Midterm Exam
Math 147B
Spring 2010
Prof. R. Ye

Your Name:
Your Signature:
Your Perm Number:

Scores:
1.
2.
3.
4.
5.
Total:

Please present detailed steps of your solutions.

10 extra credit points are included
1. (22 points) Let $\alpha(t) = (x(t), y(t), 0)$ be a unit-speed curve in the $(x, y)$-plane. Let $e_3$ denote the unit vector $(0, 0, 1)$. Consider the generalized cylinder $M$ given by the parametrization $\Phi(u, v) = \alpha(u) + ve_3$.

Now consider a generalized helix $\gamma(t) = \alpha(a_0t + a) + (b_0t + b)e_3$, where $a_0, b_0, a$ and $b$ are constants such that $a_0^2 + b_0^2 \neq 0$. Show directly that $\gamma(t)$ is a geodesic in $M$. Here “directly” means to calculate the covariant derivative of $\gamma'$ directly, or to calculate the geodesic curvature of $\gamma$ directly. So you need to calculate the quantities $\gamma'$, $\gamma''$, $n$ (unit normal of $M$) etc.

**Solution** One method we can use is to show that the geodesic equation $\nabla_{\gamma'} \gamma' = 0$ holds. Another method is to argue in terms of the geodesic curvature.

**Method 1: the geodesic equation**

There hold

$$\gamma' = a_0\alpha'(a_0t + a) + b_0e_3$$

and

$$\gamma'' = a_0^2\alpha''(a_0t + a).$$

Next we have

$$\Phi_u = \alpha'(u), \Phi_v = e_3$$

and hence

$$\Phi_u \times \Phi_v = \alpha'(u) \times e_3.$$  

Since $\alpha'$ has unit length and is perpendicular to $e_3$, it follows that

$$n = \alpha'(u) \times e_3.$$  

For a given $t$, consider $u = a_0t + a$. Since $\alpha'' \cdot \alpha' = 0$ and $\alpha$ is in the $(x, y)$-plane, $\alpha' \times \alpha''$ must be a multiple of $e_3$. There are three cases to consider.

**Case 1** $\alpha''(u) = 0$.

In this case, we have $\gamma''(t) = 0$ and hence $\nabla_{\gamma'} \gamma' = 0$. 


Case 2 $\alpha'(u) \times \alpha''(u)$ is a positive multiple of $e_3$, i.e. $\alpha'(u) \times \alpha''(u) = \kappa(u)e_3$, where $\kappa = |\alpha''|$ is the curvature of $\alpha$.

In this case we have at $t$

$$\gamma'' \cdot n = a_0^2 \alpha'' \cdot \alpha' \times e_3 = -a_0^2 |\alpha''| = -a_0^2 \kappa.$$  

It follows that

$$\nabla_{\gamma''} \gamma' = \gamma'' - (\gamma'' \cdot n)n = a_0^2 \alpha'' + a_0^2 \kappa n.$$  

But $n = \alpha' \times e_3 = -\kappa^{-1} \alpha''$. It follows that

$$\nabla_{\gamma''} \gamma' = 0.$$  

Case 3 $\alpha' \times \alpha'' = -\kappa e_3$.

This is similar to Case 2. It can also be reduced Case 2 by reversing the orientation of $\alpha$.

So, in all the 3 cases, we have $\nabla_{\gamma''} \gamma' = 0$. Hence $\gamma$ is a geodesic.

**Method 2: geodesic curvature**

By the above calculation of $\gamma'$ we see that $|\gamma'| = \sqrt{a_0^2 + b_0^2}$. Denote this number by $\lambda$. Then the unit-speed reparametrization of $\gamma$ is given by $\tilde{\gamma}(t) = \gamma(\lambda^{-1} t)$. The geodesic curvature of $\tilde{\gamma}$ (and hence of $\gamma$) is given by

$$\kappa_g = \tilde{\gamma}'' \cdot (n \times \tilde{\gamma}') = \lambda^{-3} \gamma'' \cdot (n \times \gamma').$$

To show that $\kappa_g = 0$, it suffices to show that $\gamma'' \cdot (n \times \gamma') = 0$. We have

$$n \times \gamma' = (\alpha' \times e_3) \times (a_0 \alpha' + b_0 e_3).$$

There holds $\alpha' = (x', y', 0)$ and hence $\alpha' \times e_3 = (y', -x', 0)$. We also have $(y', -x', 0) \times (x', y', 0) = (x'^2 + y'^2)e_3 = e_3$ and $(x', y', 0) \times e_3 = (-x', -y', 0) = -\alpha'$. It follows that

$$n \times \gamma' = a_0 e_3 - b_0 \alpha'.$$
Hence we deduce
\[ \gamma'' \cdot (n \times \gamma') = a_0^2 \alpha'' \cdot (a_0 e_3 - b_0 \alpha') = -a_0^2 b_0 \alpha'' \cdot \alpha'. \]
Since \( \alpha \) has unit speed, there holds \( \alpha'' \cdot \alpha' = 0 \). We infer that
\[ \gamma'' \cdot (n \times \gamma') = 0 \]
and hence \( \gamma \) is a geodesic.

2. (22 points) Calculate the geodesic curvature of the latitude circles on the unit sphere w. r. t. the outward unit normal. Recall that the geodesic curvature of a unit speed curve \( \gamma \) is defined to be \( \kappa_g = \gamma'' \cdot (n \times \gamma') \), where \( n \) denotes the (chosen) unit normal.

**Solution** We employ the parametrization
\[ \Phi(u, v) = (\sin u \cos v, \sin u \sin v, \cos u). \]
There holds
\[ n(u, v) = \Phi(u, v). \]
Now consider the latitude circle \( \gamma(t) = \Phi(u, t) \) for a given \( u \). There holds
\[ \gamma' = (-\sin u \sin t, \sin u \cos t, 0) \]
Hence we have \( |\gamma'| = |\sin u| \). Set \( a = |\sin u| \). We present the case \( \sin u \geq 0 \). The case \( \sin u < 0 \) is similar. The unit-speed reparametrization of \( \gamma \) is then given by
\[ \tilde{\gamma}(t) = \gamma(a^{-1} t). \]
We have
\[ \tilde{\gamma}' = a^{-1} \gamma' = (-\sin(a^{-1} t), \cos(a^{-1} t), 0) \]
and then
\[ \tilde{\gamma}'' = -a^{-1}(\cos(a^{-1} t), \sin(a^{-1} t), 0). \]
It follows that
\[ n \times \tilde{\gamma}' = (-\cos(a^{-1} t) \cos u, -\sin(a^{-1} t) \cos u, a) \]
and
\[ \kappa_g = \hat{\gamma}'' \cdot (n \times \gamma') = a^{-1} \cos u = \cot u. \]

3. (22 points) 1) Let \( M \) be a surface. Recall that the geodesic equation for a curve \( \gamma \) in \( M \) is \( \nabla_{\dot{\gamma}} \dot{\gamma} = 0 \) or \( \nabla_{\gamma'} \gamma' = 0 \). This is the same as saying that the geodesic curvature of \( \gamma \) is zero. Derive the equation for the geodesic curvature of a curve \( \gamma \) in \( M \) to be a given constant \( c \). We call it the “constant geodesic curvature equation”.

2) Show that a solution of the constant geodesic curvature equation must have constant speed and constant geodesic curvature. Hint: Utilize the unit-speed reparametrization.

**Solution** 1) For a unit-speed curve \( \gamma \) in \( M \) the geodesic curvature is given by \( \kappa_g = \gamma'' \cdot (n \times \gamma') \). Moreover, \( \gamma'' \) is perpendicular to \( \gamma' \). So the tangential projection of \( \gamma'' \), i.e. \( \nabla_{\gamma'} \gamma' \) is proportional to \( n \times \gamma' \). Hence the geodesic curvature being a constant \( c \) means
\[ \nabla_{\gamma'} \gamma' = c n \times \gamma'. \]

For a general curve, we need to modify the equation. Arguing in terms of the unit-speed reparametrization we then arrive at the following equation for a general curve \( \gamma \) in \( M \)
\[ \nabla_{\gamma'} \gamma' = c |\gamma'| (n \times \gamma'). \]

This is the desired constant geodesic curvature equation.

Note that \( \gamma' \) appears twice both on the LHS and RHS of the equation. Hence this equation is invariant under rescaling. The rescaling invariance should hold for the equation of constant geodesic curvature, because a curve of constant geodesic curvature will have constant geodesic curvature after any rescaling.

2) Let \( \gamma \) be a curve in \( M \) satisfying the above equation. We have
\[ \frac{d}{dt} |\gamma'|^2 = 2 \gamma'' \cdot \gamma' = 2 \nabla_{\gamma'} \gamma' \cdot \gamma' = 2 c |\gamma'| (n \times \gamma') \cdot \gamma' = 0. \]

Hence \( \gamma \) has constant speed.
Let \( a = |\gamma'| \) be the speed of \( \gamma \). We consider its unit-speed reparametrization \( \tilde{\gamma}(t) = \gamma(a^{-1}t) \). We readily verify that \( \tilde{\gamma} \) satisfies the above “constant geodesic curvature equation”. Since \( |\tilde{\gamma}'| = 1 \), we have

\[
\nabla_{\tilde{\gamma}'}\tilde{\gamma}' = c n \times \tilde{\gamma}'.
\]

Now we deduce

\[
\kappa_g = \tilde{\gamma}'' \cdot (n \times \tilde{\gamma}') = \nabla_{\tilde{\gamma}'}\tilde{\gamma}' \cdot (n \times \tilde{\gamma}') = (cn \times \gamma') \cdot (n \times \tilde{\gamma}') = c.
\]

4. (22 points) 1) Prove the following formula for the Gauss curvature in an orthogonal parametrization

\[
K = -\frac{1}{2\sqrt{EG}} \left( \left( \frac{E_v}{\sqrt{EG}} \right)_v + \left( \frac{G_u}{\sqrt{EG}} \right)_u \right).
\]

2) Assume that \( E = G = \lambda(u,v) \) for a function \( \lambda(u,v) \) and \( F = 0 \). Show that the Gauss curvature is given by

\[
K = -\frac{1}{2\lambda} \Delta \ln \lambda,
\]

where \( \Delta \) is the Laplace operator, i.e. \( \Delta f = \frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2} \).

Solution 1) From the Gauss equations we have

\[
EK = (\Gamma^v_{uu})_v - (\Gamma^v_{uv})_u + \Gamma^u_{uu} \Gamma^v_{uv} + \Gamma^v_{uv} \Gamma^u_{uv} - \Gamma^u_{uv} \Gamma^v_{uu} - (\Gamma^v_{uv})^2.
\]

In an orthogonal parametrization we have the following formulas for the Christoffel symbols, which easily follow from their matrix formulas

\[
\Gamma^v_{uu} = -\frac{1}{2} G^{-1} E_v,
\]
\[
\Gamma^v_{uv} = \frac{1}{2} G^{-1} G_u,
\]
\[
\Gamma^w_{uu} = \frac{1}{2} E^{-1} E_u,
\]
\[
\Gamma^v_{vv} = \frac{1}{2} G^{-1} G_v.
\]
\( \Gamma_{uv}^u = \frac{1}{2} E^{-1} E_v. \)

It follows that
\[
EK = -\frac{1}{2}(G^{-1} E_v)_v - \frac{1}{2}(G^{-1} G_u)_u + \frac{1}{4} E^{-1} G^{-1} E_v G_u - \frac{1}{4} G^{-2} E_v G_v + \frac{1}{4} E^{-1} G^{-2} E^2_v - \frac{1}{4} G^{-2} G^2_u.
\]

We infer
\[
K = -\frac{1}{2E} \left[ (G^{-1} E_v)_v + \frac{1}{2G^2} E_v G_v - \frac{1}{2EG} E^2_v \right] - \frac{1}{2E} \left[ (G^{-1} G_u)_u - \frac{1}{2EG} E_u G_u + \frac{1}{2G^2} G^2_u \right]
\]
\[
= -\frac{1}{2EG} [E_{vv} - \frac{1}{2E} E_v G_v - \frac{1}{2E} E^2_v] - \frac{1}{2EG} [G_{uu} - \frac{1}{2E} E_u G_u - \frac{1}{2G} G^2_u].
\]

On the other hand, we have
\[
-\frac{1}{2\sqrt{EG}} \left( \frac{E_v}{\sqrt{EG}} \right)_v = -\frac{E_{vv}}{2EG} + \frac{E_v G_u}{4EG^2} + \frac{E^2_v}{4E^2 G}
\]
and
\[
-\frac{1}{2\sqrt{EG}} \left( \frac{G_u}{\sqrt{EG}} \right)_u = -\frac{G_{uu}}{2EG} + \frac{E_u G_v}{4EG^2} + \frac{G^2_u}{4EG^2}.
\]
Hence we arrive at
\[
K = -\frac{1}{2\sqrt{EG}} \left( \left( \frac{E_v}{\sqrt{EG}} \right)_v + \left( \frac{G_u}{\sqrt{EG}} \right)_u \right).
\]

2) Substituting \( \lambda \) for \( E \) and \( G \) in the Gauss curvature equation from 1) we derive
\[
K = -\frac{1}{2\lambda} \left( \left( \frac{\lambda_v}{\lambda} \right)_v + \left( \frac{\lambda_u}{\lambda} \right)_u \right).
\]
It follows that
\[
K = -\frac{1}{2\lambda} \left( (\ln \lambda)_vv + (\ln \lambda)_{uu} \right) = -\frac{1}{2\lambda} \Delta \ln \lambda.
\]

5. (22 points) Using the framing given by the parametrization to compute the holonomy around the parallel \( u = u_0 \) on the torus \( \Phi(u, v) = \)
\((a + b \cos u) \cos v, (a + b \cos u) \sin v, b \sin u)\).

**Solution** We have

\[
\Phi_u = (-b \sin u \cos v, -b \sin u \sin v, b \cos u),
\]

\[
\Phi_v = -(a + b \cos u) \sin v, (a + b \cos u) \cos v, 0).
\]

One readily checks that \(\Phi_u \cdot \Phi_u = b^2, \Phi_u \cdot \Phi_v = 0, \Phi_v \cdot \Phi_v = (a + b \cos u)^2\).

Hence we deduce

\[
e_1 = (- \sin u \cos v, - \sin u \sin v, \cos u)
\]

and

\[
e_2 = (- \sin v, \cos v, 0).
\]

Now we consider the parallel \(u = u_0\) whose parametrization is given by

\[
\gamma(t) = ((a + b \cos u_0) \cos t, (a + b \cos u_0) \sin t, b \sin u_0).
\]

The framing along \(\gamma(t)\) is then

\[
e_1(t) = e_1(u_0, t) = (- \sin u_0 \cos t, - \sin u_0 \sin t, \cos u_0),
\]

\[
e_2(t) = e_2(u_0, t) = (- \sin t, \cos t, 0).
\]

It follows that

\[
\phi_{12} = \frac{d}{dt} e_1 \cdot e_2 = (\sin u_0 \sin t, - \sin u_0 \cos t, 0) \cdot (- \sin t, \cos t, 0) = - \sin u_0.
\]

The holonomy around the parallel is then given by

\[
\Delta \theta = - \int_0^{2\pi} \phi_{12} dt = 2\pi \sin u_0.
\]