JING GUO Part 3

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16.1

(a) Flase.

counterexample: $S_n = 1 + (-1)^n = \begin{cases} 0, & n \text{ is odd} \\ 2, & n \text{ is even} \end{cases}$

(b) True.

Definiton 16.2

(c) Flase

$S_n = 1/n$

$t_n = 2/n$

(d) True.

Theorem 16.13

16.2

(a) Flase

counterexample: $S_n = (-1)^n \frac{1}{n}$

(b) Flase

$\because$ If it holds

$\Rightarrow$ For $\forall \varepsilon > 0$ $\exists N \in \mathbb{R}$ s.t. $n > N$ implies $S_n - 0 < \varepsilon$

If $S_n > 0$ $S_n - 0 = |S_n - 0|$

If $S_n \leq 0$ $S_n < \varepsilon$ holds all the time

$|S_n - 0| = |S_n|$ may $> \varepsilon$

(c) Flase

See Theorem 16.8

(d) True.

Theorem 16.14 & 11.6

16.3

(a) $1 \quad 2^2 \quad 3^2 \quad 4^2 \quad 5^2 \quad 6^2 \quad 7^2$

(b) $-1 \quad \frac{1}{2} \quad -\frac{1}{3} \quad \frac{1}{4} \quad -\frac{1}{5} \quad \frac{1}{6} \quad -\frac{1}{7}$

(c) $\frac{1}{2} \quad -\frac{1}{2} \quad -1 \quad -\frac{1}{2} \quad \frac{1}{2} \quad 1 \quad \frac{1}{2}$

(d) $\frac{3}{2} \quad \frac{5}{5} \quad \frac{7}{8} \quad \frac{9}{11} \quad \frac{11}{14} \quad \frac{13}{17} \quad \frac{15}{20}$

16.6

(a) Claim: for $\forall k \in \mathbb{R}$. $\lim_{n \to \infty} (k/n) = 0$

We want to find some N s.t. for $\forall n > N$

$|k/n - 0| < \varepsilon$

$|k/n| < \varepsilon$

$|k|/n < \varepsilon$

If $k = 0$ It's true

If $k \neq 0$ $|k| \neq 0$

$0 < 1/n < \varepsilon / |k|$

$n > |k|/\varepsilon$

Let $N = [|k|/\varepsilon] + 1$

So for $\forall n > N$

$n > |k|/\varepsilon$

$1/n < \varepsilon/|k|$

$|k|/n < \varepsilon$

$|k/n - 0| < \varepsilon$

QED

(b) Claim: For $\forall k > 0$, $\lim_{n \to \infty} (1/n^k) = 0$.

We want an N s.t. $n > N$ makes

$|1/n^k - 0| < \varepsilon$

$|1/n^k| < \varepsilon$.

$\because n > 0 \quad k > 0 \quad \therefore n^k > 0$

$1/n^k < \varepsilon$

$n^k > 1/\varepsilon$

$n > \sqrt[k]{1/\varepsilon}$

Let $N = [\sqrt[k]{1/\varepsilon}] + 1$

For $n > N$

$n > \sqrt[k]{1/\varepsilon}$

$n^k > 1/\varepsilon$

$1/n^k < \varepsilon$

$|1/n^k - 0| < \varepsilon$ QED

1(c) $\lim \frac{3n+1}{n+2} = 3$

We want an $N \in \mathbb{R}$ s.t. $n > N$ implies

$\left| \frac{3n+1}{n+2} - 3 \right| < \varepsilon$ for $\forall \varepsilon > 0$

$\left| \frac{3n+1-3n-6}{n+2} \right| = \left| \frac{-5}{n+2} \right| = \left| \frac{5}{n+2} \right|$ ($n > 0$)

$5/n+2 < \varepsilon \quad 1/n+2 < \varepsilon/5 \quad n+2 > 5/\varepsilon$

$n > 5/\varepsilon - 2$ Let $N = [5/\varepsilon - 2] + 1$

If $n > N$

$n > 5/\varepsilon - 2$

$n+2 > 5/\varepsilon$

$n+2/5 > 1/\varepsilon$

$5/n+2 < \varepsilon$

$\left| \frac{-5}{n+2} \right| < \varepsilon \quad \left| \frac{3n+1}{n+2} - 3 \right| < \varepsilon$ for $\forall \varepsilon > 0$ QED

(d) $\lim \frac{\sin(n)}{n} = 0$

We want an N s.t. for $n > N$, $\left|\frac{\sin(n)}{n} - 0\right| < \varepsilon$

for $\forall \varepsilon > 0$.

$\left|\frac{\sin(n)}{n}\right| = \frac{|\sin(n)|}{|n|} < \frac{1}{n}$

If $\frac{1}{n} < \varepsilon$ $\quad n \geq 1/\varepsilon$. Let $N = [1/\varepsilon] + 1$

For $n > N$

$n > 1/\varepsilon$

$1/n < \varepsilon$

$\left|\frac{1}{n}\right| < \varepsilon \Rightarrow \left|\frac{\sin(n)}{n}\right| < \varepsilon \Rightarrow \left|\frac{\sin(n)}{n} - 0\right| < \varepsilon$ QED

(e) $\lim \frac{n+2}{n^2-3} = 0$

We want an N s.t. for $n > N$ $\left|\frac{n+2}{n^2-3} - 0\right| < \varepsilon$

For $\forall \varepsilon > 0$

$\left|\frac{n+2}{n^2-3}\right| < \varepsilon$

$n^2 - 3 \geq n^2/2$ for $n^2/2 \geq 3$ $\quad n^2 \geq 6$ $\quad n \geq \sqrt[2]{6}$

and $n+2 < 2n$ for $n > 2$ $\quad \sqrt[2]{6} > 2$

$\therefore$ For $n > \sqrt[2]{6}$

$\frac{n+2}{n^2-3} \leq \frac{2n}{n^2/2} = \frac{4}{n}$ let $\frac{4}{n} < \varepsilon$ $\quad n > 4/\varepsilon$

$N = [\max\{\sqrt[2]{6}, 4/\varepsilon\}] + 1$

For $n > N$, $\quad n > 4/\varepsilon$ and $n > \sqrt[2]{6}$

$\frac{2n}{n^2/2} < \varepsilon \Rightarrow \frac{n+2}{n^2-3} < \varepsilon \Rightarrow \left|\frac{n+2}{n^2-3}\right| < \varepsilon$

$\Rightarrow \left|\frac{n+2}{n^2-3} - 0\right| < \varepsilon$ for $\forall \varepsilon > 0$. QED

(5)

16.7

(a) $\lim \frac{1}{1+3n} = 0$

We want an N s.t. for $n > N$ $\left|\frac{1}{1+3n} - 0\right| < \varepsilon$ for $\forall \varepsilon > 0$.

$\left|\frac{1}{1+3n}\right| < \varepsilon$ $\quad \frac{1}{1+3n} < \varepsilon$ $\quad 1+3n > 1/\varepsilon$ $\quad n > \frac{1/\varepsilon - 1}{3}$

$N = \left[\frac{1/\varepsilon - 1}{3}\right] + 1$

for $n > N$

$n > \frac{1/\varepsilon - 1}{3}$

$3n > 1/\varepsilon - 1$ $\quad 3n+1 > 1/\varepsilon$ $\quad 1/3n+1 < \varepsilon$

$\left|\frac{1}{1+3n} - 0\right| < \varepsilon$ QED

(b) $\lim \frac{4n^2-7}{2n^3-5} = 0$

We want an N s.t. for $n > N$ $\left|\frac{4n^2-7}{2n^3-5} - 0\right| < \varepsilon$ for $\forall \varepsilon > 0$

$4n^2 - 7 < 4n^2$

$2n^3 - 5 > n^3$ for $n^3 > 5$ $\quad n > \sqrt[3]{5}$

For $n > \sqrt[3]{5}$

$\frac{4n^2-7}{2n^3-5} < \frac{4n^2}{n^3} = \frac{4}{n}$ $\quad$ Let $\frac{4}{n} < \varepsilon$ $\quad n > 4/\varepsilon$.

Let $N = [\max\{\sqrt[3]{5}, 4/\varepsilon\}] + 1$

For $n > N$ $\quad n > 4/\varepsilon$ and $n > \sqrt[3]{5}$.

$n > 4/\varepsilon \Rightarrow \frac{4}{n} < \varepsilon \Rightarrow \frac{4n^2}{n^3} < \varepsilon \Rightarrow \frac{4n^2-7}{2n^3-5} < \varepsilon$

$\Rightarrow \left|\frac{4n^2-7}{2n^3-5} - 0\right| < \varepsilon$ For $\forall \varepsilon > 0$ $\quad$ QED

(c) $\lim \frac{6n^2+5}{2n^2-3n} = 3$

We want an N s.t. for $n > N$ $\left|\frac{6n^2+5}{2n^2-3n} - 3\right| < \varepsilon$ for $\forall \varepsilon > 0$

$\left|\frac{6n^2+5}{2n^2-3n} - 3\right| = \left|\frac{6n^2+5-6n^2+9n}{2n^2-3n}\right| = \left|\frac{5+9n}{2n^2-3n}\right|$

$5+9n < 9n$

$2n^2-3n > n^2$ when $n^2 > 3n$ $n > 3$ So for $n > 3$

$\frac{5+9n}{2n^2-3n} < \frac{9n}{n^2} = \frac{9}{n}$ Let $\frac{9}{n} < \varepsilon$ $n > 9/\varepsilon$.

Let $N = [\max\{9/\varepsilon, 3\}] + 1$

For $n > N$ $n > 9/\varepsilon$ and $n > 3$

$\because n > 9/\varepsilon$ $\frac{9}{n} < \varepsilon$ $\frac{9n}{n^2} < \varepsilon$ $\because n > 3$

$\frac{5+9n}{2n^2-3n} < \frac{9n}{n^2} < \varepsilon$ $\because 5+9n > 0$ & $2n^2-3n > 0$

$\therefore \left|\frac{5+9n}{2n^2-3n}\right| < \varepsilon \Rightarrow \left|\frac{6n^2+5}{2n^2-3n} - 3\right| < \varepsilon$ for $\forall \varepsilon > 0$ QED

(d) $\lim \frac{\sqrt{n}}{n+1} = 0$

We want an N s.t. for $n > N$. $\left|\frac{\sqrt{n}}{n+1} - 0\right| < \varepsilon$ for $\forall \varepsilon > 0$

$\left|\frac{\sqrt{n}}{n+1}\right| = \frac{\sqrt{n}}{n+1}$ $n+1 > n$ $\therefore \frac{\sqrt{n}}{n+1} < \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}}$

Let $1/\sqrt{n} < \varepsilon$ $\sqrt{n} > 1/\varepsilon$ $n > 1/\varepsilon^2$

Let $N = [1/\varepsilon^2] + 1$

For $n > N$ $n > 1/\varepsilon^2$ $1/\sqrt{n} < \varepsilon$

$\Rightarrow \frac{\sqrt{n}}{n} < \varepsilon \Rightarrow \frac{\sqrt{n}}{n+1} < \varepsilon$ $\left|\frac{\sqrt{n}}{n+1} - 0\right| < \varepsilon$ for $\forall \varepsilon > 0$

QED

(e) $\lim \frac{n^2}{n!} = 0$

We want an N s.t. for $n > N$ $\left|\frac{n^2}{n!} - 0\right| < \varepsilon$ for $\forall \varepsilon > 0$

$\left|\frac{n^2}{n!}\right| = \frac{n}{(n-1)!} = \frac{n}{n-1} \frac{1}{(n-2)!}$

$\frac{n}{n-1} < 2$ for $n < 2n-2$ $n > 2$ So If $n > 2$

$\frac{n}{(n-1)!} < \frac{2}{(n-2)!}$ $\because (n-2)! \geq n-2$ for $n > 2$.

$\frac{2}{(n-2)!} \leq \frac{2}{n-2} < \varepsilon$ $n-2 > \frac{2}{\varepsilon}$ $n > 2 + \frac{2}{\varepsilon}$

Let $N = \left[\max\left\{2+\frac{2}{\varepsilon}, 2\right\}\right] + 1$

If $n > N$ $n > 2 + \frac{2}{\varepsilon}$ & $n > 2$

$n-2 > \frac{2}{\varepsilon} \Rightarrow \frac{2}{n-2} < \varepsilon \Rightarrow \frac{2}{(n-2)!} < \varepsilon \Rightarrow \frac{n}{n-1} \cdot \frac{1}{(n-2)!} < \varepsilon$

$\Rightarrow \frac{n}{(n-1)!} < \varepsilon \Rightarrow \frac{n^2}{n!} < \varepsilon \Rightarrow \left|\frac{n^2}{n!} - 0\right| < \varepsilon$ for $\forall \varepsilon > 0$ QED

(f) If $|x| < 1$

We want an N s.t. for $n > N$ $|x^n - 0| < \varepsilon$ for $\forall \varepsilon > 0$

$|x^n| = |x|^n < \varepsilon$

$n \ln^{|x|} < \ln^{\varepsilon}$ $\because |x| < 1$ $\ln^{|x|} < 0 \Rightarrow n > \log^{\varepsilon}_{|x|}$

Let $N = [\log^{\varepsilon}_{|x|}] + 1$

for $n > N$ $n > \log^{\varepsilon}_{|x|} \Rightarrow n > \ln^{\varepsilon} / \ln^{|x|}$

$n \ln^{|x|} < \ln^{\varepsilon}$

$\Rightarrow e^{n \ln^{|x|}} < e^{\ln^{\varepsilon}} \Rightarrow |x|^n < \varepsilon \Rightarrow |x^n| < \varepsilon$

$\Rightarrow |x^n - 0| < \varepsilon$ for $\forall \varepsilon > 0$ QED

16.8

(a) $a_n = 2n$

If $\{a_n\}$ is convergent

Then $\{a_n\}$ is bounded (Theorem 16.13)

If $\{a_n\}$ is bounded. then $\exists L > 0$ s.t.

$|a_n| < L \Rightarrow |2n| < L \Rightarrow |n| < L \Rightarrow n < L$

Which goes aganist to Archimedean properey.

So $\{a_n\}$ is divergent.

(b) $b_n = (-1)^n$

Assume, b_n converges to some $B \in \mathbb{R}$

Let $\varepsilon = \frac{1}{2}$ We need an N s.t. for $n > N$

$|b_n - B| < 1/2$

let n_1 be even! $n_1 > N$

let n_2 be odd $n_2 > N$

$|b_{n_1} - b_{n_2}| = |(b_{n_1} - B) - (b_{n_2} - B)| \leq |b_{n_1} - B| + |b_{n_2} - B|$

< 1

But $|b_{n_1} - b_{n_2}| = 2 < 1$

contradiction

$\Rightarrow$ b_n is divergent

(c) $c_n = \cos \frac{n\pi}{3}$

Assume c_n converges to $L \in \mathbb{R}$

let $\varepsilon = \frac{1}{2}$ We need an N s.t for $n > N$ $|c_n - L| < \frac{1}{2}$

let $n_1 = 3k_1$ $k_1 \in$ odd number

let $n_2 = 3k_2$ $k_2 \in$ even number

$|c_{n_2} - c_{n_1}| = |(c_{n_2} - L) - (c_{n_1} - L)| \leq |(c_{n_2} - L)| + |(c_{n_1} - L)|$

< 1

But $|c_{n_2} - c_{n_1}| = 2$

Contradiction

$\Rightarrow$ c_n is divergent

(d) $d_n = (-n)^2$

If $\{d_n\}$ is convergent

Than it is bounded

Assume It is bounded

Then $\exists L > 0$ s.t

$|d_n| < L$ for $\forall n$

$\Rightarrow |(-n)^2| < L \Rightarrow (n)^2 < L \Rightarrow n^2 < L$

for $n \geq 1$ $n^2 \geq n$

$\therefore n < L$ for $\forall n$

which go against to Archimedean Property

So $\{d_n\}$ is divergent

16.9

(a) True.

If $\{S_n\}$ converges to S.

$\exists N$. s.t for $n > N$. $|S_n - S| < \varepsilon$ for $\forall \varepsilon > 0$

$(|S_n - S|)^2 < \varepsilon^2$ $S_n^2 + S^2 - 2S_n S < \varepsilon^2$

$|S_n|^2 + |S|^2 - 2S_n S < \varepsilon^2$

$\because |S_n|^2 + |S|^2 - 2|S_n||S| \leq |S_n|^2 + |S|^2 - 2S_n S$

$\therefore |S_n|^2 + |S|^2 - 2|S_n||S| < \varepsilon^2$

$\Rightarrow (|S_n| - |S|)^2 < \varepsilon^2$

$||S_n| - |S|| < \varepsilon$ for $\varepsilon > 0$ QED

(b) Flase

$(|(-1)^n|)$ is convergent

$(-1)^n$ is divergent

(c) True.

Assume : $\lim S_n = 0$

$\exists N > 0$ s.t. for $n > N$ $|S_n - 0| < \varepsilon$ for $\forall \varepsilon > 0$

$|S_n| < \varepsilon$ for $\forall \varepsilon$. $|S_n| - 0 < \varepsilon$ $\because |S_n| - 0 \geq 0$

$||S_n| - 0| < \varepsilon$. Holds

So $\lim |S_n| = 0$

Assume $\lim |s_n| = 0$

Then $\exists N$ s.t. for $n > N$ $||s_n| - 0| < \varepsilon$ for $\forall \varepsilon$

$||s_n|| < \varepsilon$ $|s_n| < \varepsilon$ $|s_n - 0| < \varepsilon$

So $\lim s_n = 0$ QED

16.15

(a). If $\exists$ a sequence (s_n) of $S \setminus \{x\}$ converges to x

Then $\exists N$ s.t. for $n > N$. $|s_n - x| < \varepsilon$ for $\forall \varepsilon > 0$

Hence for $n_1 > N$. $|s_{n_1} - x| < \varepsilon$

$s_{n_1} \in S$. $s_{n_1} \in N^*(x, \varepsilon)$

So $s_{n_1} \in N^*(x, \varepsilon) \cap S$

$\Rightarrow$ x is an accumulation point of S.

Assume: x is an accumulation point of S.

Then, for $\forall \varepsilon > 0$ $N^*(x, \varepsilon) \cap S \neq \phi$

Name a point $s_n \in N^*(x, \frac{1}{n}) \cap S$

let $N = 1/\varepsilon$.

For $\forall n > N$

$\because$ $s_n \in N^*(x, \frac{1}{n}) \cap S$

$|s_n - x| < \frac{1}{n} < \frac{1}{N} = \varepsilon$.

So that $\{s_n\}$ converges to x.

(b) S is closed $\Leftrightarrow$ All the accumulation points $S' \subset S$.

$\forall x \in S'$ is an accumulation point of S.

From (a).

We know

$x \in S' \Leftrightarrow \exists \{s_n\}$ in $S \setminus \{x\}$ s.t. $s_n \to x$. $x = \lim s_n \in S' \subset S$

$\because$ For $\forall x \in S'$, the claim above holds

So, S is closed $\Leftrightarrow$ whenever (s_n) is a convergent sequence of points in S, $\lim s_n \in S$.

QED

17.1

(a) True
theorem 17.1

(b). Flase
$\{\frac{1}{n}\} \to 0$.

(c) True.
Definiton 16.2

(d). Flase
$\lim (1/(-n)) = 0$
But $\lim(-n) = -\infty$

17.2

(a) True
Theorem 17.1

(b). Flase.
$\{S_n\}$ is diverge to $+\infty$

(c) True.

(d) Flase.
$\frac{S_{n+1}}{S_n} = -2$ $S_1 = 1$ $S_2 = -2$ $S_3 = 4$ ...
$S_n = (-2)^{n-1}$ $\{S_n\}$ is not convergent

But consider If here means $|L| < 1$
$\exists N$ s.t for $n > N$ $|S_{n+1}/S_n - L| < \varepsilon$.
$|S_{n+1}/S_n| \le |L| + \varepsilon < 1$. Archimedean
Based on the density of Q $\exists C \in Q$ s.t. $|L| + \varepsilon < C < 1$
$|S_{n+1}/S_n| \le C$ $|S_{n+1}| \le C^n |S_1|$. $0 < C < 1$. $\therefore C^n \to 0$
From 16.8 $\lim S_n = 0$.

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17.4

(a) $\lim s_n = s$ Then

$\lim(k s_n) = ks$ $\lim(k+s_n) = k+s$ for $\forall k \in \mathbb{R}$

$\because \exists N$ s.t for $n>N$ $|s_n - s| < \varepsilon$ for $\forall \varepsilon > 0$.

$\exists N_1$ s.t. for $n>N_1$ $|s_n - s| < \varepsilon/|k|$ for $\forall \varepsilon > 0$

$|k||s_n - s| < \varepsilon$ for $\forall \varepsilon > 0$.

$|ks_n - ks| = |k||s_n - s| < \varepsilon/|k| \cdot |k| = \varepsilon$

$\lim(ks_n) = ks$.

(b). If (t_n) converges to t and $t_n \geq 0$ for $\forall n \in \mathbb{N}$ then $t \geq 0$.

If Not

$t<0$

$|t_n - t| < \varepsilon$ for $\forall \varepsilon > 0$ when $n > N$.

$\because t_n \geq 0$ for $\forall n \in \mathbb{N}$ $\because t < 0$

$|t_n - t| = t_n - t < \varepsilon$ $t_n < t + \varepsilon$

Let $\varepsilon = |t|/2$ $t+\varepsilon = t + |t|/2 = -|t|/2$

$t_n < -|t|/2 < 0$

Contradiction.

So $t \geq 0$.

17.5

(a) $\lim s_n = -2$

(b) $\lim s_n = 0$

(c). divergent

(d) $s_n = (\frac{8}{9})^n$

$\lim s_n = 0$