# On Uniqueness of 2-Dimensional $\kappa$-Solutions *

Rugang Ye
Department of Mathematics
University of California, Santa Barbara

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The main purpose of this paper is to present a proof of [Corollary 11.3, P1], which is stated in the following theorem.

**Theorem 1** *In dimension 2 round spheres are the only orientable $\kappa$-solutions. Consequently, round projective planes are the only nonorientable 2-dimensional $\kappa$-solutions.*

This result plays an important role in analysing structures of 3-dimensional $\kappa$-solutions and hence blow-up singularities of the Ricci flow in dimension 3, as presented in [P1] and [P2]. Namely, splitting occurs in various arguments in [P1] and [P2] regarding 3-dimensional $\kappa$-solutions, and hence 3-dimensional $\kappa$-solutions lead to 2-dimensional $\kappa$-solutions. On the other hand, in dimension 3, rescaled limits of a solution of the Ricci flow near a blow-up singularity are $\kappa$-solutions. In other words, in dimension 3 blow-up singularities of the Ricci flow are modelled by $\kappa$-solutions.

The proof in [P1] of [Corrolary 11.3, P1] is incomplete. It refers to [H1] for the statement "round sphere is the only non-flat oriented nonnegatively curved gradient shrinking soliton in dimension two". But [H1] discusses only the case of compact surfaces.

This paper has been available at the author's website (www.math.ucsb.edu/∼yer/2dkappa.pdf) and through the website and notes of B. Kleiner and J. Lott on Perelman's papers on the Ricci flow since early 2004. Our key obsevation was to employ the arguments and techniques in Section 1 of [P2] to handle noncompact solitons.

We'll follow the notations in [P1], [P2], [Y1] and [Y2]. In particular, the distance function for a given metric $g$ on a manifold will be denoted by $d_g$. For a given family of metrics $g(t)$ on a manifold depending on a time parameter $t$, $B(p, r, t)$ denotes the closed geodesic ball of center $p$ and radius $r$ with respect to $g(t)$, and $d(\cdot, \cdot, t)$ denotes the distance function with respect to $g(t)$. Similarly, $d(q, A, t)$ denotes the distance

*2000 Mathematics Subject Classification: 53C20, 53C21 from a point $q$ to a set $A$ with respect to $g(t)$. The scalar curvature of $g(t)$ at a point $p$ is denoted by $R(p,t)$. Other curvature quantities are denoted in a similar way.

For the convenience of the reader we recall the concepts of $\kappa$-noncollapseness, $\kappa$-solutions and asymptotical solitons due to Perelman [P1][P2].

**Definition 1** Let $(M,g)$ be a complete Riemannian manifold of dimension $n \geq 1$. Consider positive numbers $\kappa$ and $\rho$. We say that $(M,g)$ is $\kappa$-noncollapsed on the scale $\rho$, provided that every geodesic ball of radius $r < \rho$ in $(M,g)$, on which $|Rm| \leq r^{-2}$ holds, has volume at least $\kappa r^n$.

**Definition 2** A $\kappa$-solution (of the Ricci flow) is an ancient nonflat solution of the Ricci flow with bounded nonnegative curvature operator which is $\kappa$-noncollapsed for some $\kappa > 0$ on all scales. More precisely, a $\kappa$-solution is a smooth solution $g = g(t)$ of the Ricci flow for $-\infty < t \leq 0$ on some manifold $M$ such that for each $t$, the metric $g(t)$ is complete, nonflat, has bounded and nonnegative curvature operator, and is $\kappa$-noncollapsed on all scales.

Round spheres give rise to obvious examples of $\kappa$-solutions. Consider the sphere $S^n, n \geq 1$. Let $g_0$ denote a multiple of the standand round sphere metric $g_{S^n}$ of sectional curvature 1 on $S^n$. Then $g(t) = (1 - \frac{2}{n} R_0 t) g_0, t \in (-\infty, 0]$ is a $\kappa$-solution on $S^n$. In dimension 2, we have $\kappa = 8\pi(1 - \cos\frac{1}{\sqrt{2}})$. If $M$ is a manifold diffeomorphic to $S^n$, we can pull back $g$ to $M$ by a (time independent) diffeomorphism to obtain a $\kappa$-solution on $M$. We call these $\kappa$-solutions *round sphere $\kappa$-solutions* or simply *round spheres*. Let $(M,g)$ be a 2-dimensional round sphere $\kappa$-solution. Its quotient by a nontrivial isometric $\mathbf{Z}_2$ action is called a *round projective plane*.

By the uniqueness of the solution of the Ricci flow with a given initial value, a smooth solution of the Ricci flow on a compact, simply connected manifold which has positive constant sectional curvature extends to a round sphere $\kappa$-solution.

**Definition 3** A shrinking Ricci soliton with time origin $t_0 \in \mathbf{R}$ is a smooth family of metrics $g = g(t)$ on a manifold $M$ for $t \in (-\infty, t_0)$ which satisfies the shrinking Ricci soliton equation

$$Ric + L_X g + \frac{1}{2(t - t_0)} g = 0 \tag{0.1}$$

for a smooth time-dependent vector field $X$. One easily verifies that a shrinking Ricci soliton is a smooth solution of the Ricci flow.

A *non-collapsing Ricci soliton* with time origin $t_0 \in \mathbf{R}$ is a shrinking Ricci solition $g = g(t)$ with time origin $t_0$ such that for each $t \in (-\infty, t_0)$, $g(t)$ is complete, and $\kappa$-noncollapsed on the scale $\rho > 0$ for some $\kappa > 0$ and $\rho > 0$ (which may depend on $t$).

A shrinking Ricci soliton $g$ is called a *gradient shrinking soliton*, if $X = \nabla f$ for a time-dependent smooth function $f$ ($\nabla$ is associated with $g(t)$ at time $t$). Such a smooth function is called a *potential function* of $g$.

Let $g$ be a shrinking Ricci soliton with time origin $t_0$. Note that it can be turned into a shrinking Ricci soliton with any given time origin by a time translation. We can also rescale it by a constant factor $\lambda > 0$ to obtain another shrinking Ricci soliton. On the other hand, integrating (0.1) we infer that for each given $t' < t_0$ and all $t < t_0$ there holds

$$g(t) = \frac{t_0 - t}{t_0 - t'}\phi(t)^* g(t') \tag{0.2}$$

for a smooth family of diffeomorphisms $\phi(t)$ with $\phi(t') = id$, see [Lemma 4.7, Y1] for the simple proof.

For each $n \geq 2$, there is a noncollapsing shrinking Ricci soliton $g_*$ with time origin 0 on $S^n$ for $n \geq 2$ which is given by $g_*(t) = -2(n-1)tg_{S^n}$. It is a gradient shrinking soliton whose potential functions are the constant functions. We can shift its time origin, rescale it by a constant factor, and pull it back by a diffeomorphism $F : M \to S^n$. The shrinking Ricci solitons obtained this way are called *round sphere solitons.*

**Definition 4** Let $(M, g)$ be a $\kappa$-solution. Consider $t_0 < 0$ and a sequence of positive numbers $a_k \to \infty$. By [Proposition 11.2, P1] and [Theorem 3.1, Y2], there is a sequence of points $q_k \in M$ such that the pointed flows $(M \times (-\infty, 0], \frac{1}{a_k} g(a_k(t_0+t)), q_k)$ subconverge smoothly to pointed solutions of the Ricci flow $(M_\infty \times (-\infty, 0), g_\infty, q_\infty)$ which are noncollapsing gradient shrinking solitons. Such limit solitons are called *asymptotical solitions* of $g$, or simply asymptotical solitons.

By Lemma 1 below, $\frac{\partial R}{\partial t} \geq 0$ holds true for $\kappa$-solutions (as observed by Perelman). Hence it also holds true for asymptotical solitons.

**Lemma 1** *Let $g$ be a smooth solution of the Ricci flow on $M \times (-\Lambda, 0)$ for a manifold $M$ and some $\Lambda > 0$. Assume that for each $t \in (-\Lambda, 0)$, $g(t)$ is complete and has bounded and nonnegative curvature operator. Then $\frac{\partial R}{\partial t} \geq 0$ everywhere.*

*Proof.* The trace form of Hamilton's differential Harnack inequality says

$$\frac{\partial R}{\partial t} + \frac{R}{t} + 2\nabla R \cdot X + 2Ric(X, X) \geq 0 \tag{0.3}$$

for arbitary smooth vector fields $X$. Taking $X = 0$ we then deduce $\frac{\partial R}{\partial t} \geq -\frac{R}{t} \geq 0$. ■

By a *round sphere metric* on a manifold $M$ diffeomorphic to $S^n$ we mean $\lambda F^* g_{S^n}$, where $\lambda$ is a posiitve number and $F$ is a smooth diffeomorphism from $M$ onto $S^n$.

**Lemma 2** *Let $M$ be diffeomorphic to $S^2$ and $g$ a shrinking Ricci soliton on $M$. Then $g$ is a round sphere soliton.*

*Proof.* This is essentially equivalent to [Theorem 10.1, H1]. We present here a different argument. By a time translation we may assume that the time origin of $g$ is 0. Let $\mathcal{T}(g)(\tau)$ be the volume normaliztion of $g$ which equals $g(-1)$ at $\tau = 0$. Namely $\mathcal{T}(g)(\tau)$ is a rescaling of $g$, has constant volume, and is a solution of the volume-normalized Ricci flow with $\tau = \tau(t)$ as the time variable. We have the formula $\mathcal{T}(g)(\tau) = \lambda(t)g(t)$ with $\lambda(t) = \exp(\int_{-1}^t r)$ and $\tau = \int_{-1}^t \lambda$, where $r(t)$ denotes the average scalar curvature of $g(t)$. By (0.2) we have

$$-\frac{1}{t}g(t) = \phi(t)^* g(-1). \tag{0.4}$$

Hence $-\frac{1}{t}g(t)$ has constant volume and equals $g(-1)$ at $t = -1$. It follows that $\lambda(t) = -\frac{1}{t}$ and then $\tau = -\ln|t|$. Consequently, the $t$-interval $(-\infty, 0)$ is transformed to the $\tau$-interval $(-\infty, \infty)$. Thus $\mathcal{T}(g)$ is a smooth solution of the volume normalized Ricci flow on $M \times (-\infty, \infty)$. By the convergence theorem for the volume normalized Ricci flow on $S^2$ proved in [BSY], $\mathcal{T}(g)(\tau)$ converges smoothly to a round sphere metric as $\tau \to \infty$. It follows that $-\frac{1}{t}g(t)$ converges smoothly to a round sphere metric as $t \to 0$. Letting $t' \to 0$ in (0.2) we then deduce that $g(t)$ has constant scalar curvature for each $t$, and hence $g(t)$ is a round sphere metric for each $t$.

For each fixed $t_1 < 0$, there is a round sphere soliton $g_{t_1}$ with time origin 0 such that $g_{t_1}(t_1) = g(t_1)$. By the uniqueness of the solution of the Ricci flow with a given initial metric we then have $g_{t_1} = g$ on $[t_1, 0)$. For $t_2 < t_1$ we have $g_{t_2} = g_{t_1}$ on $[t_1, 0)$. Hence we have $g = g_{-1}$ on $(-\infty, 0)$, i.e. $g$ is a round sphere soliton.

We note that in [H1], [10.1, H1] is used for proving the convergence theorem for the volume normalized Ricc flow on $S^2$. In [BSY], this convergence theorem is proved by a different argument without using [10.1, H1]. (In [H1], [10.1, H1] is proved by using complex analysis.) ■

**Lemma 3** *Let $g$ be a smooth solution of the Ricci flow on $M \times (a, b)$ for a 2-dimensional manifold $M$ and some time interval $(a, b)$, such that for each $t \in (a, b)$, the metric $g(t)$ is complete and has nonnegative scalar curvature. Let $t_0 \in (a, b)$. Assume that $g(t_0)$ is $\kappa$-noncollapsed on the scale $\rho$ for some $\kappa > 0$ and $\rho > 0$. Then $g(t_0)$ has bounded scalar curvature.*

*Proof.* Assume that $g(t_0)$ has unbounded scalar curvature. Choose a sequence of points $p_k \in M$ such that $R(p_k, t_0) > 0$ for each $k$ and $R(p_k, t_0) \to \infty$. Choose $q_k \in B(p_k, 1, t_0)$ such that the function $d(\cdot, \partial B(p_k, 1, t_0), t_0)^2 R(\cdot, t_0)$ on $B(p_k, 1, t_0)$

achieves its maximum at $q_k$. We set $r_k = d(q_k, \partial B(p_k, 1, t_0), t_0)/2$. For $q \in B(q_k, r_k, t_0)$ we have $d(q, \partial B(p_k, 1, t_0), t_0) \geq r_k$ and hence the maximum property of $q_k$ implies

$$r_k^2 R(q, t_0) \leq d(q, \partial B(p_k, 1, t_0), t_0)^2 R(q, t_0) \leq 4r_k^2 R(q_k, t_0). \tag{0.5}$$

It follows that

$$R(q, t_0) \leq 4R(q_k, t_0) \tag{0.6}$$

on $B(q_k, r_k, t_0)$. By the maximum property of $q_k$ we also infer $r_k^2 R(q_k, t_0) \geq R(p_k, t_0)/4$, so $r_k > 0$ for each $k$ and $r_k^2 R(q_k, t_0) \to \infty$. On the other hand, it is obvious that $q_k \to \infty$. We apply Splitting Lemma in Appendix with $\lambda_k = R(q_k, t_0)$ and $C = 4$. Hence we obtain from $(M, R(q_k, t_0)g(t_0), q_k)$ a smooth limit $(g_\infty, M_\infty, q_\infty)$ which splits off a line. Since $M_\infty$ is 2-dimensional, it follows that $g_\infty$ is flat. But the scalar curvature of $R(q_k, t_0)g(t_0)$ at $q_k$ is 1, hence the scalar curvature of $g_\infty$ at $q_\infty$ is also 1. This is a contradiction. ■

**Theorem 2** *The only orientable 2-dimensional noncollapsing Ricci solitons with nonnegative scalar curvature are round sphere solitons. Consequently, the only nonorientable 2-dimensional noncollapsing Ricci solitons with nonnegative scalar curvature are nontrivial isometric* $\mathbf{Z}_2$ *quotients of round sphere solitons.*

*Proof.* Let $g$ be an orientable monotone noncollapsing Ricci soliton on a 2-dimensional manifold $M$. We may assume that its time origin is 0. By Lemma 3, $g$ has bounded scalar curvature for each $t < 0$. We also observe that $R$ is everywhere positive. Indeed, if $R$ is zero at some point $q$ and some time $t$, then the strong maximum principle (applied to the evolution equation of $R$) implies that $R$ is everywhere zero for all later times, which contradicts the nonflatness of $g$.

We claim that $M$ is compact. To prove the claim, fix a point $p_0 \in M$. Following [(1.2), P2] we have, as a consequence of (0.1) the following equation

$$dR = 2Ric(\nabla f, \cdot) = Rg(\nabla f, \cdot) = Rdf. \tag{0.7}$$

Let $d(\cdot, \cdot, t)$ denote the distance at time $t$. Let $\theta(q, t, \gamma)$ denote the (smaller) angle between $\nabla f(q, t)$ and $\gamma'(l)$, where $l = d(p_0, q, t)$ and $\gamma$ is a unit speed shortest geodesic with respect to $g(t)$ such that $\gamma(0) = p_0$ and $\gamma(l) = q$. This angle is defined to be $2\pi$ if $\nabla f(q, t) = 0$. By the arguments in the proof of [Lemma 1.2, P2] in [P2], there is a positive number $A_0$ such that

$$\theta(q, -1, \gamma) \leq \frac{\pi}{4} \tag{0.8}$$

whenever $d(p_0, q, -1) \geq A_0$.

Let $\gamma$ be a shortest geodesic from $p_0$ to a point $q$ with $d(p_0, q, -1) > A_0$. We have by (0.7) and (0.8)

$$\frac{d}{dt} R(\gamma(t), -1) = \nabla R \cdot \gamma'(t) = R \nabla f \cdot \gamma'(t) > 0, \tag{0.9}$$

as long as $t \geq A$. Thus $R(\gamma(t), -1)$ increases along the portion of $\gamma$ which lies outside of the geodesic ball $B(p_0, A_0, -1)$. Consequently, we obtain the following estimate

$$R(q, -1) \geq \alpha_{A_0} \tag{0.10}$$

for all $q \in M$, where $\alpha_{A_0} = \min\{R(q, -1) : q \in B(p_0, A_0, -1)\}$. Since $R > 0$ everywhere, $\alpha_A$ is positive. By Bonnet theorem, $M$ must be compact.

Now we apply Gauss-Bonnet theorem to infer that $M$ is diffeomorphic to $S^2$. Then we apply Lemma 2 to conclude that $(M, g)$ is a round sphere soliton. ■

*Proof of Theorem 1*

Let $(M, g^*)$ be an orientable 2-dimensional $\kappa$-solution. Consider an arbitary asymptotic soliton $(M_\infty, g_\infty, q_\infty)$ of $g^*$ (as given by [Proposition 11.2, P1] and [Theorem 3.1, Y2]). By Theorem 2, $(M_\infty, g_\infty)$ is a round sphere soliton. Consequently, $M$ is diffeomorphic to $S^2$. Moreover, modulo smooth diffeomorphisms of $M$, the metrics $\frac{1}{-t} g^*(t)$ converge smoothly on $M$ to metrics of positive constant scalar curvature as $t \to -\infty$.

Consider the volume normalization $g$ of $g^*$, i.e. $g(\tau) = \mathcal{T}(g^*)(\tau), \tau \in (\Lambda_1, \Lambda_2]$, where $\Lambda_1 = \lim_{t \to -\infty} \tau(t)$ and $\Lambda_2 = \tau(0)$, cf. the proof of Lemma 1. Then, modulo smooth diffeomorphisms of $M$, $g(\tau)$ converges to metrics of positive constant scalar curvature as $\tau \to \Lambda_1$. Following [H1], let $f$ be the solution of $\Delta f = R - r$ with mean value zero, and set $H = \nabla^2 f - \frac{1}{2} \Delta f g$ ($H_{ij}$ is the $M_{ij}$ in [H1]). By [(9.1), H1] we have

$$\frac{\partial |H|^2}{\partial \tau} = \Delta |H|^2 - 2|\nabla H|^2 - 2R|H|^2. \tag{0.11}$$

Hence the maximum principle implies that $\max |H|^2$ is nonincreasing. Since $g$ converges modulo smooth diffeomorphisms of $M$ to metrics of positive constant scalar curvature as $\tau \to \Lambda_1$, we have $H \to 0$ as $\tau \to \Lambda_1$, whence $H \equiv 0$. Now, pulling back $g$ by a family of diffeomorphisms $\phi(t)$ generated by $\nabla f$ with $\phi(-1) = id$, we obtain $\hat{g}$ which satisfies

$$\frac{\partial \hat{g}}{\partial \tau} = 2H = 0. \tag{0.12}$$

Thus $\hat{g}$ is independent of time. Since its scalar curvature approaches a positive constant as $\tau \to \Lambda_1$, we infer that $\hat{g}$ has positive constant scalar curvature. It follows that $g$ and hence $g^*$ has positive constant scalar curvature. We conclude that $g^*$ is a round sphere $\kappa$-solution. Note that we have $f \equiv 0$ because $g$ has constant scalar curvature. Consequently, $g = \hat{g}$. We deduce that $g^*(t) = \lambda(t)\hat{g}$ for positive scalars

$\lambda(t)$. This immediately yields $g^*(t) = (1 - \frac{2}{n}R_0 t)g^*(0)$, where $R_0$ denotes the scalar curvature of $g^*(0)$. ■

## Appendix

We present here a splitting lemma for the Ricci flow.

**Splitting Lemma** *Let $g = g(t)$ be a smooth solution of the Ricci flow on $M \times (a, b)$ for some manifold $M$ of dimension $n \geq 2$ and some time interval $(a, b)$, such that for each $t$, the metric $g(t)$ is complete and has nonnegative sectional curvature. Assume that the scalar curvature $R$ satisfies $\frac{\partial R}{\partial t} \geq 0$ everywhere, i.e. it is nondecreasing. Consider $t_0 \in (a, b)$. Assume that the metric $g(t_0)$ is $\kappa$-noncollapsed on the scale $\rho$ for some $\kappa > 0$ and $\rho > 0$.*

*Consider a sequence of points $q_k \in M$ and two sequences of positive numbers $\lambda_k$ and $r_k$, such that $q_k \to \infty$, $\lambda_k \geq \delta$ for some $\delta > 0$, and $\sqrt{\lambda_k} r_k \to \infty$. Moreover, assume that there is a positive constant $C$ such that $R(q, t_0) \leq C\lambda_k$ for all $k$ and all $q \in B(q_k, r_k, t_0)$. Then the rescaled pointed Riemannian manifolds $(M, \lambda_k g(t_0), q_k)$ subconverge smoothly to a smooth pointed Riemannian manifold $(M_\infty, g_\infty, q_\infty)$, which splits off a line, i.e. $M_\infty = M_{\infty,1} \times \mathbf{R}$ for some $M_{\infty,1}$ and $g_\infty$ is a product metric.*

*Proof.* For a fixed $k$, we consider the rescaled flow $g_k(t) = \lambda_k g(t_0 + \frac{t}{\lambda_k}), t \in (\lambda_k(a - t_0), 0]$. In the following arguments, the geodesic balls $B(q, r, t)$ and the distance function $d(\cdot, \cdot, t)$ will be with respect to $g_k$. We observe that $g_k(0) = \lambda_k g(t_0)$ is $\kappa$-noncollpased on the scale $\delta\rho$. Furthermore, we have for $g_k$ the scalar curvature bound $R \leq C$ on $B(q_k, \sqrt{\lambda_k} r_k, 0) \times \{0\}$. The nondecreasing property of $R$ then implies the same scalar curvature bound for $g_k$ on $B(q_k, \sqrt{\lambda_k} r_k, 0) \times (\lambda_k(a - t_0, 0]$. Since the sectional curvatures are nonnegative, we deduce for $g_k$

$$|Rm| \leq c(n)C \tag{0.13}$$

on $B(q_k, \sqrt{\lambda_k} r_k, 0) \times (-\lambda_k(t_0 - a), 0]$, where $c(n)$ is a positive constant depending only on $n$.

Next we set $I = \{t \in (-\lambda_k(t_0 - a), 0] : B(q_k, \sqrt{\lambda_k} r_k/3, t) \subset B(q_k, \sqrt{\lambda_k} r_k, 0)\}$. Obviously, $I$ is closed in $(-\lambda_k(t_0 - a), 0]$ and $0 \in I$. Consider $t \in I$. By (0.13), we have $|Rm| \leq c(n)C$ on $B(q_k, \sqrt{\lambda_k} r_k/3, t) \times (-\lambda_k(t_0 - a), 0]$. Employing the Ricci flow equation we then deduce

$$d(q, q_k, 0) \leq \frac{\sqrt{\lambda_k} r_k}{3} e^{-c_1(n)Ct} \tag{0.14}$$

for all $q \in B(q_k, \sqrt{\lambda_k} r_k/3, t)$, where $c_1(n)$ is a positive constant depending only on $n$.

It follows that $B(q_k, \sqrt{\lambda_k} r_k/3, t) \subset B(q_k, \sqrt{\lambda_k} r_k/2, 0)$, as long as $t \geq -\epsilon_0$, where

$$\epsilon_0 = \min\{\frac{\ln \frac{3}{2}}{c_1(n)C}, \delta(t_0 - a)\}. \tag{0.15}$$

Consequently, $I \cap [-\epsilon_0, 0]$ is open in $[-\epsilon_0, 0]$. We conclude that $[-\epsilon_0, 0] \subset I$. Hence we have the estimate $|Rm| \leq c(n)C$ on $B(q_k, \sqrt{\lambda_k} r_k/3, -\epsilon_0) \times [-\epsilon_0, 0]$. By Shi's local derivative estimates [S] (see also [H2]), we then deduce for all $l \geq 0$

$$|\nabla^l Rm| \leq C(l, n) \tag{0.16}$$

on $B(q_k, \sqrt{\lambda_k} r_k/3, -\epsilon_0) \times [-\epsilon_0/2, 0]$, where $C(l, n)$ is a positive constant depending only on $l$ and $n$.

Since $\sqrt{\lambda_k} r_k/3 \to \infty$, the estimates (0.16) coupled with the $\kappa$-noncollapsing property of $g_k(0)$ lead to the claimed smooth subconvergence.

It remains to prove that $(M_\infty, g_\infty)$ splits off a line. We follow the arguments presented in [Appendix G, KL].

**Claim** $(M_\infty, g_\infty)$ *contains a line passing through* $q_\infty$.

To prove the claim, we choose an arbitary $p \in M$ and set $\rho_k = d(p, q_k, t_0)/2$ and $g_k^* = \rho_k^{-2} g(t_0)$. The pointed spaces $(M, g_k^*, p)$ converge in Gromov-Hausdorff distance to the asymptotical cone $\mathcal{C}$ of $(M, g(t_0))$, see [pp.58-59, BGS]. Hence we can find for each $k$ points $\hat{q}_k, \hat{x}_k$ and $\hat{y}_k$ lying on a ray in $\mathcal{C}$ such that

$$d(\hat{x}_k, \hat{y}_k) = 2, \ \ d(\hat{x}_k, \hat{q}_k) = 1, \ \ d(\hat{q}_k, \hat{y}_k) = 1, \tag{0.17}$$

and corresponding points $x_k, y_k \in M$ such that

$$\begin{aligned} &d_{g_k^*}(x_k, y_k) - d(\hat{x}_k, \hat{y}_k) \to 0, \ \ d_{g_k^*}(x_k, q_k) - d(\hat{x}_k, \hat{q}_k) \to 0, \\ &d_{g_k^*}(q_k, y_k) - d(\hat{q}_k, \hat{y}_k) \to 0 \end{aligned} \tag{0.18}$$

as $k \to \infty$. It follows that $\tilde{\angle} x_k q_k y_k \to \pi$, where $\tilde{\angle} x_k q_k y_k$ denotes the comparison angle at $q_k$ of the triple $x_k q_k y_k$ in $(M, g_k^*)$, i.e. the angle at $q_k^*$ of a comparison Euclidean triangle $x_k^* q_k^* y_k^*$. Let $\overline{q_k x_k}$ denote a shortest geodesic in $(M, g_k^*)$ from $q_k$ to $x_k$ and $\overline{q_k y_k}$ a shortest geodesic in $(M, g_k^*)$ from $q_k$ to $y_k$. By the monotonicity of comparison angles [4.3, BBI] we infer that

$$\tilde{\angle} x_k' q_k y_k' \to \pi \tag{0.19}$$

uniformly for all $x_k' \in \overline{q_k x_k}$ and $y_k' \in \overline{q_k y_k}$.

Now observe that $g_k(0) = \lambda_k \rho_k^2 g_k^*$. So $\overline{q_k x_k}$ and $\overline{q_k y_k}$ are also shortest geodesics in $(M, g_k(0))$. Since $\lambda_k \geq \delta$ and $q_k \to \infty$, we have $\sqrt{\lambda_k} \rho_k \to \infty$. Hence (0.17) and (0.18) imply that $d_{g_k(0)}(q_k, x_k) \to \infty$ and $d_{g_k(0)}(q_k, y_k) \to \infty$. It follows that $\overline{q_k x_k}$ and

$\overline{q_k y_k}$ converge to two rays $\gamma_+$ and $\gamma_-$ in $(M_\infty, g_\infty)$ eminating from $q_\infty$. (We pass to a suitable subsequence here.) By (0.19), we have $\tilde{\angle} x q_\infty y = \pi$ in $(M_\infty, g_\infty)$ and hence

$$d_{g_\infty}(x,y) = d_{g_\infty}(x, q_\infty) + d_{g_\infty}(q_\infty, y) \tag{0.20}$$

for all $x$ on $\gamma_+$ and all $y$ on $\gamma_-$. Consequently, joining $\gamma_+$ and $\gamma_-$ yields a line in $(M_\infty, g_\infty)$.

Since the limit $g_\infty$ has nonnegative sectional curvature, Toponogov splitting theorem implies that $(M_\infty, g_\infty)$ splits off a line. ■

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Department of Mathematics, University of California, Santa Barbara, CA 93106
yer@math.ucsb.edu