κ-Solutions Of the Ricci Flow *

Rugang Ye

Department of Mathematics

University of California, Santa Barbara

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Abstract: The concept of κ-solutions of the Ricci flow plays an important role in Perelman's work on the Ricci flow, the Poincaré conjecture and the geometrization conjecture. In this paper we present a number of results on κ-solutions and a concise picture of this role .

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1 Introduction

Let M be a smooth manifold of dimension $n \geq 2$. The Ricci flow

$$\frac{\partial g}{\partial t} = -2Ric \tag{1.1}$$

was introduced by R. Hamilton in his seminar paper [H1] on 3-dimensional manifolds of positive Ricci curvature. Here, $g = g(t)$ is a smooth family of Riemannian metrics on M and Ric the Ricci curvature tensor of $g = g(t)$. (For basics and general information on the Ricci flow we refer to [H1], [H5], [CK] and [CLN].) The concept of κ-solutions of the Ricci flow was introduced by G. Perelman in his seminar papers [P1] and [P2], and plays a crucial role in the analysis of the structures of blow-up singularities of the Ricci flow, and thereby a crucial role in Perelman's work in

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[P1],[P2] and [P3] on Poincaré conjecture and Thurston's geometrization conjecture on 3-dimensional manifolds. See Section 1 for the definition of this concept. (Perelman's terminology for this concept is ancient κ-solutions.) In the Ricci flow approach to the Poincaré conjecture and the geometrization conjecture, which was initiated by Hamilton, one chooses a Riemannian metric g_0 on an arbitrary compact 3-dimensional manifold M and considers the smooth solution $g = g(t)$ of the Ricci flow (1.1) with initial value $g(0) = g_0$. This smooth solution exists on a maximal time interval $[0, T)$, where T can be finite or infinite. Without special geometric conditions on g_0, in general one expects T to be finite. If one chooses a reference point $\bar{p} \in M$ for the purpose of measuring geometric quantities with respect to distance from $\bar{p}$, then we have a *pointed Ricci flow* $(g = g(t), M \times [0, T), \bar{p})$. Assume $T < \infty$. As $t \to T$, the Ricci flow $g(t)$ will blow up, i.e. the norm of the Riemann curvature tensor $|Rm|$ will become infinite at least in some places of M. In order to gain information on the geometric and topological structures of the manifold M, it is crucial to analyze the structures of the blow singularities of the flow $g(t)$ as $t \to T$.

A general method for carrying out this analysis is to rescale the Ricci flow $g(t)$ and obtain blow-up limits from $g(t)$. Let's consider the general case $n \geq 3$. For a positive number λ and a number $\bar{T} \in (0, T)$, the rescaled flow $g_{\lambda,\bar{T}}(t)$ with scaling factor λ and scaling center time $\bar{T}$ is defined to be

$$g_{\lambda,\bar{T}}(t) = \lambda g(\bar{T} + \lambda^{-1} t) \tag{1.2}$$

for $t \in [-\lambda\bar{T}, \lambda(T - \bar{T}))$. By the scaling invariance of the Ricci flow, $g_{\lambda,\bar{T}}$ is a smooth solution of the Ricci flow on M. For a sequence of scaling factors $\lambda_k \to \infty$ and scaling center times T_k, we have a sequence of rescaled Ricci flows $g_k = g_{\lambda_k,T_k}$. If we also choose a sequence of reference points p_k, then we have a sequence of pointed Ricci flows $(g_k, M \times [-\lambda T_k, \lambda_k(T - T_k)), p_k)$. Now the basic idea is to choose λ_k, T_k and p_k suitably such that we can extract from this sequence of pointed Ricci flows subsequences which point converge (i.e. converge smoothly with the reference points p_k as centers) to smooth limits, which are called blow-up limits. These limits are pointed Ricci flows on limit manifolds with limit reference points, which we denote by $(g_\infty, M_\infty \times (-\infty, 0], p_\infty)$. They are ancient solutions of the Ricci flow, namely they are defined on the time interval $(-\infty, 0]$ and consist of complete Riemannian metrics. Moreover, they enjoy some special geometric properties.

In dimension $n = 3$, the blow-up limits turn out to be κ-solutions. Since they arise as blow-up limits of the Ricci flow $g = g(t)$, they model the structures of the blow-up singularities of $g = g(t)$ (as $t \to T$). Indeed, the results in [P1] and [P2] on structures of blow-up singularities of the Ricci flow are deduced by employing the special geometric and topological properties of κ-solutions. In the above account we assumed $T < \infty$. The general arguments of blow-up analysis as described above also work in the case $T = \infty$, with a major difference between $T < \infty$ and $T = \infty$. Namely, in the case $T < \infty$, the Ricci flow $g = g(t)$ is always κ-noncollapsed. In the case $T = \infty$, it may collapse as $t \to T$, and hence the rescaled flows may degenerate to lower dimensional objects. For this reason, the blow-up analysis is more complicated in the case $T = \infty$. Fortunately, Perelman's solution of the Poincaré conjecture and the geometrization conjecture do not require analysis of blow-up limits for $T = \infty$ in the collapsing case.

Once the structures of the blow-up singularities as $t \to T$ are well-understood, namely their topological and geometric types are classified and well under control, as presented in [P1] and [P2], one can perform surgery at $t = T$ to remove singularities and construct a manifold $\bar{M}$ with surgery out of M and a metric $\bar{g}$ with surgery out of a partial limit of g at T, as done in [P2]. (Earlier work on surgeries of the Ricci flow was done by Hamilton [H8].) Then one considers the Ricci flow on $\bar{M}$ with the initial metric $\bar{g}$. If $\bar{g}$ runs into blow-up singularities at some finite time, the above blow-up analysis can be repeated for $\bar{g}$. By induction, one then obtains a Ricci flow $g^* = g^*(t)$ with surgeries which consists of a sequence of smooth solutions of the Ricci flow on a sequence of smooth manifolds, as done in [P2]. This Ricci flow with surgeries exists on a maximal time interval $[0, T^*)$. An important special case is included in this scheme. If the Ricci flow $g = g(t)$ blows up on the whole M as $t \to \infty$, then there is nothing left from M for performing surgery. In this case, the topological structure of M is immediately classified in consistence with the Poincaré conjecture. The same kind of picture can also occur at a later stage of the Ricci flow g^* with surgeries. In either case, the Ricci flow with surgeries g^* is said to be extinct (at a finite time), and the goal of classifying the topological structure of M is achieved. For convenience, we extend g^* to be the empty solution on $[\bar{T}, \infty)$ if $\bar{T}$ is an extinction time. By an intricate argument, Perelman shows that in general $T^* = \infty$ (without assuming finite time extinction), see [P2] and the expositions in [MT] and [KL].

The next stage of Perelman's work goes as follows. On the one hand, he shows in [P3] that under the condition of the Poincaré conjecture, g^* has to be extinct at a finite time (see also [CM]). This leads to his proof of the Poincaré conjecture as presented in [P2] and [P3]. On the other hand, in the general situation he shows in [P2] that as $t \to \infty$, the Ricci flow with surgeries g^* leads to the geometrization of M, i.e. a suitable decomposition of M into finite many pieces, such that each piece carries a standard geometric structure. This includes the Poincaré conjecture as a special case. Here Perelman applies an important argument of Hamilton in [H8] for obtaining hyperbolic manifolds as limits as $t \to \infty$ from the noncollapsing part of the Ricci flow, and appeals to his own results on locally collapsing manifolds for obtaining the desired topological and geometric structures on the collapsing part of the Ricci flow. (See [SY] for relevant results on collapsed manifolds.)

It should be clear from the above accounts that κ-solutions play a crucial role in Perelman's work. Although Perelman only needs the case $n = 3$, the prospect of applying the Ricci flow in higher dimensions looks promising, and hence it is of great interests to understand κ-solutions also in higher dimensions. Indeed, many results of Perelman on κ-solutions are valid in general dimensions.

The contents of this paper are organized as follows. In Section 1, we present the concept of κ-noncollpasedness, a basic result on blow-up limits of the Ricci flow and the concept of κ-solutions. In Section 2, we present the concept of gradient shrinking solitons and Perelman's theorem on gradient shrinking solitons as blow-down limits of κ-solutions, which plays an important role for understanding the structures of κ-solutions. In [P1], a sketch of the proof for this theorem was given. The complete proof of this theorem was presented in the author's papers [Y2] and [Y3] (see also [MT] and [KL]). In this section, two important tools, the l-function (the reduced length) and the reduced volume, are also presented. In Section 3, we present the classification of 2-dimensional κ-solutions. This result is crucial for carrying out the argument of dimension reduction, in which a 3-dimensional Ricci flow splits into the product of a 2-dimensional Ricci flow with the 1-dimensional Ricci flow. This argument plays an important role in a number of places, e.g. in classifying 3-dimensional κ-solutions. The proof of the classification of 2-dimensional κ-solutions presented in [P1] is incomplete. The first complete proof was given in the author's paper [Y4]. This proof is reproduced here. (The proof in [Y4] was later adapted in [CZ]. Different arguments were presented in [KL] and [MT].) In Section 4, we present Perelman's classification of 3-dimensional κ-solutions and his compactness theorem for the space of 3-dimensional noncompact κ-solutions. In Section 5, we present Perelman's theorem on canonical neighborhoods of solutions of the Ricci flow, which is a central, culminating result. By using this result we demonstrate that the blow-up limits $(g_\infty, M_\infty \times (-\infty, 0], p_\infty)$ discussed above must be κ-solutions in dimension 3. We also present Perelman's result on obtaining bounded curvature at bounded distance and a result on obtaining suitable scaling parameters (λ_k, T_k, p_k) as discussed above such that the blow-up sequence g_k subconverges.

Our goal is to present a concise picture of the theory of κ-solutions and its role in Perelman's work on the Ricci flow, the Poincaré conjecture and the geometrization conjecture. We do not attempt to include all results on κ-solutions. In a number of places we stay away from complicated technical details of the proofs. The complete details can be found in Perelman's papers [P1] and [P2], the excellent book [MT] and notes [KL], the paper [CZ], and the papers [Y1], [Y2], [Y3], [Y4] and [Y5]. (The specific references are always given.) On the other hand, we include complete proofs in a number of other places, e.g. in Section 4, because these proofs are relatively short and can provide a good help for understanding the concepts and methods, or because we cannot find suitable references for the complete details.

2 Blow-up limits of the Ricci flow and κ-solutions

Definition 1 ([P1]) Let (M, g) be a complete Riemannian manifold of dimension $n \geq 1$. Consider positive numbers κ and ρ. We say that (M, g) is κ-noncollapsed on the scale ρ, provided that every geodesic ball $B(x, r)$ of radius $r < \rho$ in (M, g), on which $|Rm| \leq r^{-2}$ holds, has volume at least κr^n.

Definition 2 ([P1]) A κ-solution (of the Ricci flow (1.1)) is an ancient nonflat solution of the Ricci flow (1.1) on a manifold M with bounded nonnegative curvature operator which is κ-noncollapsed for some $\kappa > 0$ on all scales. More precisely, a κ-solution is a smooth solution $g = g(t)$ of the Ricci flow for $-\infty < t \leq 0$ on some manifold M such that for each t, the metric $g(t)$ is complete, nonflat, has bounded and nonnegative curvature operator, and is κ-noncollapsed on all scales.

Round spheres give rise to obvious examples of κ-solutions. Consider the sphere $S^n, n \geq 1$. Let g_0 denote a constant multiple of the standand round sphere metric g_{S^n} of sectional curvature 1 on S^n. Then $g(t) = (1 - \frac{2}{n}R_{g_0}t)g_0, t \in (-\infty, 0]$ is a κ-solution on S^n, where R_{g_0} denotes the scalar curvature of g_0. In dimension 2, we have $\kappa = 8\pi(1 - \cos\frac{1}{\sqrt{2}})$. If M is a manifold diffeomorphic to S^n, we can pull back g to M by a (time independent) diffeomorphism to obtain a κ-solution on M. We call these κ-solutions *round sphere κ-solutions* or simply *round spheres*. By the uniqueness of the solution of the Ricci flow with a given initial value, a smooth solution of the Ricci flow on a compact, simply connected manifold which has positive constant sectional curvature extends to a round sphere κ-solution.

By the following lemma and nonnegativity of curvature operator, the scalar curvature R of a κ-solution g at $t = 0$ controls the norm of its curvature operator for all time $t \leq 0$. Hence g has uniformly bounded curvature operator on the entire time interval $(-\infty, 0]$.

Lemma 2.1 *Let g be a smooth solution of the Ricci flow on $M \times (-\Lambda, 0)$ for a manifold M and some $\Lambda > 0$. Assume that for each $t \in (-\Lambda, 0)$, $g(t)$ is complete and has bounded and nonnegative curvature operator. Then $\frac{\partial R}{\partial t} \geq 0$ everywhere.*

Proof. The trace form [1.2, H4] of Hamilton's differential Harnack inequality says

$$\frac{\partial R}{\partial t} + \frac{R}{t} + 2\nabla R \cdot X + 2Ric(X, X) \geq 0 \tag{2.1}$$

for arbitary smooth vector fields X. Taking $X = 0$ we then deduce $\frac{\partial R}{\partial t} \geq -\frac{R}{t} \geq 0$. ■

The cornerstone for performing blow-up analysis for the Ricci flow is Perelman's non-collapsing result in [P1] for the Ricci flow for finite time. The original κ-noncollapsing result of Perelman in [P1] is formulated relative to bounds for $|Rm|$. Later, a κ-noncollapsing result for bounded time measured relative to upper bounds of the scalar curvature R was obtained independently by Perelman (see [KL]) and the present author (see [Y1]). More recently, the present author obtained in [Y5] new κ-noncollapsing estimates which improve these earlier results. We state these new estimates below.

Theorem 2.2 *Let $g = g(t)$ be a smooth solution of the Ricci flow (1.1) on $M \times [0,T)$ for a compact manifold M of dimension $n \geq 3$ and some $T < \infty$. There are positive constants A and B depending only on the initial metric $g_0 = g(0)$ in terms of its logarithmic Sobolev inequality and an upper bound of T with the following properties. Let $L > 0$ and $t \in [0,T)$. Consider the Riemannian manifold (M,g) with $g = g(t)$. Assume $R \leq \frac{1}{r^2}$ on a geodesic ball $B(x,r)$ with $0 < r \leq L$. Then there holds*

$$vol(B(x,r)) \geq \left(\frac{1}{2^{n+3}A + 2BL^2}\right)^{\frac{n}{2}} r^n. \tag{2.2}$$

Next let $\lambda_0(g_0)$ denote the first eigenvalue of the operator $-\Delta + \frac{R}{4}$ for the initial metric. Assume $\lambda_0(g_0) > 0$. Then the above estimate (2.2) can be improved to (under the same condition on $B(x,r)$)

$$vol(B(x,r)) \geq \left(\frac{1}{2^{n+3}A}\right)^{\frac{n}{2}} r^n, \tag{2.3}$$

where $A > 0$ depends only on the initial metric g_0 in terms of its logarithmic Sobolev inequality and $\lambda_0(g_0)$ (without depending on T) and the assumption $r < L$ is not needed.

In other words, the flow $g = g(t), t \in [0,T)$ is κ-noncollapsed relative to upper bounds of the scalar curvature on all scales.

Next we present a general result on blow-up limits of the Ricci flow based on the κ-noncollapsing results described above.

Theorem 2.3 *Let $g = g(t)$ be a smooth solution of the Ricci flow (1.1) on $M \times [0,T)$ for a compact manifold M of dimension $n \geq 3$ and some $T < \infty$. Let $\lambda_k \to \infty$ be a sequence of scaling factors, $T_k \in (0,T)$ a sequence of scaling center times, and $p_k \in M$ a sequence of reference points. Consider the rescaled flows $g_k(t) = g_{\lambda_k, T_k}$. Assume that for each $L > 0$ there are numbers $K_L > 0$ and $0 < T_L < T$ such that $|Rm| \leq K_L$ holds true for $g_k(t)$ with $T_k > T_L$ on the geodesic ball of center p_k and radius L, where $t \in (-\lambda_k(T_k - T_L), 0]$ is arbitrary. Then a subsequence of $(g_k, M \times (-\lambda_k(T_k - T_L), 0], x_k)$ point converges smoothly to a pointed Ricci flow $(g_\infty, M_\infty \times (-T_\infty, 0], p_\infty)$ for some manifold M_∞ and $p_\infty \in M_\infty$, such that $g_\infty(t)$ is complete for each t. The flow g_∞ is κ-noncollapsed relative to upper bounds of the scalar curvature on all scales, where $\kappa = 2^{-\frac{n(n+3)}{2}} A^{-\frac{n}{2}}$ and A is from Theorem 2.2. Moreover, there holds for g_∞ at all t*

$$\left(\int_{M_\infty} |u|^{\frac{2n}{n-2}} dvol\right)^{\frac{n-2}{n}} \leq A \int_{M_\infty} (|\nabla u|^2 + \frac{R}{4} u^2) dvol \tag{2.4}$$

for all $u \in W^{1,2}(M_\infty)$. (By [Y5], this constant A is the same as the A in Theorem 2.2.) In other words, blow-up limits of the Ricci flow always satisfy the Sobolev inequality (2.4).

Proof. Note that pointed convergence means convergence on geodesic balls of center p_k and any given radius. By pulling back of metrics, this means convergence on geodesic balls of center p_∞ and any given radius with respect to g_∞. By Theorem 2.2 the rescaled flow g_k satisfies for any given $L > 0$ and any given time t the volume estimate

$$vol(B(p,r)) \geq \left(\frac{1}{2^{n+3}A + 2B\lambda_k L^2}\right)^{\frac{n}{2}} r^n, \tag{2.5}$$

provided that $r \leq L$ and $R \leq r^{-2}$ on $B(p_k, r)$ at time t. By the assumption on $|Rm|$ we have for any given time $T_L \leq t \leq T_k$ the estimate $R \leq c(n)K_L$ on $B(p_k, L)$ for a positive constant $c(n)$ depending on n. Hence the volume estimate (2.5) holds true for $p = p_k$, provided that $r \leq \min\{L, \sqrt{c(n)K_L}\}$. By Bishop-Gromov volume comparison, we then obtain at any $t \in [-\lambda_k(T_k - T_L), 0]$

$$vol(B(p,r)) \geq C(L,n) \left(\frac{1}{2^{n+3}A + 2B\lambda_k L^2}\right)^{\frac{n}{2}} r^n \tag{2.6}$$

for all $p \in B(p_k, L/2)$, where $C(L,n)$ is a positive constant depending on L and n. By [CGT] (see also [Lemma B.1, Y1]), we obtain a positive constant $\delta(n, L, A, B)$ depending on n, L, A and B such that at any $t \in [-\lambda_k(T_k - T_L), 0]$

$$i(p) \geq \delta(n, L, A, B) \tag{2.7}$$

for all $p \in B(p_k, L/2)$, where $i(p)$ denotes the injectivity radius at p.

Now we have for g_k upper bounds for $|Rm|$ and positive lower bounds for the injectivity radius on balls of center p_k and radius $L/2$, over the time interval $[-\lambda_k(T_k - T_L), 0]$, where $L > 0$ is arbitrary. Note that these bounds hold uniformly for all g_k for a given L. Moreover, the time interval approaches $(-\infty, 0]$ for each L. Now it is quite easy to apply the basic arguments of Gromov-Cheeger-Hamilton compactness theorem [H6] to obtain a subsequence of $(g_k, M \times (-\lambda_k T_k, 0], p_k)$ which point converges smoothly to a smooth pointed Ricci flow $g_\infty, M_\infty \times (-\infty, 0], p_\infty)$.

The stated κ-noncollapsing property of g_∞ follows from (2.2) by passing to the limit. The Sobolev inequality (2.4) follows from the Sobolev inequalities established in [Y5]. ■

Two important issues are prompted by this general result. On the one hand, Theorem 2.3 allows one to obtain smooth blow-up limits for the Ricci flow at a blow-up time, provided that one can choose λ_k, T_k and p_k such that the needed curvature bounds hold. There is a simple situation in which the needed curvature bounds are obviously valid. This is the situation when we choose p_k such that the $|Rm|$ of g achieves at (p_k, T_k) its maximal value on the domain $M \times [0, T_k]$ (one can also use a suitable subinterval of $[0, T_k]$), and choose $\lambda_k = |Rm|(p_k, T_k)^{-1}$. Of course, this can only yield information about blow-up singularities at maximal curvature points. There can be blow-up singularities in other places, however. Indeed, one needs to understand the structures of the Ricc flow in any region of large curvature. This issue is handled by Perelman in [P1] by using some delicate arguments. His main result in dimension 3 is the canonical neighborhood theorem, which will be presented in Section 6.10, along with a result on obtaining parameters λ_k, T_k and p_k with the properties assumed in Theorem 2.3.

On the other hand, Theorem 2.3 says that the blow-up limits are ancient solutions and κ-noncollapsed on all scales. A natural and important question is when the blow-up limits are κ-solutions, namely when they have bounded nonnegative curvature operator and are non-flat. (Note that the curvature bounds in Theorem 2.3 are not assumed to be uniform with respect to L, so they only lead to bounded curvature at bounded distance, and do not yield directly globally bounded curvature.) The following theorem follows from Perelman's results.

Theorem 2.4 *Assume* $n = 3$ *and*

$$|Rm|(p_k, T_k) = 1 \tag{2.8}$$

for each g_k*. then* g_∞ *has bounded nonnegative curvature operator and is nonflat. Consequently,* g_∞ *is a* κ*-solution.*

This theorem is proved by using the canonical neighborhood theorem and the Hamilton-Ivey pinching, or the Hamilton pinching, see Section 6.10.

3 Asymptotical solitons, the l-function and the reduced volume

As we have seen in the last section, blow-up singularities of the Ricci flow in dimension 3 are modeled by κ-solutions. Hence it is important to analyze structures of κ-solutions. It turns out that to a large degree κ-solutions can be understood in terms of gradient shrinking Ricci solitons.

Definition 3 Let g be a smooth solution of the Ricci flow on $M \times I$ for a smooth manifold M and an interval I. We say that g is a *shrinking Ricci soliton* (or simply *shrinking soliton*) with *time origin* $t_0 \geq \sup I$ on an open subset O of $M \times I$, provided that g satisfies the gradient shrinking soliton equation

$$Ric + \frac{1}{2(t-t_0)} g + \frac{1}{2} L_X f = 0 \tag{3.1}$$

in O for a smooth (time-dependent) vector filed X on O. (Obviously, a translation in time produces from a given shrinking soliton a new shrinking soliton with a different time origin.) We say that g is complete, if $g(t)$ is a complete a metric for some each $t \in I$ (equivalently, for some $t \in I$). If $X = \nabla f$ for a smooth (time-dependent) function f on O, then g is called a *gradient shrinking Ricci soliton* (or simply *gradient shrinking soliton*) and f is called a *potential function* of f. Note that in this case the equation (3.1) becomes

$$Ric + \frac{1}{2(t-t_0)} g + \nabla^2 f = 0. \tag{3.2}$$

A fixed metric g_0 satisfying

$$Ric + \frac{1}{2} L_X g_0 = \lambda g_0 \tag{3.3}$$

for a fixed vector field X and a constant $\lambda > 0$ will be called a *slice shrinking soliton,* with " slice" meaning space slice. (It is also called a shrinking soliton in the literature. Our terminology is for the sake of clarity and convenience.) If $X = \nabla f$, i.e.

$$Ric + \nabla^2 g_0 = \lambda g_0, \tag{3.4}$$

then g_0 will be called a *slice gradient shrinking soliton.* It is easy to see that a slice gradient shrinking soliton generates a gradient shrinking soliton g which equals g_0 at a time determined by λ and t_0. (A similar statement holds true for general slice shrinking solitons on compact manifolds.)

A shrinking soliton g evolves by the pullback of a family of diffeomorphisms coupled with scaling. More precisely, we have

$$g(t) = \frac{t-t_0}{\bar{t}-t_0} \phi^* g(\bar{t}), \tag{3.5}$$

where $\bar{t}$ is an arbitary point in I and ϕ is the solution of the equation

$$\frac{\partial \phi}{\partial t} = -X \tag{3.6}$$

with $\phi(\bar{t}) = id$ (id denotes the identity map of M). (Conversely, any $g(t)$ given this way is a shrinking soliton.) Indeed, we have

$$\frac{\partial g}{\partial t} = -2Ric = \frac{1}{t-t_0} g + L_X g. \tag{3.7}$$

Hence

$$\frac{\partial}{\partial t} \phi^* g = \frac{1}{t-t_0} \phi^* g. \tag{3.8}$$

The equation (3.5) follows.

As will be seen later (Theorem 6.4), κ-solutions will be used to model large curvature regions of solutions of the Ricci flow in dimension 3. To establish this modeling, one needs to understand the asymptotical structures of κ-solutions as $t \to -\infty$. Let g be a κ-solution on a manifold M of dimension $n \geq 2$. To analyze its asymptotical structure as $t \to -\infty$, we blow it down as $t \to -\infty$. The idea of blowing-down is suggested by the Ricci flow equation (1.1). Since g has nonnegative curvature operator, this equation says that g has a non-positive rate of change for forward time, hence $g(t)$ shrinks as t increases. This means the same as saying that $g(t)$ expands as t decreases. To obtain a smooth limit from $g(t)$ near $-\infty$ one then needs to blow it down. For $a > 0$ we consider the rescaled flow $g_a(t) = a^{-1}g(at)$, which is also a κ-solution. Let a_k be a sequence of positive numbers approaching ∞. Then we consider the blow-down flows $g_{a_k}(t)$. In order to extract smooth limits from g_{a_k}, we need geometric estimates for κ-solutions. A basic tool here is the reduced distance, or the l-function of Perelman. In [P1], the l-function is formulated for solutions of the backward Ricci flow

$$\frac{\partial g}{\partial t} = 2Ric. \tag{3.9}$$

By the time reversal $t \to -t$, solutions of the Ricci flow can be converted into solutions of the backward Ricci flow. Hence the theory of the l-function developed in [P1] can be applied. On the other hand, we can formulate the theory of the l-function directly for solutions of the Ricci flow. For simplicity we'll do this in the context of ancient solutions.

Consider an ancient solution $g = g(t)$ of the Ricci flow on M for some manifold M of dimension $n \geq 2$. Thus, for each $t \leq 0$, $g(t)$ is defined and complete. For $t < 0$, we consider Perelman's $\mathcal{L}$-energy for piecewise C^1 curves $\gamma : [t, 0] \to M$,

$$\mathcal{L}(\gamma) = \int_t^0 \sqrt{-\tau}(R(\gamma(\tau), \tau) + |\dot{\gamma}|^2)d\tau, \tag{3.10}$$

where $|\cdot| = |\cdot|_{g(\tau)}$.

Next we choose a reference point $p \in M$ and define $L(q,t) = L_g(q,t)$ to be the infimum of $\mathcal{L}(\gamma)$ for $\gamma : [t, 0] \to M$ with $\gamma(0) = p$ and $\gamma(t) = q$. (We write $L_g(q,t)$ if we need to indicate the dependence on g.)

Definition 4 We define the *reduced distance* or the *l-function* (of Perelman) to be

$$l(q,t) = l_g(q,t) = \frac{L(q,t)}{2\sqrt{-t}}. \tag{3.11}$$

The reference point p will be called the *l-base*.

We have the following geometric estimates for κ-solutions. Let $d(\cdot,\cdot,t)$ denote the distance with respect to $g(t)$.

Theorem 3.1 *There is a positive constant C depending only on the dimension n such that*

$$R \leq \frac{Cl}{|t|} \tag{3.12}$$

everywhere on $M \times (-\infty, 0]$,

$$|\nabla l|^2 \leq \frac{Cl}{|t|} \tag{3.13}$$

almost everywhere in M for each $t \in (-\infty, 0]$,

$$|\sqrt{l}(q_1, \tau) - \sqrt{l}(q_2, t)| \leq \sqrt{\frac{C}{4|t|}} d(q_1, q_2, t) \tag{3.14}$$

for all $t \in (-\infty, 0]$ and all $q_1, q_2 \in M$, and

$$|l_t| \leq \frac{Cl}{|t|} \tag{3.15}$$

almost everywhere in $(-\infty, 0]$ for each $q \in M$. Moreover, we have the following Harnack inequality

$$(\frac{t_1}{t_2})^C \leq \frac{l(q, t_2)}{l(q, t_1)} \leq (\frac{t_2}{t_1})^C \tag{3.16}$$

for all $q \in M$ and $t_1, t_2 \in (-\infty, 0]$ with $|t_2| > |t_1|$.

Proof. This follows from [Theorem 2.18, Y2], which is based on Perelman's estimates in [P1] and the analytic properties of the l-function established in [Y2]. ■

Lemma 3.2 *For each $t \in (-\infty, 0]$, there is a minimum point $p(t)$ of $l(\cdot, t)$. Moreover, there holds $l(p(t), t) \leq n/2$.*

Proof. The upper bound follows from Section 7.1 of [P1], see also [Lemma 3.1, Y2]. The existence of p_t follows from [Lemma 2.3, Y2]. ■

Based on the above geometric estimates we can now extract smooth blow-down limits from a κ solution $g(t)$. The following theorem is formulated in [Y3] as part of [Proposition 11.2, P1].

Theorem 3.3 *Let $g = g(t)$ be a κ-solution on a manifold M of dimension $n \geq 2$ as before. Let $t_k \to -\infty$ be given. For each t_k, let $p(t_k)$ be a minimum point of $l(\cdot, t)$. Then the pointed flows $(g_{|t_k|}, M \times (0, \infty), p(t_k))$ subconverge smoothly to pointed smooth solutions $(g_\infty, M_\infty \times (-\infty, 0), p_\infty)$ of the Ricci flow, which will be called asymptotical limits of g. These limits are κ-noncollapsed on all scales.*

Proof. We reproduce the proof given in [Y3]. The various quantities associated with $g_{|t_k|}$ will be indicated by the subscript k or $g_{|t_k|}$, e.g. $l_k = l_{g_{|t_k|}}$ and $d_k = d_{g_{|g_k|}}$. By Lemma 3.2 and the scaling invariance of the l-function we have

$$l_k(p(t_k), -1) \leq \frac{n}{2}. \tag{3.17}$$

By this estimate and [Lemma 3.2, Y2] we infer

$$l_k(q, -1) \leq C_2 d_k^2(x(\tau_k), q, -1) + n \tag{3.18}$$

for all $q \in M$, where C_2 is a positive constant depending only on the dimension n. Then it follows from the Harnack inequality (3.16) that

$$l_k(q, t) \leq |t|^{\pm C}(C_2(d_k^2(p(t_k), q, -1) + n), \tag{3.19}$$

where $\pm = +$ if $|t| \geq 1$, $\pm = -$ if $|t| < 1$, and C is a positive constant depending only on n. Consequently, we obtain from (3.12) and the nonnegativity of curvature operator the estimate

$$|Rm|_k(q, t) \leq C|t|^{-1\pm C}(d_k^2(p(t_k), q, -1) + 1). \tag{3.20}$$

By the κ-noncollapsing property of g_k we then obtain the desired pointed smooth convergence to pointed solutions $(g_\infty, M_\infty \times (0, \infty), x_\infty)$ of the Ricci flow as in the proof of Theorem 2.3. The κ-noncollapsing property of g_∞ follows from the κ-noncollapsing property of g_k and the smooth convergence. ■

The next stage of the process of understanding κ-solutions is to identify asymptotical limits to be gradient shrinking solitons. A crucial tool for this purpose is the reduced volume of Perelman. For convenience we again formulate it for the Ricci flow rather than the backward Ricci flow as in [P1], and restrict the discussion to ancient solutions.

Definition 5 Let g be an ancient solution of the Ricci flow on a manifold M of dimension $n \geq 2$. Choose a point p as the l-base. We define the *reduced volume* (of Perelman) to be

$$\tilde{V}(t) = \tilde{V}_g(t) = \int_M |t|^{-\frac{n}{2}} e^{-l(q,t)} dvol_{g(t)}. \tag{3.21}$$

It is easy to see that $\tilde{V}$ is invariant under the rescaling $g \to g_a$. The key properties of the reduced volume are its monotonicity, upper bounds, and the associated rigidities. The following results were obtained in [Y2] (in the formulation of the backward Ricci flow), which include Perelman's results on the reduced volume in [P1] as special cases. The basis for these results is Perelman's differential inequality [P1]

$$\frac{d}{dt}(|t|^{-\frac{n}{2}} e^{-l(v,t)} J(t)(v)) \geq 0 \tag{3.22}$$

and the analytic properties of the l-function established in [Y2]. Here v is a tangent vector at p which lies in the injectivity domain of the $\mathcal{L}$-exponential map (the exponential map associated with the $\mathcal{L}$-geodesics), $l(v,t) = l(\gamma_v(t), t)$ with γ_v denoting the $\mathcal{L}$-geodesic determined by the initial tangent vector v, and $J(t)$ denotes the Jacobian of the $\mathcal{L}$-exponential map.

Theorem 3.4 *If the Ricci curvature is bounded from below on $[T, 0]$ for each $T < 0$, then $\tilde{V}(t)$ is a nondecreasing function.*

Theorem 3.5 *Assume that the Ricci curvature is nonnegative for $s \in [t, 0]$. Then $\tilde{V}(t) < (4\pi)^{\frac{n}{2}}$ unless $(M, g(0))$ is isometric to $\mathbf{R}^n$ and $g(s) = g(0)$ for all $s \in [t, 0]$, in which case $\tilde{V}(t) = (4\pi)^{\frac{n}{2}}$.*

The next theorem is formulated in [Y3] as the remaining part of [Proposition 11.2, P1].

Theorem 3.6 *Let $(g_\infty, M_\infty \times (0, \infty), p_\infty)$ be an asymptotical limit of a κ-solution g. Then g_∞ is a nonflat gradient shrinking soliton with time origin 0. Moreover, the limit l-function l_∞ is a potential function. Henceforth asymptotical limits will be called "asymptotical solitons".*

Here the limit l-function l_∞ refers to the limit of the sequence of l-functions $l_{g_{|t_k|}}$ for the associated sequence of rescaled flows $g_{|t_k|}$ (as in Theorem 3.3). The application of the reduced volume in the proof of this theorem as presented in [Y3] is in terms of the *asymptotical reduced volume* $\tilde{V}_\infty$ of g_∞, which is the limit of the reduced volume of $g_{|t_k|}$. By the monotonicity of the reduced volume (Theorem 3.4) and a delicate convergence argument one deduces that $\tilde{V}_\infty(t)$ is independent of time t. On the other hand, one has

$$\tilde{V}_\infty(t_2) - \tilde{V}_\infty(t_1) = -\int_{t_1}^{t_2} \int_{M_\infty} (\frac{\partial l_\infty}{\partial t} + R_\infty + \frac{n}{2t}) e^{-l_\infty} |t|^{-\frac{n}{2}} dq dt \tag{3.23}$$

for $t_1 < t_2$, which follows from another delicate argument about absolute convergence of improper integrals. (The subscript ∞ for various quantities refers to g_∞.) Thus we have

$$\int_{t_1}^{t_2} \int_{M_\infty} (\frac{\partial l_\infty}{\partial t} + R_\infty + \frac{n}{2t}) e^{-l_\infty} |t|^{-\frac{n}{2}} dq d dt = 0. \tag{3.24}$$

Besides this identity, another important tool employed in the proof of Theorem 3.6 is the following lemma.

Lemma 3.7 *The equation*

$$\frac{\partial l_\infty}{\partial t} + \frac{R_\infty}{2} - \frac{|\nabla l_\infty|^2}{2} + \frac{l_\infty}{2t} = 0 \tag{3.25}$$

holds true almost everywhere on $M_\infty \times (0, \infty)$. *The inequality*

$$\Delta l_\infty - \frac{|\nabla l_\infty|^2}{2} + \frac{R_\infty}{2} + \frac{l_\infty - n}{2\tau} \le 0 \tag{3.26}$$

holds true for each $t < 0$ *in the weak sense, i.e.*

$$\int_{M_\infty} \{-\nabla l_\infty \cdot \nabla\phi + \frac{1}{2}(-|\nabla l_\infty|^2 + R_\infty - \frac{l_\infty - n}{t})\phi\} dq \le 0 \tag{3.27}$$

for all nonnegative Lipschitz functions ϕ *with compact support. Finally, the inequality*

$$\frac{\partial l_\infty}{\partial t} + \Delta l_\infty - |\nabla l_\infty|^2 + R_\infty + \frac{n}{2t} \le 0 \tag{3.28}$$

holds true on $M_\infty \times (0, \infty)$ *when* Δ *is interpreted in the weak sense, i.e.*

$$Q_{t_1,t_2}(\phi) \le 0 \tag{3.29}$$

for arbitray $t_1 < t_2 < 0$ *and nonnegative Lipschitz functions* ϕ *on* $M_\infty \times [t_1, t_2]$ *with compact support, where*

$$Q_{t_1,t_2}(\phi) = \int_{t_1}^{t_2}\int_{M_\infty} \{-\nabla l_\infty \cdot \nabla\phi + (\frac{\partial l_\infty}{\partial t} - |\nabla l_\infty|^2 + R_\infty + \frac{n}{2t})\phi\} dq dt. \tag{3.30}$$

This lemma is derived in two stages. First a similar lemma is derived for g_k, based on Perelman's differential inequalities in [P1] and the analytic properties of the l-function established in [Y2]. Then Lemma 3.7 is derived via a convergence argument. Here, a lemma about strong convergence of Sobolev functions established in [Y3] is needed. (see [Y3] for details.) (In [Y3], this lemma is formulated in terms of the variable $\tau = -t$.)

Now we have all the ingredients needed to finish the proof of Theorem 3.6. Note that the identity (3.24) means $Q_{t_1,t_2}(|t|^{-\frac{n}{2}} e^{-l_\infty}) = 0$. Combining this with Lemma 3.7 one infers that the inequalities in Lemma 3.7 become equalities, and the function l_∞ is smooth (as a consequence of parabolic regularity). Then one appeals to the characterization of gradient shrinking solitons in terms of these differential equalities (see [P1] and [Y2]) to conclude that g_∞ is a gradient shrinking soliton with potential functional l_∞. The nonflat property of l_∞ is derived by using the strict upper bound for the reduced volume of g_k provided by Theorem 3.5. We refer to [Y3] for details.

4 Classification of 2-dimensional κ-solutions

As shown in [P1][P2] and will be discussed in Sections 5 and 6, the following result plays an important role in analyzing structures of 3-dimensional κ-solutions and blow-up singularities of the Ricci flow in dimension 3.

Theorem 4.1 *In dimension 2 round spheres are the only orientable κ-solutions.*

This theorem is precisely [Corrolary 11.3, P1]. Its proof given in [P1] is incomplete, since it leaves out the case of noncompact 2-dimensional κ-solutions. Here we reproduce the first complete proof of this result which is presented in the present author's paper [Y4]. (This paper has been available at the author's website and through the website of B. Kleiner and J. Lott and the references in their notes on Perelman's papers on the Ricci flow [KL] since early 2004.)

There is a gradient shrinking Ricci soliton g_* with time origin 0 on S^2 which is given by $g_*(t) = -2tg_{S^2}$. Its potential functions are the constant functions. We can shift its time origin, rescale it by a constant factor, and pull it back by a diffeomorphism. The shrinking Ricci solitons obtained this way will be called *round sphere solitons*. By a *round sphere metric* on a manifold M diffeomorphic to S^2 we mean $\lambda F^* g_{S^2}$, where λ is a positive number and F is a smooth diffeomorphism from M onto S^2.

Lemma 4.2 *Let M be diffeomorphic to S^2 and g a shrinking Ricci soliton on M. Then g is a round sphere soliton.*

Proof. This follows from [Theorem 10.1, H1]. For a different proof based on [BSY], we refer to [Y4]. ∎

Lemma 4.3 *Let g be a smooth solution of the Ricci flow on $M \times (a, b)$ for a 2-dimensional manifold M and some time interval (a, b), such that for each $t \in (a, b)$, the metric $g(t)$ is complete and has nonnegative scalar curvature. Moreover, assume $\frac{\partial R}{\partial t} \geq 0$. Let $t_0 \in (a, b)$. Assume that $g(t_0)$ is κ-noncollapsed on the scale ρ for some $\kappa > 0$ and $\rho > 0$. Then $g(t_0)$ has bounded scalar curvature.*

Proof. The proof is along the lines of arguments in several places in [P1]. Our argument for point picking is more direct. We'll add the notation t explicitly to various quantities to indicate the metric $g(t)$ at time t, e.g. $B(p, r, t)$ is the geodesic ball of center p and radius r with respect to the metric $g(t)$. Assume that $g(t_0)$ has unbounded scalar curvature. Choose a sequence of points $p_k \in M$ such that $R(p_k, t_0) > 0$ for each k and $R(p_k, t_0) \to \infty$. Choose $q_k \in B(p_k, 1, t_0)$ such that the function $d(\cdot, \partial B(p_k, 1, t_0), t_0)^2 R(\cdot, t_0)$ on $B(p_k, 1, t_0)$ achieves its maximum at q_k. We set $r_k = d(q_k, \partial B(p_k, 1, t_0), t_0)/2$. For $q \in B(q_k, r_k, t_0)$ we have $d(q, \partial B(p_k, 1, t_0), t_0) \geq r_k$ and hence the maximum property of q_k implies

$$r_k^2 R(q, t_0) \leq d(q, \partial B(p_k, 1, t_0), t_0)^2 R(q, t_0) \leq 4r_k^2 R(q_k, t_0). \tag{4.1}$$

It follows that

$$R(q, t_0) \leq 4R(q_k, t_0) \tag{4.2}$$

on $B(q_k, r_k, t_0)$. By the maximum property of q_k we also infer $r_k^2 R(q_k, t_0) \geq R(p_k, t_0)/4$, so $r_k > 0$ for each k and $r_k^2 R(q_k, t_0) \to \infty$. On the other hand, it is obvious that $q_k \to \infty$. Now we consider the rescaled flows $g_k(t) = R(q_k, t_0) g(t_0 + R(q_k, t_0)t)$. Set $\rho_k = r_k \sqrt{R(q_k, t_0)}$. There holds $R(q_k, 0) = 1$ and $R(\cdot, 0) \leq 4$ on $B(q_k, \rho_k, 0)$ for g_k. By the property $\frac{\partial R}{\partial t} \geq 0$ we can control the curvature for $t \leq 0$ as well. Combining this with the κ-noncollapsing condition we then obtain from (M, g_k, q_k) a smooth limit $(g_\infty, M_\infty, p_\infty)$. By Splitting Lemma in [Appendix, Y4] or [Appendix G, KL] this limit splits off a line, i.e. is the isometric product of a lower dimensional manifold with **R**. (We only need the splitting at $t = 0$, although it holds for the flow.) Since M_∞ is 2-dimensional, it follows that g_∞ is flat. But the scalar curvature of $R(q_k, t_0) g(t_0)$ at q_k is 1, hence the scalar curvature of $g_\infty(0)$ at q_∞ is also 1. This is a contradiction. ■

Note that by Lemma 2.1 and smooth convergence asymptotical solitons have non-decreasing scalar curvature, i.e. satisfy $\frac{\partial R}{\partial t} \geq 0$.

Lemma 4.4 *Let g be a 2-dimensional asymptotical soliton, or, more generally, a complete nonflat κ-noncollapsed gradient shrinking soliton with non-negative and non-decreasing scalar curvature. Then it is a round sphere soliton.*

Proof. By time translations we may assume that 0 is the time origin of g. Hence g satisfies the soliton equation

$$Ric + \frac{1}{2t} g + \nabla^2 f = 0. \tag{4.3}$$

By Lemma 4.3, g has bounded scalar curvature for each $t < 0$. We also observe that R is everywhere positive. Indeed, if R is zero at some point p and some time t, then the strong maximum principle applied to the evolution equation of R

$$\frac{\partial R}{\partial t} = \Delta R + R^2 \tag{4.4}$$

implies that R is everywhere zero, which contradicts the nonflatness of g.

We claim that M is compact. To prove the claim, fix a point $p_0 \in M$. Following [(1.2), P2] we have, as a consequence of (4.3) the following equation

$$dR = 2Ric(\nabla f, \cdot) = Rg(\nabla f, \cdot) = Rdf. \tag{4.5}$$

Let $\theta(q, t, \gamma)$ denote the (smaller) angle between $\nabla f(q, t)$ and $\gamma'(l)$, where $l = d(p_0, q, t)$ and γ is a unit speed shortest geodesic with respect to $g(t)$ such that $\gamma(0) = p_0$ and $\gamma(l) = q$. This angle is defined to be 2π if $\nabla f(q, t) = 0$. By the arguments in the proof of [Lemma 1.2, P2] in [P2], there is a positive number A_0 such that

$$\theta(q, -1, \gamma) \leq \frac{\pi}{4} \tag{4.6}$$

whenever $d(p_0, q, -1) \geq A_0$.

Let γ be a shortest geodesic from p_0 to a point q with $d(p_0, q, -1) > A_0$. We have by (6.8) and (4.6)

$$\frac{d}{dt} R(\gamma(t), -1) = \nabla R \cdot \gamma'(t) = R \nabla f \cdot \gamma'(t) > 0, \tag{4.7}$$

as long as $t \geq A$. Thus $R(\gamma(t), -1)$ increases along the portion of γ which lies outside of the geodesic ball $B(p_0, A_0, -1)$. Consequently, we obtain the following estimate

$$R(q, -1) \geq \alpha_{A_0} \tag{4.8}$$

for all $q \in M$, where $\alpha_{A_0} = \min\{R(q, -1) : q \in B(p_0, A_0, -1)\}$. Since $R > 0$ everywhere, α_A is positive. By Bonnet theorem, M must be compact.

Now we apply Gauss-Bonnet theorem to infer that M is diffeomorphic to S^2. Then we apply Lemma 4.2 to conclude that (M, g) is a round sphere soliton. ■

Remark This result extends to other classes of shrinking solitons, see [Y4].

Proof of Theorem 4.1

Let (M, g^*) be an orientable 2-dimensional κ-solution. Consider an arbitary asymptotic soliton $(M_\infty, g_\infty, q_\infty)$ of g^* given by Theorem 3.3 and Theorem 3.6. By Lemma 4.4, (M_∞, g_∞) is a round sphere soliton. Consequently, M is diffeomorphic to S^2. Moreover, modulo smooth diffeomorphisms of M, the metrics $\frac{1}{|t|} g^*(t)$ converge smoothly on M to metrics of positive constant scalar curvature as $t \to -\infty$.

We set $g(\tau) = \lambda(t) g^*(t)$ with $\lambda(t) = \exp(\int_{-1}^{t} r^*)$ and $\tau = \int_{-1}^{t} \lambda$, where $r^*(t)$ denotes the average scalar curvature of $g^*(t)$. Then $g = g(\tau)$ satisfies the volume-normalized Ricci flow

$$\frac{\partial g}{\partial \tau} = (r - R) g \tag{4.9}$$

with r denoting the average scalar curvature of g. There holds $\tau \in (\Lambda_1, \Lambda_2]$, where $\Lambda_1 = \lim_{t \to -\infty} \tau(t)$ and $\Lambda_2 = \tau(0)$, Then, modulo smooth diffeomorphisms of M, $g(\tau)$ converges to metrics of positive constant scalar curvature as $\tau \to \Lambda_1$. Following [H1], let f be the solution of $\Delta f = R - r$ with mean value zero, and set $H = \nabla^2 f - \frac{1}{2} \Delta f g$ (H_{ij} is the M_{ij} in [H1]). By [(9.1), H1] we have

$$\frac{\partial |H|^2}{\partial \tau} = \Delta |H|^2 - 2|\nabla H|^2 - 2R|H|^2. \tag{4.10}$$

Hence the maximum principle implies that $\max |H|^2$ is nonincreasing. Since g converges modulo smooth diffeomorphisms of M to metrics of positive constant scalar curvature as $\tau \to \Lambda_1$, we have $H \to 0$ as $\tau \to \Lambda_1$, whence $H \equiv 0$. Now, pulling back

g by a family of diffeomorphisms $\phi(t)$ generated by ∇f with $\phi(-1) = id$, we obtain $\hat{g}$ which satisfies

$$\frac{\partial \hat{g}}{\partial \tau} = 2H = 0. \tag{4.11}$$

Thus $\hat{g}$ is independent of time. Since its scalar curvature approaches a positive constant as $\tau \to \Lambda_1$, we infer that $\hat{g}$ has positive constant scalar curvature. It follows that g and hence g^* has positive constant scalar curvature. We conclude that g^* is a round sphere κ-solution. Note that we have $f \equiv 0$ because g has constant scalar curvature. Consequently, $g = \hat{g}$. We deduce that $g^*(t) = \lambda(t)\hat{g}$ for positive scalars $\lambda(t)$. This immediately yields $g^*(t) = (1 - \frac{2}{n}R_0 t)g^*(0)$, where R_0 denotes the scalar curvature of $g^*(0)$. ∎

5 Classification of 3-dimensional κ-solutions and Perelman's compactness theorem

In [P2], Perelman obtained the classification of 3-dimensional κ-solutions based on a classification of 3-dimensional gradient shrinking solitons. One ingredient in this classifications is the classification of 2-dimensional κ-solutions and gradient shrinking solitons presented in the last section. Perelman's classification of 3-dimensional gradient shrinking solitons in [P2] is as follows.

Theorem 5.1 *Let g be a complete, κ-noncollapsed gradient shrinking soliton with bounded nonnegative sectional curavture on a 3-dimensional manifold M. Then the universal cover of (M, g) is isometric to $\mathbf{R}^3, S^3$ or $S^2 \times \mathbf{R}$, where S^2 is equipped with a constant multiple of the round sphere metric of sectional curvature 1.*

The key ingredient for establishing this theorem is Lemma 1.2 in [P2] which we state below as a theorem.

Theorem 5.2 *There is no 3-dimensional complete, noncompact and nonflat gradient shrinking soliton which is κ-noncollapsed and has bounded positive sectional curvature.*

One argument in the proof of this theorem in [P2] uses Theorem 4.1. (The above proof of Lemma 4.4 offers some aspect of this argument.) Another argument is in terms of the level surfaces of the potential function of the soliton. We refer to [P2] for details.

Proof of Theorem 5.1 By Theorem 5.2, one is left to handle a compact universal cover or a noncompact universal cover whose sectional curvature is not strictly positive everywhere. In the former case, the Ricci curvature must be positive. Otherwise, Hamilton's strong maximum principle [H2] implies a splitting, i.e. the universal cover $(\tilde{M}, \tilde{g})$ is the isometric product of a 1-dimensional factor and a 2-dimensional factor. This is impossible because the only possible 1-dimensional factor is S^1, which is excluded by the simple connectedness. Then the conclusion follows from Hamilton's theorem on the Ricci flow on compact 3-manifolds of positive Ricci curvature in [H1]. In the latter case, by Hamilton's strong maximum principle the universal cover $(\tilde{M}, \tilde{g})$ splits off a line. If the 2-dimensional factor is flat, then it must be $\mathbf{R}^2$. It follows that $(\tilde{M}, \tilde{g})$ is isometric to $\mathbf{R}^3$. If the 2-dimensional factor is nonflat, then it is a round sphere soliton by Lemma 4.4 and Lemma 2.1. Thus $(\tilde{M}, \tilde{g})$ is isometric to $S^2 \times \mathbf{R}$. ■

Recently, Perelman's classification was extended by L. Ni-N. Wallach [NiW] and A. Naber [Na] to 3-dimensional complete shrinking solitons of bounded nonnegative Ricci curvature by using different methods. Ni and Wallach used an evolution equation of curvature quantities associated with the Ricci flow and an integration argument, while Naber used the l-function, the reduced volume and the potential function. (Ni-Wallach only treated gradient shrinking solitons. But they allow unbounded curvature which satisfies a certain growth condition.) We state their result in the following theorem.

Theorem 5.3 *Let g be a complete shrinking soliton of bounded nonnegative Ricci curvature on a 3-dimensional manifold M. Then the universal cover of (M, g) is isometric to $\mathbf{R}^3, S^3$ or $S^2 \times \mathbf{R}$.*

Based on the 3-dimensional classification of gradient shrinking solitons and the results on asymptotical solitons (namely Theorem 3.3 and Theorem 3.6) Perelman obtained the following classification of 3-dimensional κ-solutions.

Theorem 5.4 *Let (M, g) be a 3-dimensional κ-solution. Then it falls into one of the following three mutually exclusive cases:*
1. It has an asymptotical soliton which is isometric to a metric quotient of the round 3-dimensional sphere. In this case, (M, g) is isometric to a metric quotient of the round 3-dimensional sphere. (This case is the same as the case that (M, g) has a compact aymptotical soliton.)
2. It has an asymptotical soliton which is isometric to the nontrivial $\mathbf{Z}_2$ quotient of $S^2 \times \mathbf{R}$. In this case, a nontrivial isometric $\mathbf{Z}_2$ cover of (M, g) has an asymptotical soliton which is isometric to the round cylinder $S^2 \times \mathbf{R}$. (This case is the same as the case that (M, g) has an asymptotical soliton which contains the one-sided projective plane.)
3. It has an asymptotical soliton which is isometric to the round cylinder $S^2 \times \mathbf{R}$. In this case, M can be noncompact or compact.
Finally, if (M, g) is compact, but has a noncompact asymptotical soliton, then M is diffeomorphic to S^3 or $\mathbf{RP}^3$.

Proof. The three types of asymptotical solitons follow from Theorem 5.1. In Case 1, M is obviously compact. By Theorem 3.3 and Theorem 3.6, there is a sequence of times $t_k \to \infty$, such that $|t_k|^{-1}g(t_k)$ approaches a metric of constant positive sectional curvature. By Hamilton's results in [H1] the Ricci curvature is positive for all $t \leq t_k$. Moreover, the pinching of the Ricci curvature improves along g in the forward time direction. Hence the Ricci curvature is no less pinched at $t \geq t_k$ than at t_k. For a fixed t we let $k \to \infty$ and conclude that the Ricci curvature is zero pinched at t. It follows that (M, g) is isometric to a metric quotient of the round 3-sphere.

The last statement does not seem to be in Perelman's papers. It is [Lemma 59.3, KL]. We refer to [KL] for its proof. ■

Examples in Case 3 include the round cylinder itself and Bryant soliton which is a steady Ricci soliton on $\mathbf{R}^3$. In [1.4, P2] Perelman presented a construction of compact examples. More detailed classifications of κ-solutions are certainly desirable, but Theorem 5.4 is sufficient for the purposes in [P1], [P2] and [P3].

Next we would like to present Perelman's compactness theorem for the space of 3-dimensional noncompact κ-solutions. This result is very useful for deriving geometric estimates for κ-solutions themselves and general solutions of the Ricci flow in dimension 3. Before stating the compactness theorem, we state Perelman's result on the asymptotical volume ratio, which is a tool for proving the compactness theorem. Let (M, g) be an n-dimensional noncompact, complete Riemannian manifold of nonnegative Ricci curvature and $p \in M$. By Bishop-Gromov volume comparison, the ratio function $vol(B(p, r))/r^n$ is nonincreasing in r. Let $\mathcal{V} = \mathcal{V}(M, g)$ denote its limit as $t \to \infty$. It is independent of p and called the *asymptotical volume ratio* of (M, g).

Theorem 5.5 *Let (M, g) be a κ-solution of dimension $n \geq 3$. Then $\mathcal{V}(M, g(t)) = 0$ for each t.*

The proof of this theorem given in [P1] is by induction on the dimension. It uses the concept of asymptotical scalar curvature ratio, rescaling limits and splitting arguments. We refer to [P1], [KL] and [MT] for details.

Theorem 5.6 *The space of 3-dimensional noncompact κ-solutions is compact modulo scaling. More precisely, let (M_k, g_k) be a sequence of 3-dimensional noncompact κ-solutions, and let p_k be a point of M_k for each k. Let λ_k denote the scalar curvature of $g_k(0)$ at p_k. (By the strong maximum principle, $\lambda_k > 0$.) Then a subsequence of the pointed κ-solutions $(M_k, \bar{g}_k, p_k)$ with $\bar{g}_k = \lambda_k g_k$ point converges smoothly to a κ-solution $(M_\infty, \bar{g}_\infty, p_\infty)$.*

Proof. It suffices to consider orientable κ-solutions, because we can pass to the orientable double cover for an unorientable κ-solution. Note that $R(p_k, 0) = 1$ for $\bar{g}_k$. In the first part of the proof given in [P1] one shows that for each given (finite) radius L the scalar curvature R is bounded on $B(p_k, L, 0)$. This is done by a contradiction argument involving Theorem 5.5. We refer to [P1], [KL] and [MT] for details.

Given the result of bounded scalar curvature at bounded distances, we also have bounded curvature operator at bounded distance because the curvature operator is nonnegative. Hence we can argue as in Theorem 2.3 to obtain a pointed limit $(M_\infty, \bar{g}_\infty, p_\infty)$ from a subsequence of (M_k, g_k, p_k). This limit is κ-noncollapsed on all scales and has nonnegative curvature operator. Moreover, it satisfies $\frac{\partial R}{\partial t} \geq 0$. To obtain the desired κ-solution property, it remains to show that it has bounded scalar curvature at $t = 0$. Assume the contrary. Then we can find a sequence of points p_k in M_∞ going to ∞, such that $R(p_k, 0) \to \infty$. Then we choose points q_k as in the proof of Lemma 4.3. Now we set $\lambda_k = R(q_k, 0)$ and consider the rescaled flows $g_{\infty,k}(t) = \lambda_k g_\infty(\lambda_k^{-1} t)$. Then we have $R(q_k, 0) = 1$ and $R(\cdot, 0) \leq 4$ on $B(q_k, \rho_k, 0)$ for g_k, where $\rho_k \to \infty$. By the property $\frac{\partial R}{\partial t} \geq 0$ we can control R for $t < 0$ as well. Combining this with the κ-noncollapsing property we can then obtain a smooth pointed limit flow $(M_\infty^\infty, g_{\infty,\infty})$, which is obviously a κ-solution. By the splitting argument alluded to in the proof of Lemma 4.3, this limits splits off a line. By Theorem 4.1, the 2-dimensional factor must be a round sphere. It follows that this limit is the round cylinder $S^2 \times \mathbf{R}$. Consequently, q_k is contained in a neck-like region Z_k which approaches the round cylinder after the dilation by the factor λ_k, which approaches ∞. (So Z_k is getting thinner and thinner, while getting longer and longer relative to the cross size.) This is impossible in a noncompact, complete manifold of nonnegative sectional curvature, see e.g. [46.1, KL]. (A result in [S] is used here.) ∎

6 Achieving bounded curvature and the canonical neighborhood theorem

As a preparation for the main content of this seciton, we first present the concept of Hamilton-Ivey pinching and Hamilton pinching, which is a key property of the Ricci flow in dimension 3. The Hamilton pinching condition is an improvement of the Hamilton-Ivey pinching condition and contains additional useful information for large time. For bounded time, it suffices to use the Hamilton-Ivey pinching condition.

Defintion 6 Let M be a manifold of dimension 3.
A. Let g be a smooth metric M. Let $\nu(p)$ denote the smallest eigenvalue of the curvature operator of g at p. We say that g satisfies the Hamilton-Ivey pinching condition, if $R(p) \geq -1$, and

$$f^{-1}(R) \geq -\nu(p), \tag{6.1}$$

for all $p \in M$, where f denotes the function $x \ln x - x$ defined for $x \geq 1$.
B. Let g be a smooth metric on M and t a nonnegative number. We say that g satisfies the Hamilton pinching condition at time t, if there holds at each p

$$R(p) \geq -\frac{3}{1+t} \tag{6.2}$$

and one of the following two inequalities holds true
1) $\nu(p) \geq 0$,
2)

$$R(p) \geq |\nu(p)| \left(\ln |\nu(p)| + \ln(1+t) - 3\right). \tag{6.3}$$

The first theorem below is due to Hamilton [H5] and Ivey [I]. The second is due to Hamilton [H5].

Theorem 6.1 *The Ricci flow preserves the Hamilton-Ivey pinching condition on closed 3-manifolds. It also preserves the Hamilton-Ivey pinching condition among complete metrics with bounded sectional curvatures on noncompact 3-manifolds.*

Theorem 6.2 *The Ricci flow preserves the Hamilton pinching condition on closed 3-manifolds. More precisely, let g_0 be a metric on a closed manifold M satisfying the Hamilton time-t_0 pinching condition. Let $g = g(t)$ be a smooth solution of the Ricci flow on $M \times [t_0, T)$ for some $T > t_0$ with $g(0) = g_0$. Then $g = g(t), t \in [t_0, T)$ satisfies the Hamilton pinching condition.*

The Ricci flow also preserves the Hamilton pinching condition among complete metrics with bounded sectional curvatures on noncompact 3-manifolds.

These two results are proved by employing the maximum principle. They depend in a strong way on the special structures of the curvature operator in dimension 3. The present author is not aware of any counterexample in higher dimensions. So one may wonder to what extent they can be extended to higher dimensions. The following simple lemma demonstrates why Hamilton-Ivey pinching and Hamilton pinching are so useful.

Lemma 6.3 *For each k let g_k be a smooth metric on a 3-manifold M_k satisfying the Hamilton-Ivey pinching condition. Let $\lambda_k \to \infty$ and $p_k \in M_k$. Assume that $(\lambda_k g_k, M_k, p_k)$ point converge smoothly to a pointed Riemannian manifold $(M_\infty, g_\infty, p_\infty)$. Then g has nonnegative curvature operator. We have the same conclusion if the Hamilton-Ivey pinching is replaced by the Hamilton pinching at a fixed t.*

Proof. We treat the case of the Hamilton-Ivey pinching. The case of the Hamilton pinching is similar. Note that the pinching condition (6.1) means

$$R \geq |\nu|(\ln|\nu| - 1) \tag{6.4}$$

when $\nu \leq -1$. So the ratio of R over $|\nu|$ for g_k goes to ∞ when ν is negative and $|\nu|$ goes to ∞ along g_k and some points. After scaling g_k by the factor λ_k, and hence scaling its curvature quantities by λ_k^{-1}, R converges and hence is bounded by the assumption. Consequently, the rescaled $|\nu|$ must go to zero because it is scaled by the same factor as R. On the other hand, if ν is negative but does not go to ∞, then the rescaled ν must go to zero because $\lambda_k \to \infty$. In the remaining case, ν is nonnegative, then the limit of the rescaled ν is obviously nonnegative. These three cases may happen along different subsequences, and this argument is applied to each situation. ■

Next we present Perelman's theorem on canonical neighborhoods of solutions of the Ricci flow in dimension 3. It is based on two results, one is Perelman's theorem of modeling large curvature regions of the Ricci flow by κ-solutions. The other is Perelman's theorem on the canonical neighborhood property of κ-solutions. The former is [Theorem 12.1, P1], with a slight modification given in [Theorem 52.3, KL], the latter is a result presented in Section 1.5 of [P2]. We state them below in this order. (In the statements of Theorem 6.4 we make a simplification by dropping the T which appears in [Theorem 12.1, P1] and [Theorem 52.3, KL].)

Theorem 6.4 *Given $\epsilon > 0$, $\kappa > 0$ and $\rho > 0$, one can find $r_0 > 0$ with the following property. If $g = g(t), 0 \le t \le t_0$ with $t_0 \ge 1$ is a smooth solution of the Ricci flow on a closed 3-manifold M, which satisfies the Hamilton-Ivey pinching condition (or the Hamilton pinching condition) and is κ-noncollapsed on scales $< \rho$, then for any point p_0 with $Q = R(p_0, t_0) \ge r_0^{-2}$, the solution in $\{(p, t) : d^2(p, p_0, t_0) < (\epsilon Q)^{-1}, t_0 - (\epsilon Q)^{-1} \le t \le t_0\}$, is, after scaling by the factor Q, ϵ-close to the corresponding subset of some κ-solution.*

This theorem is proved by an intricate contradiction argument involving delicate limiting arguments. Here two metrics are said to be ϵ-close if the C^{ϵ}-norm (measured with respect to one of the two metrics) of the difference of the two metrics does not exceed ϵ. (For two metrics on two different manifolds this involves pulling back of metrics.) Note that e.g. $C^{3.14}$ means the Hölder space $C^{3,0.14}$. Perelman's arguments for proving this theorem extend to noncompact, complete manifolds with an assumption on curvature bounds, see e.g. [KL]. It is for simplicity of statements that this theorem is only formulated for closed manifolds. For convenience, we state the implication of Theorem 6.4 for the case $0 < t_0 < 1$ as a theorem.

Theorem 6.5 *Given $\epsilon > 0$, $\kappa > 0$ and $\rho > 0$, one can find $r_0 > 0$ with the following property. If $g = g(t), 0 \le t \le t_0$ with $0 < t_0 \le 1$ is a smooth solution of the Ricci flow on a closed 3-manifold M, which satisfies the Hamilton-Ivey pinching condition (or the Hamilton pinching condition) and is κ-noncollapsed on scales $< \rho\sqrt{t_0}$, then for any point p_0 with $Q = R(p_0, t_0) \ge t_0^{-1} r_0^{-2}$, the solution in $\{(p, t) : d^2(p, p_0, t_0) < (\epsilon Q)^{-1}, t_0 - (\epsilon Q)^{-1} \le t \le t_0\}$, is, after scaling by the factor Q, ϵ-close to the corresponding subset of some κ-solution.*

Proof. Consider the rescaled solution $\bar{g} = t_0^{-1} g(t_0 t)$. Applying Theorem 6.4 and scaling the result back to g. ■

Theorem 6.6 Part A. *(the universal κ property) There is $\kappa_0 > 0$ such that every 3-dimensional κ-solution is either a κ_0-solution or a metric quotient of the round sphere.*
Part B. *(the universal derivative estimates) There is a universal constant $\eta > 0$, such that each 3-dimensional κ-solution satisfies*

$$|\nabla R| < \eta R^{\frac{3}{2}}, |R_t| < \eta R^2 \tag{6.5}$$

everywhere.
Part C. *(the canonical neighborhood property) There is a positive constant ϵ_0 with the following property. For each $0 < \epsilon \leq \epsilon_0$ one can find $C_1 = C_1(\epsilon) > 0$ and $C_2 = C_2(\epsilon) > 0$ such that for each point (p, t) in every 3-dimensional κ-solution there is a radius $r, 0 < r < C_1 R(p, t)^{-\frac{1}{2}}$, and a neighborhood B, $B(p, r, t) \subset B \subset B(p, 2r, t)$ ($B(p, r, t)$ is the open geodesic ball of radius r with respect to $g(t)$), which falls into one of the four categories:*
(a) B is the slice of a strong ϵ-neck at its maximal time,
(b) B is an ϵ-cap,
(c) B is a closed manifolds diffeomorphic to $\mathbf{S}^3$ or $\mathbf{RP}^3$, or
(d) B is a closed manifold of constant positive sectional curvature.
Furthermore, the scalar curvature in B at time t is between $C_2^{-1} R(p, t)$ and $C_2 R(p, t)$, its volume in cases (a), (b) or (c) is greater than $C_2^{-1} R(p, t)^{-\frac{3}{2}}$, and in case (c) the sectional curvature in B at time t is greater than $C_2^{-1} R(p, t)$.

This theorem is a consequence of the result on asymptotical solitons of κ-solutions (Theorem 3.3 and Theorem 3.6), the classification result Theorem 5.4 and the compactness theorem (Theorem 5.6. Here, ϵ-caps and strong ϵ-necks are defined as follows. Consider a smooth family of metrics $g = g(t)$ on a 3-dimensional manifold, e.g. a smooth solution of the Ricci flow. Let Ω be a domain in M. The domain with metric $(\Omega, g(t))$ for some t is called an ϵ-neck, if, after scaling the metric by the factor r^{-2}, it is ϵ-close to the standard neck $S^2 \times (-\epsilon^{-1}, \epsilon^{-1})$, where S^2 is equipped with the round sphere metric of sectional curvature 1. Let Ω be a domain diffeomorphic to the open ball $\mathbf{B}^3$ in $\mathbf{R}^3$ or diffeomorphic to $\mathbf{RP}^3$ minus a closed 3-ball in the interior. The domain with metric $(\Omega, g(t))$ for some t is called an ϵ-cap, if each point outsides some compact subset is contained in an ϵ-neck, and the scalar curvature stays bounded on the end of Ω. Next consider a parabolic neighborhood $P(p, r, t, \Delta t)$, which is the set of all points (q, s) with $q \in B(p, r, t)$ and $s \in [t, t + \Delta t]$ or $s \in [t + \Delta t, t]$, depending on the sign of Δt. A parabolic neighborhood $P(p, \epsilon^{-1} r, t, r^2)$ (with the metric $g(s), t \leq s \leq t + r^2$) is called a strong ϵ-neck, if, after scaling by the factor r^{-2} and a time shift, it is ϵ-close to the evolving standard neck $S^2 \times (-\epsilon^{-1}, \epsilon^{-1})$ on the time interval $[-1, 0]$, which is part of a shrinking soliton isometric to $S^2 \times \mathbf{R}$. Here we also assume that the scalar curvature of the evolving standard neck equals $(1 - s)^{-1}$ for each $s \in [-1, 0]$.

Definition 7 The above neighborhood B with the properties described in Part C and the property (6.5) is called a "canonical neighborhood" of (p, t), or more precisely, an $(\epsilon, C_1, C_2, \eta)$-canonical neighborhood of size r.

Thus Part B and Part C of Theorem 6.6 can be restated as follows. Some authors call this result the canonical neighborhood theorem. In our terminology, the canonical neighborhood theorem refers to Theorem 6.8 below.

Theorem 6.7 *(the canonical neighborhood property of κ-solutions) There is a positive number ϵ_0 with the following property. For each $0 < \epsilon \leq \epsilon_0$, every point (p, t) in every 3-dimensional κ-solution has an $(\epsilon, C_1, C_2, \eta)$-canonical neighborhood of size r, where C_1 and C_2 depend on ϵ and $0 < r < C_1 R(p, t)^{-\frac{1}{2}}$.*

The number ϵ_0 will be used in the results below. Combining these two theorems one arrives at the canonical neighborhood theorem of Perelman, cf. the first paragraph in Section 3 of [P2]. This theorem and the analysis around it play a central role in Perelman's work on the Ricci flow, the Poincaré conjecture and the geometrization conjecture. Again, we state only the case of closed manifolds for the sake of simplicity.

Theorem 6.8 *(the canonical neighborhood theorem) Given $0 < \epsilon \leq \epsilon_0$, $\kappa > 0$ and $\rho > 0$, one can find $r_0 > 0$ with the following property. If $g = g(t), 0 \leq t \leq r_0$ with $t_0 \geq 1$ is a smooth solution of the Ricci flow on a closed 3-manifold M, which satisfies the Hamilton-Ivey pinching condition (or the Hamilton pinching condition) and is κ-noncollapsed on scales $< \rho$, then any point (p_0, t_0) with $R(p_0, t_0) \geq r_0^{-2}$ has an $(\epsilon, C_1, C_2, 2\eta)$-canonical neighborhood of size r, where C_1 and C_2 depend on ϵ and $0 < r < C_1 R(p_0, t_0)^{-\frac{1}{2}}$. We'll say that g has the canonical neighborhood property, or g has the (ϵ, C_1, C_2)-canonical neighborhood property.*

As before, we state the implication for the case $0 < t_0 \leq 1$ as a theorem.

Theorem 6.9 *Given $0 < \epsilon \leq \epsilon_0$, $\kappa > 0$ and $\rho > 0$, one can find $r_0 > 0$ with the following property. If $g = g(t), 0 \leq t \leq t_0$ with $0 < t_0 \leq 1$ is a smooth solution of the Ricci flow on a closed 3-manifold M, which satisfies the Hamilton-Ivey pinching condition (or the Hamilton pinching condition) and is κ-noncollapsed on scales $< \rho\sqrt{t_0}$, then any point (p_0, t_0) with $R(p_0, t_0) \geq t_0^{-1} r_0^{-2}$ has an $(\epsilon, C_1, C_2, 2\eta)$-canonical neighborhood of size r, where C_1 and C_2 depend on ϵ and $0 < r < C_1 R(p_0, t_0)^{-\frac{1}{2}}$. We'll say that g has the canonical neighborhood property, or g has the (ϵ, C_1, C_2)-canonical neighborhood property.*

Now we employ the canonical neighborhood theorem to prove Theorem 2.4.

Proof of Theorem 2.4 Consider a blow-up limit flow g_∞ of a smooth solution g of the Ricci flow as given in Theorem 2.3. By Theorem 6.1 (or Theorem 6.2) and Lemma

6.3, g_∞ has nonnegative curvature operator. By the curvature assumption (2.8) we have $|Rm|(p_\infty, 0) = 1$ for g_∞. By Theorem 2.3, g_∞ is κ-noncollapsed on all scales. It remains to show that g_∞ has bounded curvature at every time. But the condition $\frac{\partial R}{\partial t} \geq 0$ is not available, in contrast to the situation in the proof of Theorem 5.6 (or Theorem 4.3). So we have to appeal to a different argument.

By Theorem 6.8 or Theorem 6.9, g has the canonical neighborhood property. As a blow-up limit of g, g_∞ also has the canonical neighborhood property, as is easy to see. Now we assume that for some t_0, $g_\infty(t_0)$ does not have bounded curvature. Then M_∞ is noncompact and we can find sequence p_k such that $R(p_k, t_0) \to$ for g_∞. Then each p_k has a canonical neighborhood. The cases (c) and (d) of the canonical neighborhood are excluded. In the remaining two cases we can find an ϵ-neck inside of the canonical neighborhood. Hence we can find a sequence of ϵ-necks Z_k going off to ∞, with R on Z_k going to ∞ as $k \to \infty$. This can be ruled out as in the proof of Theorem 5.6.

Note that in the above contradiction argument we can assume positive curvature operator for g_∞. Indeed, if the curvature operator of g_∞ has a zero eigenvector at some point. Then the strong maximum principle of Hamilton [H2] implies that (M_∞, g_∞) splits locally as an isometric product of a 2-dimensional Ricci flow and the 1-dimensional Ricci flow. Passing to a double cover if needed, we can assume that this splitting is global. Then the 2-dimensional factor is a κ-solution. By Theorem 4.1 it must be the round sphere. It follows that g_∞ has bounded curvature. ■

Finally, we present Perelman's theorem on bounded curvature at bounded distance.

Theorem 6.10 *(bounded curvature at bounded distance, cf. [Theorem 10.2, MT]) Fix $0 < \epsilon \leq \epsilon_0, C_1 > 0$ and $C_2 > 0$. For each number $\rho > 0$ there are numbers $K > 0$ and $\kappa > 0$ with the following property. Let $g = g(t)$ be a smooth solution of the Ricci flow on $M \times (a, b)$ for a 3-dimensional manifold M and $0 \leq a < b$, which satisfies the Hamilton-Ivey pinching or the Hamilton pinching condition. Let $p_0 \in M$ and $t_0 \in (a, b)$ such that $R(p_0, t_0) \geq \kappa$. Assume that every point (p, t) with $R(p, t) \geq 4R(p_0, t_0)$ and $t \leq t_0$ has an $(\epsilon, C_1, C_2, 2\eta)$-canonical neighborhood. Then we have $R(p, t_0) \leq KR(p_0, t_0)$ for all $p \in B(p_0, \rho R(p_0, t_0)^{-1/2}, t_0)$.*

For its proof we refer to [MT]. This result enables one to find suitable parameters $\bar{T}_k, p_k$ and λ_k for obtaining blow-up limits in Theorem 2.3, as formulated in the following theorem.

Theorem 6.11 *(achieving bounded curvature for rescaled flows) Let $g = g(t)$ be a smooth solution of the Ricci flow on $M \times [0, T)$ for a closed manifold M of dimension 3 and some finite $T > 0$. Let $p_0 \in M$.*
Part A $\limsup_{t \to T} |Rm|(p_0, t) = \infty$ *if and only if* $\lim_{t \to T} R(p, t) = \infty$.

Part B *Assume* $\limsup_{t\to T} |Rm|(p_0,t) = \infty$. *Then we can choose* $T_k \to T$, *and set* $\lambda_k = |Rm|(p_0,T_k)$ *(or* $\lambda_k = R(p_0,T_k)$*) and* $p_k = p_0$ *for each* k. *Then the scaling factors* λ_k, *scaling time centers* T_k *and reference points* p_k *have the properties assumed in Theorem 2.3. Consequently, the rescaled flows* $(g_{\lambda_k,T_k}, M\times(-\lambda_k T_k,0],p_0)$ *point converge smoothly to a* κ*-solution.*
Part C *Set* $\Omega = \{p\in M : \sup_{0\le t<T} |Rm|(p,t) < \infty\}$. *Then* Ω *is an open subset of* M.

Proof. **Part A** We only need to handle the direction from $|Rm|$ to R, because the other direction is obvious. Assume $\limsup_{t\to T} |Rm|(p,t) = \infty$. Then we have $|Rm|(p,t_k)\to\infty$ for a sequence $t_k\to T$. This is equivalent to the statement

$$|\lambda| + |\mu| + |\nu| \to \infty \tag{6.6}$$

along (p,t_k), where $\lambda\ge\mu\ge\nu$ are the eigenvalues of the curvature operator. If $\nu\ge -C$ for a finite number C along (p,t_k), then we have

$$R = 2(\lambda+\mu+\nu) = 2(|\lambda+C| + |\mu+C| + |\nu+C| - 3C) \ge 2(|\lambda|+|\mu|+|\nu| - 6C)$$

and hence goes to ∞. If $\nu\to-\infty$ along (p,t_k), then we have (6.4) and hence R also goes to ∞. We can apply this argument to subsequences of (p,t_k), and conclude that $R\to\infty$ along (p,t_k).

By Theorem 2.2, Theorem 6.8 and Theorem 6.9, g has the canonical neighborhood property. It follows that there is a positive number r_0 such that each (p,t) has a canonical neighborhood and hence satisfies

$$|R_t|(p,t) < 2\eta R(p,t)^2, \text{ i.e. } |\frac{d}{dt}R^{-1}|(p,t) < 2\eta \tag{6.7}$$

and

$$|\nabla R|(p,t) < 2\eta R(p,t)^{\frac{3}{2}}, \text{ i.e. } |\nabla R^{-\frac{1}{2}}| < \eta, \tag{6.8}$$

whenever

$$R(p,t) \ge (\max\{t,1\})^{-1} r_0^{-2}. \tag{6.9}$$

Integrating (6.7) leads to

$$R(p,\tau_2) > \frac{R(p,\tau_1)}{1+2\eta|\tau_2-\tau_1|R(p,\tau_1)} \tag{6.10}$$

whenever (6.9) holds true on the interval $[\tau_1,\tau_2]$ or $[\tau_2,\tau_1]$. We apply (6.10) to $p=p_0$ and $\tau_1 = t_k$. Since $T-t_k\to 0$, we can use a continuity argument to infer that

$\lim_{t \to T} R(p_0, t) = \infty$.

Part B By Theorem 6.8, Theorem 6.9 and Theorem 6.10, the following curvature bound holds true for g: *For each number $L > 0$ there are numbers $K_L > 0$ and $\kappa_L > 0$ such that $R(p, t_0) \le K_L R(p_0, t_0)$ for all $p \in B(p_0, LR(p_0, t_0)^{-1/2}, t_0)$ whenever $R(p_0, t_0) \ge \kappa_L$.*

Now assume $\limsup_{t \to T} |Rm|(p_0, t) = \infty$. Then we can choose $T_k \to T$ such that

$$|Rm|(p_0, T_k) = \sup_{0 \le t \le T_k} |Rm|(p_0, t). \tag{6.11}$$

For each $L > 0$, let K_L and κ_L be given by the above curvature bound statement. By Part A we have $\lim_{t \to T} R(p, t) = \infty$. Hence there is a time $T(L)$ such that $R(p_0, t) \ge K$ for all $T(L) \le t < T$. Applying the above curvature bound for each $T(L) \le t_0 < T_k$ (with $T_k \ge T(L)$) and (6.11) we then deduce for $g_{\lambda_k, T_k}(t) = \lambda_k g(T_k + \lambda_k^{-1} t)$ (with $\lambda_k = |Rm|(p_0, T_k)$)

$$R(p, t) \le K_L \tag{6.12}$$

for all $-\lambda_k(T_k - T(L)) \le t \le 0$ and $p \in B(p_0, L, t)$. By the Hamilton-Ivey pinching or the Hamilton pinching property (cf. the above arguments utilizing these properties) we then deduce for g_{λ_k, T_k}

$$|Rm|(p, t) \le C K_L \tag{6.13}$$

for all $-\lambda_k(T_k - T(L)) \le t \le 0$ and $p \in B(p_0, L, t)$, where $C > 0$ is universal. This is exactly the property needed in Theorem 2.3.

Part C Let p_0 be the limit of a sequence of points $p_k \in M - \Omega$. We claim that $p_0 \in M - \Omega$. Let $A > 8\eta r_0^{-2}$ be given. Since $R(p_k, t) \to \infty$ as $t \to T$, We can find times $0 < t_k < T$ such that $R(p_k, t_k) \ge A(2\eta)^{-1} > 2r_0^{-1}$ and $t_k \to T$, where $r_0 > 0$ is given in Part A of the proof. For convenience, we can assume $T > 1$ by using a rescaling if necessary. Then we can assume $t_k > 1$ for all k. Now we apply (6.10) to infer

$$R(p_k, \tau) > \frac{R(p_k, t_k)}{1 + 2\eta|\tau - t_k| R(p_k, t_k)} \tag{6.14}$$

whenever

$$R(p_k, t) \ge r_0^{-2} \tag{6.15}$$

for $t \in [\tau, t_k]$. By a continuity argument we then deduce for $\tau_k = t_k - A^{-1}$, since $2\eta A^{-1} R(p_k, t_k) \ge 1$,

$$R(p_k, \tau_k) > \frac{R(p_k, t_k)}{1 + 2\eta A^{-1} R(p_k, t_k)} \ge \frac{R(p_k, t_k)}{4\eta A^{-1} R(p_k, t_k)} = \frac{A}{4\eta}, \tag{6.16}$$

which is greater than $2r_0^{-2}$. (Indeed, we obtain $R(p_k, t) \geq A(4\eta)^{-1}$ for all $t \in [\tau_k, t_k]$.) Now we note that $\tau_k \in [0, T - A^{-1}]$, on which g is smooth. Next let γ_k be a unit speed shortest geodesic of $g(\tau_k)$ from p_k to p_0. We integrate (6.8) to deduce

$$\begin{aligned} \sqrt{R(\gamma_k(s), \tau_k)} &> \frac{\sqrt{R(p_k, \tau_k)}}{1 + \eta d(\gamma_k(s), p_k, \tau_k)\sqrt{R(p_k, \tau_k)}} \geq \frac{\frac{A}{4\eta}}{1 + \eta d(p_0, p_k, \tau_k)\frac{A}{4\eta}} \\ &= \frac{1}{\frac{4\eta}{A} + \eta d(p_0, p_k, \tau_k)} \end{aligned} \tag{6.17}$$

for $0 < s \leq d(p_0, p_k, \tau_k)$, as along as $R(\gamma_k(s'), \tau_k) \geq r_0^{-1}$ for $s' \in [0, s]$. By the smoothness of g on $[0, T - A^{-1}]$ we have $d(p_0, p_k, \tau_k) \to 0$ as $k \to \infty$. So (6.17) leads to

$$\sqrt{R(\gamma_k(s), \tau_k)} > \frac{A}{6\eta} \tag{6.18}$$

for large k. By a continuity argument we then arrive at

$$\sqrt{R(p_0, \tau_k)} > \frac{A}{6\eta} \tag{6.19}$$

for large k. Obviously, this implies that $\limsup_{t \to T} R(p_0, t) = \infty$. Consequently, $p_0 \in M - \Omega$. It follows that Ω is open. ∎

We would like to remark that in Perelman's surgery procedure, one does not employ blow-up limits directly as given by Theorem 2.3 and Theorem 6.11. Indeed, one performs surgery on (Ω, g_T), where g_T denotes the smooth limit of $g(t)$ on Ω as $t \to T$. Besides the basic information about (Ω, g_T) contained in Theorem 6.11, the fine structure of (Ω, g_T) needed for performing surgery is provided by the canonical neighborhood theorem. (In [P2], Perelman also uses an additional result on improved ϵ-necks deep down ϵ-horns.) On the other hand, blow-up limits such as provided by Theorem 2.3 and Theorem 6.11 (and more general versions of them) are used in many places in Perelman's papers [P1] and [P2] (including the proof of Theorem 6.4, which is a basis of the canonical neighborhood theorem), and play a key role there. For presentations and improvements of the surgery results of Perelman we refer to [P2], [MT], [KL], [Y6] and [Y7].

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Department of Mathematics, University of California, Santa Barbara, CA 93106
yer@math.ucsb.edu