Final Exam  
Math 147A  
Winter 2010  
Prof. R. Ye  

Your Name: 
Your Signature: 
Your Perm Number: 

Scores: 
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1. The final is due at 5:00 pm on Thursday, March 18, 2010. Please slide your final underneath the door into Prof. Ye’s office at SH6509. You can also leave it in Prof. Ye’s mailbox in the mail room of the math. department. 
2. No team work is allowed on the final. No help from anyone is allowed. 
3. Please present detailed steps of your solutions. 
4. Please write clearly and cleanly. Please write with enough spaces between the lines. 
5. Be sure to staple your final. 
6. Be sure to keep a backup copy of your final.
You can apply all theorems and formulas in the text or given in the lectures. (Be sure to state clearly what you are using.) If you appeal to a result in an exercise, you need to establish that result first.

1. (14 points) Enneper’s surface \( M \) is given by the parametrization

\[
\Phi(u, v) = (u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + vu^2, u^2 - v^2).
\]  

(0.1)

1) Show that Enneper’s surface is a minimal surface.

2) Find the Gauss curvature of Enneper’s surface.

**Solution** We have

\[
\Phi_u = (1 - u^2 + v^2, 2uv, 2u), \quad \Phi_v = (2uv, 1 - v^2 + u^2, -2v), \quad \Phi_{uu} = (-2u, 2v, 2), \quad \Phi_{uv} = (2v, 2u, 0), \quad \Phi_{vv} = (2u, -2v, -2).
\]

(0.2)

It follows that

\[
E = \Phi_u \cdot \Phi_u = (1 + u^2 + v^2)^2, \\
F = \Phi_u \cdot \Phi_v = 2uv - 2u^3v + 2uv^3 + 2uv + 4u^2v - 2u^2v + 2v^3 + 4u^2v^3, \\
G = \Phi_v \cdot \Phi_v = (1 + u^2 + v^2)^2, \\
|\Phi_u \times \Phi_v|^2 = EG - F^2 = (1 + u^2 + v^2)^4,
\]

(0.5)

and

\[
\Phi_{uu} \cdot \Phi_v = 0, \\
\Phi_{uv} \cdot \Phi_v = \frac{4u^2}{1 + u^2 + v^2} + \frac{4v^2}{1 + u^2 + v^2} + \frac{2 - 2(u^2 + v^2)^2}{(1 + u^2 + v^2)^2} = 2.
\]

(0.8)

Then we deduce

\[
m = \Phi_{uv} \cdot \Phi_v = 0, \\
l = \Phi_{uu} \cdot \Phi_v = \frac{4u^2}{1 + u^2 + v^2} + \frac{4v^2}{1 + u^2 + v^2} + \frac{2 - 2(u^2 + v^2)^2}{(1 + u^2 + v^2)^2} = 2.
\]

(0.9)

and similarly,

\[
n = \Phi_{vv} \cdot \Phi_v = -2.
\]
We finally arrive at
\[ H = \frac{Gl - 2Fm + En}{2(EG - F^2)} = \frac{(1 + u^2 + v^2)^2 \cdot 2 - 0 + (1 + u^2 + v^2)^2 \cdot (-2)}{2(1 + u^2 + v^2)^4} = 0 \] (0.10)
and
\[ K = \frac{ln - m^2}{EG - F^2} = \frac{-4}{(1 + u^2 + v^2)^4}. \] (0.11)

By (0.10), Enneper’s surface is a minimal surface.

2. (12 points) Consider a ruled surface \( M \) given by the parametrization \( \Phi(u, v) = \alpha(t) + v\beta(t) \), where \( \alpha(t) \) and \( \beta(t) \) are two parametrized curves. We assume that \( \Phi(u, v) \) is a regular parametrization. Assume that the Gauss curvature of \( M \) is zero everywhere. Show that the unit normal of \( M \) is independent of \( v \). Hint: the cross product of two vectors is perpendicular to them.

**Solution 1** We prove a stronger statement: \( K = 0 \) everywhere iff the unit normal \( n \) is independent of \( v \). First we have
\[ \Phi_u = \alpha'(u) + v\beta'(u), \Phi_v = \beta(u), \]
\[ \Phi_{uu} = \alpha''(u) + v\beta''(u), \Phi_{uv} = \beta'(u), \Phi_{vv} = 0. \] (0.12)
It follows that
\[ n = \Phi_{vv} \cdot n = 0. \] (0.13)

Then the Gauss curvature is given by
\[ K = \frac{ln - m^2}{EG - F^2} = \frac{-m^2}{EG - F^2}. \] (0.14)
It follows that \( K = 0 \) iff \( m = 0 \). But
\[ n_v \cdot \Phi_u = -n \cdot \Phi_{uv} = -m, \]
\[ n_v \cdot \Phi_v = -n \cdot \Phi_{vv} = 0. \] (0.15)

Hence we infer that \( K = 0 \) iff \( n_v \) is perpendicular to both \( \Phi_u \) and \( \Phi_v \). Consequently, \( K = 0 \) iff \( n_v \) is parallel to \( \Phi_u \times \Phi_v \), i.e., parallel to \( n \). But \( n_v \cdot n = 0 \). If follows that \( K = 0 \) iff \( n_v = 0 \). Hence \( K = 0 \) everywhere iff \( n_v = 0 \) everywhere, i.e., \( n \) is independent of \( v \). In particular, if \( K = 0 \) everywhere, then \( n \) is independent of \( v \).

**Solution 2** First we have the formulas (0.12) and (0.14). Since \( K = 0 \), we infer that \( m = 0 \). But \( m = \Phi_{uv} \cdot n = \beta'(u) \cdot n \). Hence we have \( \beta'(u) \cdot n = 0 \). Now we handle
two possible cases.

**Case 1** $\beta'(u) \times \beta(u) \neq 0$.

Since $\Phi_v = \beta(u)$, there holds $\beta(u) \cdot n = 0$. Hence it follows that

$$n = \frac{\beta'(u) \times \beta(u)}{|\beta'(u) \times \beta(u)|},$$

which is independent of $v$.

**Case 2** $\beta'(u) \times \beta(u) = 0$.

We infer from (0.12)

$$\Phi_u \times \Phi_v = \alpha'(u) \times \beta(u) + v\beta'(u) \times \beta(u) = \alpha'(u) \times \beta(u),$$

which is independent of $v$. \hfill \blacksquare

3. (12 points) Let $\gamma$ be a curve in a surface $M$. Assume that it is also contained in the sphere $S$ of radius 1.

1) Assume that $\gamma$ is a line of curvature in $M$. Show that the angle between the normal direction of $M$ and the normal direction of $S$ is constant along $\gamma$.

2) Assume that the angle between the normal direction of $M$ and the normal direction of $S$ is constant along $\gamma$. Show that $\gamma$ is a line of curvature in $M$. Hint: treat the case of nonzero angle and the case of zero angle separately.

**Solution** 1) We have

$$n' = -\kappa \gamma'$$

along $\gamma$, where $\kappa = \kappa(t)$. (Here $n = n(t)$ stands for $n(\gamma(t))$. similar notations are used below.) On the other hand, every curve in the sphere $S$ is a line of curvature in $S$. Hence we also have

$$n'_S = -\lambda \gamma'$$

for some $\lambda = \lambda(t)$, where $n_S$ denotes the (e.g. inward) unit normal of $S$. It follows that

$$(n \cdot n_S)' = n' \cdot n_S + n \cdot n'_S = -\kappa \gamma' \cdot n - \lambda \gamma' \cdot n_S = 0.$$

We infer that $\cos \theta = n \cdot n_S$ is a constant $\lambda$ along $\gamma$, where $\theta$ denotes the angle between $n$ and $n_S$ along $\gamma$. It follows that $\theta$ equals the constant $\cos^{-1} \lambda$ along $\gamma$. (Note that $\theta$ is always defined to be in the interval $[0, \pi]$, which is the range for $\cos^{-1}$.)

2) Changing the orientation of $S$ if necessary, we can assume that the angle $\theta$ between $n$ and $n_S$ along $\gamma$ is in the interval $[0, \frac{\pi}{2}]$.\hfill 4
**Case 1** $\theta = 0$
In this case, $n = n_S$ along $\gamma$. Hence
\[ n' = n_S' = -\lambda \gamma'. \]  
(0.21)
It follows that $\gamma$ is a line of curvature in $M$.

**Case 2** $\theta \neq 0$
We have
\[ n' \cdot n_S = (n \cdot n_S)' - n \cdot n_S' = 0 + n \cdot \lambda \gamma' = 0. \]  
(0.22)
We also have $n' \cdot n = 0$ because $n$ has constant length 1. Hence $n'$ is perpendicular to both $n$ and $n_S$. Thus $n'(t)$ lies in the intersection of $T_{\gamma(t)} S$ and $T_{\gamma(t)} M$. But these two tangent planes are distinct because of the angle assumption. Hence their intersection is a line. Since $\gamma'(t)$ is in this intersection, it follows that $n'(t)$ is proportional to $\gamma'(t)$. Hence $\gamma$ is a line of curvature in $M$. 

4. (12 points) Consider a curve $\gamma$ in a surface $M$. Assume that $\gamma$ has positive curvature. Show that $\gamma$ is an asymptotic curve if and only if its binormal $B$ is parallel to the unit normal of $M$ at each point.

**Solution** There holds
\[ II_{\gamma}(\gamma', \gamma') = \gamma'' \cdot n = \kappa N \cdot n, \]  
(0.23)
where $N$ denotes the principal normal of $\gamma$, $n$ the unit normal of $M$, and $\kappa$ the curvature of $\gamma$. We have $\kappa \neq 0$, otherwise the Frenet frame of $\gamma$ is not defined. Hence $\gamma$ is asymptotic iff $N \cdot n \equiv 0$. But $\gamma' \cdot n = 0$. Hence $\gamma$ is asymptotic iff $n$ is perpendicular to both $N$ and $T = \gamma'$, i.e. $n$ is parallel to the binormal $B$.

5. (12 points) Consider a surface of revolution $M$ given by the parametrization
\[ \Phi(u, v) = (f(u) \cos v, f(u) \sin v, g(u)) \]
for some functions $f$ and $g$. Show that every point on $M$ is parabolic if and only if $M$ is part of a circular cylinder or a circular cone.

**Solution** Performing a reparametrization if necessary, we can assume that the curve $(f(t), g(t))$ has unit speed. (This leads to a reparametrized $\Phi(u, v)$, which represents the same surface as before.) Then the principal curvatures are given by
\[ \kappa_1 = f'' g' - f' g'', \kappa_2 = \frac{g'}{f}. \]  
(0.24)
Furthermore, the Gauss curvature is given by

\[ K = -\frac{f''}{f}. \quad (0.25) \]

**Part 1, Method 1** We assume that every point on \( M \) is parabolic. Then \( K = 0 \) everywhere, and hence \( f'' = 0 \) everywhere. We infer \( f(u) = au + b \) for some constants \( a \) and \( b \).

**Case 1** \( a = 0 \).

Then \( f = b \). It follows that \( M \) is part of the circular cylinder of radius \( |b| \). (Note that \( b \) cannot be zero, otherwise \( \Phi(u, v) \) would not be regular.)

**Case 2** \( a \neq 0 \).

Then \( u = a^{-1}f(u) + a^{-1}b \). On the other hand, there holds

\[ K = \kappa_1 \kappa_2 = (f''g' - f'g'') \frac{g'}{f} = -a \frac{g''g'}{f}. \quad (0.26) \]

It follows that \( g''g' = 0 \). But \( g''g' = \frac{1}{2}((g')^2)' \). Hence \( ((g')^2)' = 0 \) and then \( g' = c \) for a constant \( c \). (We first infer that \( (g')^2 \) is a constant.) We deduce \( g = cu + d \) for a constant \( d \).

Now we conclude \( g(u) = ca^{-1}f(u) + C \) with \( C = ca^{-1}f(u) + d + ca^{-1}b \). Thus the curve \( y = f(t), z = g(t) \) is part of the straight line \( z = ca^{-1}y + C \). Note that \( c = 0 \), otherwise the surface \( M \) would be contained in the plane \( z = C \), contradicting the parabolic assumption. It follows that the line is non-horizontal. Obviously, it is also non-vertical. Consequently, \( M \) is part of the circular cone generated by the line.

**Part 1, Method 2** Again we assume that every point on \( M \) is parabolic. We claim that \( \kappa_1 = 0 \) everywhere. Suppose that \( \kappa_1(p) \neq 0 \) at some \( p \). Then \( \kappa_1 \neq 0 \) in a neighborhood of \( p \) by the continuity of \( \kappa_1 \). Hence \( \kappa_2 = 0 \) in this neighborhood. It follows that \( g' = 0 \) and hence \( g \) equals a constant \( c \) in this neighborhood. But this neighborhood of \( M \) is part of the plane \( z = c \), contradicting the parabolic assumption.

Now we know that \( \kappa_1 = 0 \) everywhere. Hence \( f''g' - f'g'' = 0 \) everywhere. Since \( f'' = 0 \), we infer that \( f'g'' = 0 \) everywhere. Now we can argue in the above two cases. The first case is handled in the same way as before. In the second case, we infer that \( g'' = 0 \), and again arrive at \( g(u) = cu + d \). The remaining argument is the same as before.

Alternatively, we can argue without using the equation \( f'' = 0 \). Note that \( \kappa_2 \neq 0 \), because \( \kappa_1 = 0 \) and every point is parabolic. Hence \( g' \neq 0 \). Since \( f''g' - f'g'' = 0 \), there holds

\[ \left( \frac{f'}{g'} \right)' = \frac{f''g' - f'g''}{(g')^2} = 0. \quad (0.27) \]

It follows that \( f' = cg' \) for a constant \( c \). Hence \( f = cg + d \) for a constant \( d \). Thus the curve \( y = f(t), z = g(t) \) is part of the straight line \( y = cz + d \). If \( c = 0 \), this line is a
vertical line at the distance $|d|$ from the $z$-axis, and hence $M$ is part of the circular cylinder of cross radius $|d|$. If $c \neq 0$, this line is non-vertical. Obviously, it is also non-horizontal. Hence $M$ is part of the circular cone generated by this line.

**Part 2** Assume that $M$ is a circular cone or circular cylinder. Then, in the both cases, $\kappa_1 = 0$ because $f$ and $g$ are linear, and $\kappa_2 \neq 0$ because $g$ is linear and not a constant. Indeed, in the case of a circular cylinder, we have $f(t) = d, g(t) = t$. In the case of a circular cone, we have $f(t) = \frac{c}{c^2+1}t + d, g(t) = \frac{1}{c^2+1}t$. It follows that every point of $M$ is parabolic. \hfill \blacksquare

6. (13 points) Use a suitable parametrization to find the principal curvatures, the Gauss and mean curvatures of the cone $z^2 = 2(x^2 + y^2)$ with respect to the downward normal.

**Solution 1** We consider the general case of a vertical circular cone with vertex at the origin:

$$z^2 = a^2(x^2 + y^2) \quad (0.28)$$

with a constant $a > 0$. We have the following natural parametrization of the cone

$$\Phi(u, v) = (u \cos v, u \sin v, au) \quad (0.29)$$

for $u \neq 0, 0 \leq v < 2\pi$, which is suggested by polar coordinates for $(x, y)$. There holds

$$\begin{align*}
\Phi_u &= (\cos v, \sin v, a), \\
\Phi_v &= (-u \sin v, u \cos v, 0), \\
\Phi_{uu} &= (0, 0, 0), \\
\Phi_{uv} &= (-\sin v, \cos v, 0), \\
\Phi_{vv} &= (-u \cos v, -u \sin v, 0) \quad (0.30)
\end{align*}$$

and

$$\Phi_u \times \Phi_v = (-au \cos v, -au \sin v, u), \quad |\Phi_u \times \Phi_v| = \sqrt{a^2 + 1}|u|. \quad (0.31)$$

Note that $\Phi_u \times \Phi_v$ is upward for the positive part of the cone (the part with $z > 0$), and downward for the negative part of the cone (the part with $z < 0$). From the above formula we derive the desired downward unit normal

$$n = \left(\frac{a}{\sqrt{a^2 + 1}} \cos v, \frac{a}{\sqrt{a^2 + 1}} \sin v, -\frac{1}{\sqrt{a^2 + 1}}\right). \quad (0.32)$$

We deduce

$$E = \Phi_u \cdot \Phi_u = a^2 + 1, \quad F = \Phi_u \cdot \Phi_v = 0, \quad G = \Phi_v \cdot \Phi_v = u^2, \quad (0.33)$$

$$l = \Phi_{uu} \cdot n = 0, \quad m = \Phi_{uv} \cdot n = 0, \quad n = \Phi_{vv} \cdot n = -\frac{a}{\sqrt{a^2 + 1}}u. \quad (0.34)$$

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Hence the matrix of the shape operator is given by
\[
\begin{pmatrix}
  a^2 + 1 & 0 \\
  0 & u^2
\end{pmatrix}^{-1}
\begin{pmatrix}
  0 & 0 \\
  -\frac{a}{\sqrt{a^2 + 1}} u
\end{pmatrix}
= 
\begin{pmatrix}
  \frac{1}{a^2 + 1} & 0 \\
  0 & -\frac{a}{\sqrt{a^2 + 1}} u
\end{pmatrix}
= 
\begin{pmatrix}
  0 & -\frac{a}{\sqrt{a^2 + 1}} u^{-1}
\end{pmatrix}.
\]

Consequently, we obtain the principal curvatures
\[
\kappa_1 = 0, \kappa_2 = -\frac{a}{\sqrt{a^2 + 1}} u^{-1} = -\frac{a^2}{\sqrt{a^2 + 1}} z^{-1}.
\]

The Gaus curvature and mean curvature are then given by
\[
K = 0, H = -\frac{a^2}{2\sqrt{a^2 + 1}} z^{-1}.
\]

In the given case \(a = \sqrt{2}\) we obtain
\[
\kappa_1 = 0, \kappa_2 = -\frac{2}{\sqrt{3}} z^{-1}, K = 0, H = \frac{1}{\sqrt{3} z}.
\]

**Solution 2** We can consider the cone as a surface of revolution generated by the curve \(z^2 = 2y^2\), i.e. \(z = \sqrt{2}y\), which can be expressed as \(y = t, z = \sqrt{2}t\). We can reparametrize to get unit speed:
\[
y = f(t) = \frac{1}{\sqrt{3}} t, z = g(t) = \sqrt{\frac{2}{3}} t.
\]

Then we obtain the natural parametrization
\[
\Phi(u, v) = \left( \frac{1}{\sqrt{3}} u \cos v, \frac{1}{\sqrt{3}} u \sin v, \sqrt{\frac{2}{3}} u \right).
\]

By the formulas for surfaces of revolution with unit-speed generating curve in the text we obtain the principal curvatures with respect to the orientation induced from the above parametrization
\[
\kappa_1 = f'g'' - f''g' = \frac{1}{\sqrt{3}} \cdot 0 - 0 \cdot \sqrt{\frac{2}{3}} = 0, \kappa_2 = \frac{g'}{f} = \frac{\sqrt{2}}{\sqrt{3} u} = \frac{\sqrt{2}}{u} = \frac{2}{\sqrt{3}} z^{-1}.
\]

It follows that
\[
K = 0, H = \frac{1}{\sqrt{3} z}.
\]
Note that the above formulas hold true under the condition that \( f > 0 \), which means the positive part of the cone. As in the above first method, the normal \( \Phi_u \times \Phi_v \) turns out to be upward on the positive part of the cone. Hence we obtain the curvatures on the positive part of the cone for the downward orientation

\[
\kappa_1 = 0, \kappa_2 = -\frac{2}{\sqrt{3}}z^{-1}, K = 0, H = -\frac{1}{\sqrt{3}z}. \tag{0.43}
\]

We can modify the curvature formulas in the case \( f < 0 \) and arrive at (0.43) on the negative part of the cone.

7. (13 points) Let \( S \) be the graph of the function \( z = 2x^2 + 3y^2 + 1 \). Choose the upward normal direction. Consider the point \( p = (0, 0, 1) \) on \( S \). Let \( v_1 \) denote the unit tangent vector in the \( x \)-direction at \( p \), and \( v_2 \) the unit tangent vector in the \( y \)-direction at \( p \).

1) Find the normal curvature \( k_v \) of \( S \) at \( p \) in the direction of \( v = av_1 + bv_2 \) with \( a^2 + b^2 = 1 \).

2) Let \( \gamma \) be a regular curve on \( S \) with \( \gamma(0) = p \) and \( \gamma'(0) = v \). What is the relation between \( k_v \) and the curvature of \( \gamma \) at time 0? Why?

**Solution**

1) We adopt the natural parametrization

\[
\Phi(u, v) = (u, v, 2u^2 + 3v^2 + 1). \tag{0.44}
\]

There hold

\[
\Phi_u = (1, 0, 4u), \Phi_v = (0, 1, 6v), \Phi_u \times \Phi_v = (-4u, -6v, 1), \tag{0.45}
\]

and

\[
n = \left( \frac{4u}{1 + 16u^2 + 36v^2}, \frac{-6v}{1 + 16u^2 + 36v^2}, \frac{1}{1 + 16u^2 + 36v^2} \right). \tag{0.46}
\]

For a given tangent vector \( v = av_1 + bv_2 \) we can choose the following curve

\[
\gamma(t) = (at, bt, 2a^2t^2 + 3b^2t^2 + 1) \tag{0.47}
\]

in \( S \). It is easy to see that \( \gamma'(0) = v \). It follows that

\[
\kappa_v = n(p) \cdot \gamma''(0) \tag{0.48}
\]

Since \( p = \Phi(0, 0) \) we infer \( n(p) = (0, 0, 1) \). On the other hand, there holds

\[
\gamma'' = (0, 0, 4a^2 + 6b^2). \tag{0.49}
\]

It follows that

\[
\kappa_v = 4a^2 + 6b^2. \tag{0.50}
\]
2) There hold
\[ \kappa_V = \mathbf{n}(p) \cdot \gamma''(0) = (0, 0, 1) \cdot \gamma''(0) \]  
and \[ \gamma''(0) = \kappa(0)\mathbf{N}(0). \]  
Hence
\[ \kappa_V = \kappa(0)(0, 0, 1) \cdot \mathbf{N}(0) = \kappa(0) \cos \theta, \]  
where \( \theta \) is the angle between the principal normal of \( \gamma \) at 0 and the \( z \)-direction.

8. (12 points) Let \( \gamma \) be a unit speed curve with positive curvature. Define a new curve \( \alpha \) by the formula
\[ \alpha(t) = \int_0^t (aT(s) + bB(s))ds, \]
where \( T \) denotes the tangent vector of \( \gamma \), \( B \) denotes the binormal of \( \gamma \), and \( a, b \) are constants such that \( a^2 + b^2 = 1 \). Assume that \( a\kappa - b\tau \) is nonzero along \( \gamma \), where \( \kappa \) and \( \tau \) are the curvature and torsion of \( \gamma \).

1) Find the curvature, torsion, the principal normal and binormal of \( \alpha \).

2) Suppose that \( \gamma \) is a circular helix. Show that \( \alpha \) is also a circular helix.

**Solution**

1) Let \( \kappa_\alpha \) and \( \tau_\alpha \) denote the curvature and torsion of \( \alpha \) respectively. Let \( T_\alpha, N_\alpha \) and \( B_\alpha \) denote the Frenet frame of \( \alpha \). There holds
\[ \alpha' = aT + bB. \]  
It has length 1 because \( a^2 + b^2 = 1 \). Hence \( \alpha \) is a unit-speed curve and there holds
\[ T_\alpha = \alpha' = aT + bB. \]

Next we have
\[ T'_\alpha = aT' + bB' = a\kappa N - b\tau N = (a\kappa - b\tau)N. \]
It follows that
\[ \kappa_\alpha = |a\kappa - b\tau|, \]  
\[ N_\alpha = N \]
if \( a\kappa - b\tau > 0 \), and
\[ N_\alpha = -N \]  
if \( a\kappa - b\tau < 0 \).
if \(a\kappa - b\tau < 0\).

**Case 1** \(a\kappa - b\tau > 0\).

In this case, we have

\[
B_\alpha = T_\alpha \times N_\alpha = (aT + bB) \times N = aB - bT
\]

(0.59)

and

\[
\tau_\alpha = N'_\alpha \cdot B_\alpha = N' \cdot B_\alpha = (-\kappa T + \tau B) \cdot (aB - bT) = b\kappa + a\tau.
\]

(0.60)

**Case 2** \(a\kappa - b\tau < 0\).

Arguing as in Case 1 we deduce

\[
B_\alpha = -aB + bT
\]

(0.61)

and

\[
\tau_\alpha = N'_\alpha \cdot B_\alpha = -N' \cdot B_\alpha = -(-\kappa T + \tau B) \cdot (-aB + bT) = b\kappa + a\tau.
\]

(0.62)

2) Since \(\gamma\) is a circular helix, its curvature and torsion are nonzero constants. By 1) we then infer that the curvature and torsion of \(\alpha\) are also constants. It also follows that \(\kappa_{\alpha}\) is nonzero, because \(a\kappa - b\tau\) is assumed to be nonzero. Now, if \(b\kappa + a\tau \neq 0\), then \(\tau_\alpha \neq 0\) and hence \(\alpha\) is a circular helix. (Recall that a regular curve of constant speed is a circular helix if it has nonzero constant curvature and torsion.) Otherwise, \(\alpha\) is a circle, which can be considered to be a degenerate circular helix. (The claim that \(\alpha\) must be a circular helix is not entirely precise.)