by the standard Poisson group structure on G ("the product structure").
- one induced by the double Poisson structure on $G\times G$ ("the double structure").

One possible explanation for this correlation would be a positive answer to Question 2. A cluster algebra structure on the varieties $\mathcal{P}^u_{v,w}$ would endow them with compatible Poisson and positive structures.

In fact, there are many different Poisson structures on $\mathcal{B}\times\mathcal{B}$, in particular, one for each pair of transverse Lagrangian subalgebras $\mathfrak{u}$ and $\mathfrak{u}'$ of $\mathfrak{g}\times\mathfrak{g}$. For more on such subalgebras, see the papers of Evens and Lu, [EL01, EL04]. It is proved in [LY06] that the intersection of each $N(\mathfrak{u})$ and $N(\mathfrak{u}')$ orbits is a regular Poisson subvariety of $\mathcal{B}\times\mathcal{B}$, where $N(\mathfrak{u})$ and $N(\mathfrak{u}')$ are the normalizers of $\mathfrak{u}$ and $\mathfrak{u}'$ in $G\times G$. This provides the analog of the partition (1) of $\mathcal{B}\times\mathcal{B}$ in the general case.

Question 3. *Is there a generalization of the finer partition* (4) *to general choices of transverse Lagrangian subalgebras which has good Poisson properties?*

Let us first collect a few useful pieces of information about Poisson algebraic groups.

Proposition 8.1. *The subgroups* B^+, B^- *are Poisson subgroups of* G*, and* $B^+\times B^-$ *and* G_Δ *are Poisson subgroups of* $G\times G$*.*

This implies that the standard Poisson structure on G and the double structure on $G\times G$ can be pushed forward to well defined Poisson structures on $\mathcal{B}$ and $\mathcal{B}\times\mathcal{B}$. The latter become Poisson homogeneous spaces for $\mathcal{B}$ and $\mathcal{B}\times\mathcal{B}$, respectively, of so called group type.

Proposition 8.2. *Let X be a Poisson H-variety for any Poisson algebraic group H, and assume the Poisson tensor vanishes at $x \in X$. Then the map $h \mapsto h \cdot x$ is Poisson. In particular, $H \cdot x$ is a Poisson subvariety.*

Proof. The Poisson tensor vanishes at x, so $g \mapsto (g, x)$ is a Poisson map. The rest is clear. Here again $H \cdot x$ becomes a Poisson homogeneous H-space of group type. □

This proposition implies that $\mathcal{R}_{v,w}$ and $\mathcal{P}^u_{v,w}$ are Poisson subvarieties of $\mathcal{B}$ and $\mathcal{B} \times \mathcal{B}$. In fact, they are torus orbits of symplectic leaves (see [EL04, GY05]).

It also implies that many projection maps between orbits are Poisson.

Proposition 8.3. *The projection maps*

$$\pi^w_{w'} : \mathcal{B}_w \to \mathcal{B}_{w'}$$
$$\bar{\pi}^w_{w'} : \mathcal{B}^w \to \mathcal{B}^{w'}$$

for $w, w' \in W$ with $\ell(w^{-1}w') = \ell(w) - \ell(w')$ are Poisson submersions, as are the analogous maps for $B^+ \times B^-$ orbits on $\mathcal{B} \times \mathcal{B}$.

Since the Poisson structure on $\mathcal{B} \times \mathcal{B}$ is not the product of Poisson structures on $\mathcal{B}$, the second half of the proposition does not follow immediately from the first, though it does have the same proof. Similar techniques were used in [GY05, Proposition 1.6, §3.1].

Proof. If $f : X \to Y$ and $g : Y \to Z$ are surjective maps between Poisson varieties, and f and $g \circ f$ are both Poisson then g is Poisson as well.

Now, let f be the projection map $B^+ \to B^+ w \cdot B^+$ given by action on $w \cdot B^+$. Since the Poisson tensor vanishes at $w \cdot B^+$, this map is Poisson.

Let $g = \pi^w_{w'}$. In this case, $g \circ f$ is just the map $B^+ \to B^+ w' \cdot B^+$ given by action on $w' \cdot B^+$, which as we argued above, is Poisson. Thus, $\pi^w_{w'}$ is Poisson as well.

The same argument works on $\mathcal{B} \times \mathcal{B}$. □

We might hope that there is some sort of compatibility between the Poisson structure of $\mathcal{P}^u_{v,w}$ and our subvarieties $\mathcal{P}^{\mathbf{u}}_{\mathbf{v},\mathbf{w}}$. They cannot be Poisson since $\mathcal{P}^u_{v,w}$ are the minimal T-invariant Poisson subvarieties of $\mathcal{B} \times \mathcal{B}$. But there is a weaker notion of compatibility with Poisson structures which it is more reasonable to expect.

One calls a subscheme Y of an algebraic Poisson variety X **coisotropic** if its ideal sheaf $\mathcal{I}_Y$ is closed under Poisson bracket. That is, if f and g are two rational functions on X, which both vanish on Y, $\{f, g\}$ does as well. Note that this is a much weaker condition than being a Poisson subvariety, which requires $\{f, g\}$ to vanish on Y if either of f or g does.

Proposition 8.4. *If $Y \subset X$ is coisotropic and $\psi : Z \to X$ is a Poisson map, $\psi^{-1}(Y)$ is coisotropic in Z.*

Some care must be taken when applying this proposition to algebraic maps which are not reduced. In this case, one may pullback the reduced subscheme structure on a subset of X, and obtain a non-reduced subscheme $\psi^{-1}(Y)$ which is coisotropic, even though its reduced counterpart $\psi^{-1}(Y)_{red} \subset Z$ is not. This will not be a problem as long as the map ψ is smooth (in the algebraic sense, which is roughly equivalent to being a submersion in the differential category).

Proof. Obviously, this only needs to be checked for affine varieties, so let $\varphi^* : A \to B$ is a homomorphism of Poisson algebras, and $I \subset A$ is an ideal such that $\{I, I\} \subset I$, then $\{\varphi(I), \varphi(I)\} \subset \varphi(I)$, and so

$$\{B\varphi(I), B\varphi(I)\} \subset B\{\varphi(I), \varphi(I)\} + B\varphi(I)\{\varphi(I), B\} + \varphi(I) \cdot \{B, B\} \subset B\varphi(I) \quad \square$$

Theorem 8.5. *The Deodhar components* $\mathcal{R}_{\mathbf{v},\mathbf{w}}$ *and* $\mathcal{P}^{\mathbf{u}}_{\mathbf{v},\mathbf{w}}$ *are coisotropic subvarieties of* $\mathcal{B}$ *and* $\mathcal{B} \times \mathcal{B}$ *respectively.*

Proof. We will first consider the case of $\mathcal{R}_{\mathbf{v},\mathbf{w}}$. By Proposition 8.4, coisotropy is preserved under pullback by Poisson maps. Thus, consider the Poisson map $\pi_{n-1} : \mathcal{B}_{w_{(n)}} \to \mathcal{B}_{w_{(n-1)}}$. Each Deodhar component of $\mathcal{R}_{v,w}$ is the intersection of the pull-back of a Deodhar component of $\mathcal{R}_{vs_{i_n}, w_{(n-1)}}$ or $\mathcal{R}_{v,w_{(n-1)}}$ with $\mathcal{B}^{w_0 v}$ (here we use the fact that π_{n-1} is smooth (in the sense of algebraic geometry), to see that the subscheme structures match up). By induction, these are coisotropic subvarieties of $\mathcal{B}_{w_{(n-1)}}$, and so their pullbacks are coisotropic in $\mathcal{B}_{w_{(n)}}$, which is a Poisson subvariety of $\mathcal{B}$.

The proof for $\mathcal{P}^u_{v,w}$ is precisely the same. $\square$

We complete this section with a proof of the fact that our partition is in fact a stratification. While the dimension calculations we have done suggest that this is the case, it needs to be carefully checked. In some other, superficially similar situations, e.g. certain moduli spaces of flat connections, similar partitions are not stratifications. Luckily, on $\mathcal{B} \times \mathcal{B}$, this is not the case.

Theorem 8.6. *The partition in equation* (4) *is a stratification of* $\mathcal{B}_v \times \mathcal{B}^w$ *(and thus of* $\mathcal{P}^u_{v,w}$*).*

Proof. Let $\mathcal{X}$ be the sheaf of all algebraic vector fields on $\mathcal{B}_v \times \mathcal{B}^w$ and let $\mathcal{G}_i$ the subsheaf of vector fields such X such that the pushforward $T\pi_k(X)$ is well defined and Hamiltonian on $\mathcal{B}_{v_{(k)}} \times \mathcal{B}^{w_{(k)}}$. In particular, $\mathcal{G}_n$ is simply the sheaf of Hamiltonian vector fields. Now let $\mathcal{G}$ be the intersection of these sheaves. This sheaf is H invariant, the maps π_k and the Poisson structure on each variety are H-invariant, so $[\mathfrak{h}, \mathcal{G}] \subset \mathcal{G}$, and $\mathcal{G}' = \mathfrak{h} + \mathcal{G}$ is a Lie algebra subsheaf of $\mathcal{X}$. Consider the orbits of $\mathcal{G}'$ on $\mathcal{B}_v \times \mathcal{B}^w$, that is, the foliation one obtains from integrating it. This is a partition of $\mathcal{B}_v \times \mathcal{B}^w$ which respects the partition $\mathcal{B}_v \times \mathcal{B}^w = \sqcup \mathcal{P}^u_{v,w}$ (since these are torus orbits of symplectic leaves). We claim that these are precisely the subvarieties $\mathcal{P}^{\mathbf{u}}_{\mathbf{v},\mathbf{w}}$. Since the varieties $\mathcal{P}^{\mathbf{u}}_{\mathbf{v},\mathbf{w}}$ are connected, we need only check that the image of the evaluation map $\mathrm{ev}_x : \mathcal{G}' \to T_x(\mathcal{B}_v \times \mathcal{B}^w)$ has image precisely $T_x\mathcal{P}^{\mathbf{u}}_{\mathbf{v},\mathbf{w}}$. That the image is contained in this tangent space is clear, since $T_x\pi_k \circ \mathrm{ev}_x$ lands in the tangent space to the symplectic leaf in each case, since all push forwards are Hamiltonian. On the other hand, assume that the claim is true for $k < n$. Then if $u > u_{(n-1)}$,

$$T_x\pi_{n-1} : T_x\mathcal{P}^{\mathbf{u}}_{\mathbf{v},\mathbf{w}} \to T_x\mathcal{P}^{\mathbf{u_{(n-1)}}}_{\mathbf{v_{(n-1)}},\mathbf{w_{(n-1)}}}$$

is an isomorphism, and the hamiltonian vector fields $X_{\pi_k^* f}$ for all functions f on $\mathcal{B}_{v_{(n-1)}} \times \mathcal{B}^{w_{(n-1)}}$ span the tangent space to $\mathcal{P}^{\mathbf{u}}_{\mathbf{v},\mathbf{w}}$. Otherwise, there is a 1 dimensional kernel, which the Hamiltonian vector fields of pullbacks form a complement to in $T_x\mathcal{P}^{\mathbf{u}}_{\mathbf{v},\mathbf{w}}$. Since evaluation by the Poisson form $\Pi_\sharp$ vanishing on ker $T_x\pi_{n-1}$ is an open condition, on some (analytic) neighborhood of x, $\Pi_\sharp$ does not vanish on $\ker T_x\pi_{n-1}$, and there is a 1-form σ such that $\Pi_\sharp\sigma \in \ker T_x\pi_{n-1}$. The Hamiltonian vector field X_σ is in $\mathcal{G}$, since it is killed by $T_x\pi_k$ for all $k < n$.

Thus, we have proved that the subvarieties $\mathcal{P}^{\mathbf{u}}_{\mathbf{v},\mathbf{w}}$ are precisely the leaves of the foliation corresponding to $\mathcal{G}'$. Since the closure $\overline{\mathcal{P}^{\mathbf{u}}_{\mathbf{v},\mathbf{w}}}$ is also invariant under $\mathcal{G}'$, it is a union of leaves of the corresponding foliation, i.e. of varieties $\mathcal{P}^{\mathbf{u}'}_{\mathbf{v},\mathbf{w}}$ where u' ranges over some set of reduced double subexpressions of (v, w). Since the varieties $\mathcal{P}^{\mathbf{u}}_{\mathbf{v},\mathbf{w}}$ are smooth, connected and locally closed, this partition is, in fact, a stratification. □

Consider the poset structure on double distinguished subexpressions given by $\mathbf{u}' \geq \mathbf{u}$ if and only if $u'_{(k)} \geq u_{(k)}$ for all k. Since Bruhat order gives the closure relations on $\mathcal{B} \times \mathcal{B}$, the subvariety

$$Q^{\mathbf{u}}_{\mathbf{v},\mathbf{w}} = \bigsqcup_{\mathbf{u}' \geq \mathbf{u}} \mathcal{P}^{\mathbf{u}'}_{\mathbf{v},\mathbf{w}}$$

is closed. By definition, $\overline{\mathcal{P}^{\mathbf{u}}_{\mathbf{v},\mathbf{w}}} \subset Q^{\mathbf{u}}_{\mathbf{v},\mathbf{w}}$.

Question 4. *Are these sets equal? If not, what are the closure relations for the stratification of $\mathcal{B}_v \times \mathcal{B}^w$ by $\mathcal{P}^{\mathbf{u}}_{\mathbf{v},\mathbf{w}}$?*

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