Solutions for the Midterm Exam

Midterm Exam
Math 5A
Winter 2007
Prof. R. Ye

Your Name:
Your Signature:
Your Perm Number:

Scores:
1.
2.
3.
Total:

Please present detailed steps of your solutions.

Do as many problems and work out as much as time allows.
1. (40 points) Consider a mass-spring system with mass \( m = 2 \) and spring constant \( k = 2 \). Its damping force is given by \(-b\) times the velocity.
1) Find the location \( x = x(t) \) of the mass at time \( t \) when \( b = \sqrt{15} \), \( x(0) = 3 \) and \( \dot{x}(0) = -\frac{\sqrt{15}}{4} \).
2) Find the general formula for \( x = x(t) \) when \( b = 4 \).
3) Find the general formula for \( x = x(t) \) when \( b = \sqrt{17} \).

Solution

The equation for \( x = x(t) \) is

\[
2\ddot{x} + b\dot{x} + 2x = 0. \tag{0.1}
\]

The associated equation for the characteristic roots is then

\[
2\ddot{x} + b\dot{x} + 2x = 0. \tag{0.2}
\]

We obtain the roots:

\[
\lambda = \frac{-b \pm \sqrt{b^2 - 4 \cdot 2 \cdot 2}}{2 \cdot 2} = -\frac{b}{4} \pm \frac{\sqrt{b^2 - 16}}{4}. \tag{0.3}
\]

1) For \( b = \sqrt{15} \) we obtain

\[
\lambda_1 = -\frac{\sqrt{15}}{4} + \frac{i}{4}, \quad \lambda_2 = -\frac{\sqrt{15}}{4} - \frac{i}{4}. \tag{0.4}
\]

So we obtain the general solution

\[
x = e^{-\frac{\sqrt{15}}{4} t}(c_1 \cos \frac{t}{4} + c_2 \sin \frac{t}{4}). \tag{0.5}
\]

It follows that

\[
\dot{x} = -\frac{\sqrt{15}}{4} e^{-\frac{\sqrt{15}}{4} t}(c_1 \cos \frac{t}{4} + c_2 \sin \frac{t}{4}) + e^{-\frac{\sqrt{15}}{4} t}(-\frac{c_1}{4} \sin \frac{t}{4} + \frac{c_2}{4} \cos \frac{t}{4}). \tag{0.6}
\]

Setting \( t = 0 \) we deduce

\[
x(0) = c_1, \quad \dot{x}(0) = -\frac{\sqrt{15}}{4} c_1 + \frac{c_2}{4}. \tag{0.7}
\]

By the initial conditions we then infer

\[
c_1 = 3, \quad -\frac{\sqrt{15}}{4} c_1 + \frac{c_2}{4} = -\frac{\sqrt{15}}{4}. \tag{0.8}
\]
We arrive at $c_1 = 3, c_2 = 2\sqrt{15}$ and hence

$$x = e^{-\sqrt{15}t}(3\cos \frac{t}{4} + 2\sqrt{15}\sin \frac{t}{4}).$$

(0.9)

2) For $b = 4$ we obtain one

$$\lambda = -\frac{4}{4} = -1$$

(0.10)

One solution of the differential equation is

$$x_1(t) = e^{-t}.$$  

(0.11)

Another solution is

$$x_2(t) = te^{-t}.$$  

(0.12)

We arrive at the general solution

$$x = (c_1 + c_2t)e^{-t}.$$  

(0.13)

3) For $b = \sqrt{17}$ we obtain roots

$$\lambda = \frac{-\sqrt{17}}{4} \pm \frac{1}{4}.$$  

(0.14)

Hence $\lambda_1 = \frac{-\sqrt{17}+1}{4}, \lambda_2 = \frac{-\sqrt{17}-1}{4}$ and the corresponding solutions are

$$x_1(t) = e^{-\sqrt{17+1}t}, x_2(t) = e^{-\sqrt{17-1}t}.$$  

(0.15)

The general solution is then given by

$$x(t) = c_1e^{-\sqrt{17+1}t} + c_2e^{-\sqrt{17-1}t}.$$  

(0.16)
2. (30 points) Find the general solution of the equation

\[ 4\ddot{x} + 3x = -4\sin\left(\frac{\sqrt{3}}{2}t\right) + \cos t. \]  

(0.17)

Solution

We break the problem into three parts.

Part 1 Consider the homogeneous equation

\[ 4\ddot{x} + 3x = 0. \]  

(0.18)

It can be rewritten as

\[ \ddot{x} + \frac{3}{4}x = 0 \]  

(0.19)

and we obtain the general solution

\[ x_0(t) = c_1 \cos\left(\frac{\sqrt{3}}{2}t\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}t\right). \]  

(0.20)

Part 2 Consider the equation

\[ 4\ddot{x} + 3x = \cos t. \]  

(0.21)

We look for a particular solution. We set \( x_1(t) = a\cos t + b\sin t \). By inspection we can see that we can set \( x_1(t) = a\cos t \). Then

\[ \ddot{x}_1 = -a\cos t. \]  

(0.22)

and we obtain

\[ 4(-a\cos t) + 3a\cos t = \cos t. \]  

(0.23)

It follows that \(-4a + 3a = 1\) and hence \(a = -1\) and

\[ x_1(t) = -\cos t. \]  

(0.24)

Part 3 Consider the equation

\[ 4\ddot{x} + 3x = -4\sin\left(\frac{\sqrt{3}}{2}t\right). \]  

(0.25)

We look for a particular solution. Since the root for the homogeneous equation equals \(\frac{\sqrt{3}}{2}\), we are dealing with the case of resonance. We set

\[ x_2(t) = t \left[ a\cos\left(\frac{\sqrt{3}}{2}t\right) + b\sin\left(\frac{\sqrt{3}}{2}t\right) \right]. \]  

(0.26)
For convenience, we set
\[ P(t) = a \cos\left(\frac{\sqrt{3}}{2} t\right) + b \sin\left(\frac{\sqrt{3}}{2} t\right). \] (0.27)

Then \( x_2(t) = tP(t) \). We have
\[ \dot{x}_2 = P + t\dot{P}, \quad \ddot{x}_2 = 2\dot{P} + t\ddot{P}. \] (0.28)

Hence
\[ 4\ddot{x}_2 + 3x_2 = 8\dot{P} + t(4\dddot{P} + 3P) = 8\dot{P}, \] (0.29)
because \( 4\dddot{P} + 3P = 0 \). Hence the equation \( 4\ddot{x}_2 + 3x_2 = -4\sin\left(\frac{\sqrt{3}}{2} t\right) \) becomes
\[ 8\dot{P} = -4\sin\left(\frac{\sqrt{3}}{2} t\right), \text{ i.e. } \dot{P} = -\frac{1}{2} \sin\left(\frac{\sqrt{3}}{2} t\right). \] (0.30)

But
\[ \dot{P} = -a \frac{\sqrt{3}}{2} \sin\left(\frac{\sqrt{3}}{2} t\right) + b \frac{\sqrt{3}}{2} \cos\left(\frac{\sqrt{3}}{2} t\right). \] (0.31)

It follows that \( b = 0 \) and \( a = \frac{1}{\sqrt{3}} \). Hence
\[ x_2(t) = \frac{1}{\sqrt{3}} t \cos\left(\frac{\sqrt{3}}{2} t\right). \] (0.32)

**Part 4** Putting the above results together we obtain the desired general solution
\[ x(t) = c_1 \cos\left(\frac{\sqrt{3}}{2} t\right) + c_2 \sin\left(\frac{\sqrt{3}}{2} t\right) - \cos t + \frac{1}{\sqrt{3}} t \cos\left(\frac{\sqrt{3}}{2} t\right). \] (0.33)
3. (30 points) Find a particular solution of the equation

\[ y'' - 6y' + 9y = 2e^{-3t} + e^{3t} + 2t + 1. \]  

(0.34)

**Solution**

1) We first figure out the solutions of the homogeneous equation

\[ y'' - 6y' + 9y = 0. \]  

(0.35)

Its characteristic equation is \( \lambda^2 - 6\lambda + 9 = 0 \), i.e. \( (\lambda - 3)^2 = 0 \). So we obtain a double root \( \lambda_1 = \lambda_2 = 3 \). Then we obtain two linearly independent solutions

\[ y_1(t) = e^{3t}, \quad y_2(t) = te^{3t}. \]  

(0.36)

2) We look for a particular solution of the equation

\[ y'' - 6y' + 9y = 2e^{-3t}. \]  

(0.37)

We set \( y_P(t) = Ae^{-3t} \). Then \( y_P' = -3Ae^{-3t} \) and \( y_P'' = 9Ae^{-3t} \). So

\[ y''_P - 6y'_P + 9y_P = (9A + 18A + 9A)e^{-3t} = 36Ae^{-3t}. \]  

(0.38)

So we arrive at \( 36A = 2 \), i.e. \( A = \frac{1}{18} \) and \( y_P = \frac{1}{18}e^{-3t} \).

3) We look for a particular solution of the equation

\[ y'' - 6y' + 9y = e^{3t}. \]  

(0.39)

Since both \( e^{3t} \) and \( te^{3t} \) are solutions of the homogeneous equation, we set

\[ y_p(t) = At^2e^{3t}. \]  

(0.40)

Then \( y'_p = A(2t + 3t^2)e^{3t} \) and \( y''_p = A(2 + 12t + 9t^2)e^{3t} \). It follows that

\[ y''_p - 6y'_p + 9y_p = A(2 + 12t + 9t^2 - 12t - 18t^2 + 9t^2)e^{3t} = 2Ae^{3t}. \]  

(0.41)

Hence we arrive at \( 2A = 1 \), i.e. \( A = \frac{1}{2} \) and \( y_P = \frac{1}{2}t^2e^{3t} \).

4) We look for a particular solution for

\[ y'' - 6y' + 9y = 2t + 1. \]  

(0.42)

We set \( y_P(t) = A + Bt \). Then \( y'_P = B \) and \( y''_P = 0 \). So

\[ y''_P - 6y'_P + 9y_p = -6B + 9(A + Bt) = (9A - 6B) + 9Bt. \]  

(0.43)
We arrive at

\[(9A - 6B) + 9Bt = 2t + 1\]  
\[(0.44)\]

which means

\[9A - 6B = 1, 9B = 2.\]  
\[(0.45)\]

We conclude that \(B = \frac{2}{9}, A = \frac{1}{9} + \frac{12}{81} = \frac{21}{81}\) and \(y_p = \frac{21}{81} + \frac{2}{9}t\).

5) Putting the above together we obtain a desired particular solution of the equation (??)

\[y_p = \frac{1}{18}e^{-3t} + \frac{1}{2}t^2e^{3t} + \frac{21}{81} + \frac{2}{9}t.\]  
\[(0.46)\]