

Midterm Exam  
Math 240B  
Prof. R. Ye  
Winter 2011

Your Name:  
Score:

Each problem is worth 20 points.

1. Problem 11-7 on page 286 of the textbook.
2. 1) Problem 12-3 on page 319 of the textbook. 2) Problem 12-4 on the same page.
3. Problem 12-6 on page 320 of the textbook.
4. Problem 12-17 on page 323 of the textbook.
5. Recall the following definition of the Lie derivative of covariant tensor fields. Let  $X$  be a smooth vector field on a smooth manifold  $M$  and  $\sigma$  a smooth covariant tensor field on  $M$ . Consider the one-parameter family of local diffeomorphisms  $\Phi(p, t)$  generated by  $X$ , i.e.

$$\frac{d\Phi}{dt}(p, t) = X(\Phi(p, t)), \Phi(p, 0) = p.$$

Set  $\Phi_t = \Phi(\cdot, t)$ . Then

$$(L_X\sigma)(p) = \frac{d}{dt}(\Phi_t)_p^*\sigma_{\Phi(p,t)}|_{t=0}.$$

Prove the following.

- 1)  $L_X\sigma$  is smooth.
- 2) There hold

$$L_X(\sigma_1 \otimes \sigma_2) = L_X\sigma_1 \otimes \sigma_2 + \sigma_1 \otimes L_X\sigma_2$$

for smooth covariant tensor fields  $\sigma_1$  and  $\sigma_2$ , and

$$L_X(\alpha \wedge \beta) = L_X\alpha \wedge \beta + \alpha \wedge L_X\beta$$

for smooth differential forms  $\alpha$  and  $\beta$ .

- 3) There holds

$$L_{fX}\alpha = df \wedge i_X\alpha + fL_X\alpha$$

for smooth functions  $f$ , where the *interior product*  $i_X$  is defined as follows. Let  $k$  denote the degree of  $\alpha$ . Then

$$i_X \alpha(X_1, \dots, X_{k-1}) = \alpha(X, X_1, \dots, X_{k-1})$$

if  $k \geq 0$ , and  $i_X \alpha = 0$  if  $k = 0$ . (In particular,  $i_X \alpha$  is a  $(k - 1)$ -form.)