Please present detailed steps of your solutions.

Each problem is worth 15 points. In particular, 5 extra credit points are included.

Please email your final to Prof. Ye: yer@math.ucsb.edu
(You can type up or scan your final.)
The final is due on June 10, 2010.
1. Suppose that $M$ is a surface with no umbilic points and one constant principal curvature $\kappa_1 \neq 0$. Prove that $M$ is (a subset of) a tube of radius $r = 1/|\kappa_1|$ about a curve. That is, there is a curve $\gamma$ so that $M$ is (a subset of) the union of circles of radius $r$ in each normal plane, centered along the curve. Hints: As usual, work with a parametrization where the $u$-curves are lines of curvature with principal curvature $\kappa_1$ and the $v$-curves are lines of curvature with principal curvature $\kappa_2$. Use the Codazzi equations to show that the $u$-curves have curvature $|\kappa_1|$ and are planar. Then define $\gamma$ appropriately and check that it is a regular curve.

2. Prove or give a counterexample:
1) A line lying in a surface is both an asymptotic curve and a geodesic.
2) If a curve is both an asymptotic curve and a geodesic, then it must be a line.
3) If a curve is both a geodesic and a line of curvature, then it must be planar.

3. Let $M$ be a surface with a given orientation. Show that the geodesic curvature at $p$ of a regular curve $\gamma$ in $M$ is equal (in absolute value) to the curvature at $p$ of the orthogonal projection of $\gamma$ into $T_pM$.

4. Consider the catenoid $M$ given by the parametrization
$$
\Phi(u, v) = (\cos v \cosh u, \sin v \cosh u, u)
$$
for $-\infty < u < \infty$, $0 \leq v \leq 2\pi$. Choose the outward orientation. For $r > 0$ denote by $M_r$ the portion of the catenoid defined by $-r \leq u \leq r$.
1) Calculate the holonomy around each boundary circle of $M_r$ in the positive direction.
2) Calculate the geodesic curvature of each boundary circle of $M_r$.
3) Use Gauss-Bonnet theorem to calculate the integral $\int_{M_r} KdA$. Find the total Gauss curvature $\int_M KdA$ by taking the limit as $r \to \infty$.

5. Let $M$ be a surface with a given orientation. Assume that the holonomy around each closed curve of $M$ w. r. t. each framing is zero. Show that $M$ is flat, i. e. the Gauss curvature of $M$ is zero everywhere.

6. Let $M$ be a parametrized surface such that all $u$-curves and $v$-curves are geodesics. Moreover, assume that the angle between any $u$-curve and any
$v$-curve is $\pi/2$ whenever they intersect. Show that $M$ is flat.

7. Let $M$ be a parametrized surface such that all $u$-curves and $v$-curves are straight lines. Moreover, assume that the angle between any $u$-curve and any $v$-curve is $\pi/2$ whenever they intersect. Show that $M$ is a plane.