Midterm Exam
Math 117
Winter 2013
Prof. R. Ye

Your Name:
Your Signature:
Your Perm Number:

Scores:
1.
2.
3.
4.
Total: (out of 100)

Please present detailed steps of your solutions.
1. (25 points) For each \( n \in \mathbb{N} \), let \( P(n) \) be a statement concerning \( n \) which is either true or false. Assume the following two conditions:

1) \( P(1) \) and \( P(2) \) are true.

2) For each \( n \in \mathbb{N} \), if \( P(n) \) is true, then \( P(n + 2) \) is also true.

Prove that \( P(n) \) is true for all \( n \in \mathbb{N} \). Hint: You can use the well-ordering axiom, the principle of mathematical induction, or Peano axioms.

**Proof 1** Define \( S = \{ n \in \mathbb{N} : P(n) \text{ is false} \} \). We claim that \( S = \emptyset \). Assume that \( S \) is nonempty. By the well-ordering axiom of natural numbers, there is a smallest number \( n_0 \in S \). Since \( P(1) \) and \( P(2) \) are true, there holds \( n_0 \geq 3 \). Then \( n_0 - 2 \in \mathbb{N} \). Since \( n_0 - 2 < n_0 \), there holds \( n_0 - 2 \notin S \), otherwise \( n_0 \) would not be the smallest number in \( S \). Hence \( P(n_0 - 2) \) is true. By the assumption 2) we infer that \( P(n_0) \) is true, contradicting the fact that \( n_0 \in S \). We conclude that \( S \) is empty, and hence \( P(n) \) is true for all \( n \in \mathbb{N} \). 

**Proof 2** Set \( Q(n) = P(2n) \) for \( n \in \mathbb{N} \). Then \( Q(1) = P(2) \) is true. If \( Q(n) = P(2n) \) is true, then \( Q(n + 1) = P(2n + 2) \) is also true by the assumption 2). By the principle of mathematical induction, \( Q(n) = P(2n) \) is true for all \( n \in \mathbb{N} \), i.e., \( P(n) \) is true for all even natural numbers \( n \).

Next set \( H(n) = P(2n - 1) \). Then \( H(1) = P(1) \) is true. If \( H(n) = P(2n - 1) \) is true, then \( H(n + 1) = P((2n - 1) + 2) \) is true by the assumption 2). By the principle of mathematical induction we infer that \( H(n) = P(2n - 1) \) is true for all \( n \in \mathbb{N} \). This means that \( P(n) \) is true for all odd natural numbers \( n \).

Combining the above two conclusions we infer that \( P(n) \) is true for all \( n \in \mathbb{N} \). ■
2. 25 points) 1) Assume \(0 < x < y\). Show \(0 < 1/y < 1/x\).
2) Assume \(x^2 = y^2\). Determine the relation between \(x\) and \(y\). (A detailed proof is required.)

**Proof.**

1) First we show that \(1/x > 0\). Assume \(1/x \neq 0\). Since \(x\) is positive, we can multiply this inequality by \(x\) to deduce 
\[0 \cdot x > 1/x \cdot x.\]
Hence \(0 > 1\). This contradicts the fact that \(1 > 0\), which has been established in the lectures. It follows that \(1/x > 0\), Similarly, we have \(1/y > 0\).

(The proof for \(1 > 0\) is as follows. Assume \(1 > 0\). Adding \(-1\) to both sides of this inequality we deduce 
\[−1 > 0.\] Multiplying with \(-1\) we infer \((-1)^2 > 0:\)
Hence \(1 > 0\), contradicting the assumption \(0 > 1\).

Now we multiply the inequality \(x < y\) by \(1/x\) to deduce \(1 < y/x\). Then we multiply by \(1/y\) to arrive at \(1/y < 1/x\).)

2) Assume \(x^2 = y^2\). Then \(x^2 - y^2 = 0\), and hence \((x + y)(x - y) = 0\). It follows that 
\(x + y = 0\) or \(x - y = 0\). Conversely, if \(x + y = 0\) or \(x - y = 0\), then \((x + y)(x - y) = 0\), and hence \(x^2 - y^2 = 0\), i.e. \(x^2 = y^2\). We conclude that the relation between \(x\) and \(y\) is 
\(x = y\) or \(x = -y\). In other words, the relation is \(|x| = |y|\).

**An Alternative Argument:**

Assume \(x \neq 0\). Then \(y \neq 0\). (Otherwise \(x^2 = 0^2 = 0\), and hence \(x = 0\).) We first claim \(|x| \leq |y|\). Assume \(|x| > |y|\). Multiplying this inequality with \(|x|\) we infer 
\[|x|^2 > |x||y|.\] Multiplying the same inequality with \(|y|\) we deduce \(|x||y| > |y|^2\). It follows that \(|x|^2 > |y|^2\), which means \(x^2 > y^2\), contradicting the assumption \(x^2 = y^2\).
Hence we conclude that \(|x| \leq |y|\). In a similar way we can prove the claim \(|x| \geq |y|\).
Hence we infer that \(|x| = |y|\).

If \(x = 0\), then \(y^2 = x^2 = 0\) and hence \(y = 0\). Hence we also have \(|x| = |y|\).
Conversely, if \(|x| = |y|\), then we infer \(|x|^2 = |y|^2\), which means \(x^2 = y^2\).

In conclusion, the equality \(x^2 = y^2\) is equivalent to the equation \(|x| = |y|\), which is equivalent to the following relation: \(x = y\) or \(x = -y\).
3. (25 points) Let \( x \in \mathbb{R} \). Prove that \( x = \sup \{ q \in \mathbb{Q} : q < x \} \).

**Proof** 1) Define \( S = \{ q \in \mathbb{Q} : q < x \} \). It is bounded above because \( x \) is an upper bound for it. By the Archimedean property, we can find a natural number \( n > -x \). Then \( -n < x \), and hence \( -n \in S \). Thus \( S \) is nonempty. By the completeness axiom, \( y = \sup S \) exist.

2) Since \( x \) is an upper bound for \( S \) and \( y \) is the least upper bound for \( S \), there holds \( y \leq x \).

3) We claim that \( y = x \). Assume \( y \neq x \). By 2) we then have \( y < x \). By the density of \( \mathbb{Q} \), there exists a number \( r \in \mathbb{Q} \) such that \( y < r < x \). Then \( r \in S \), and hence \( r \leq y \), contradicting the fact that \( y < r \). \( \blacksquare \)
4. (25 points) Let \( S = \{ \frac{1}{n} + \frac{1}{m} - \frac{1}{k} \text{ for all } n, m, k \in \mathbb{N} \} \).

1) Find \( \text{sup } S \) and \( \text{inf } S \).

2) Does \( \text{max } S \) exist? Why?

3) Does \( \text{min } S \) exist? Why?

**Proof**

1) For a fixed \( k \), the maximum of \( \frac{1}{n} + \frac{1}{m} - \frac{1}{k} \) is \( 1 + 1 - \frac{1}{k} = 2 - \frac{1}{k} \). Letting \( k \) become bigger and bigger we see that \( \text{sup } S = 1+1-0 = 2 \). To get the infimum, we take \( k = 1 \) and let \( n \) and \( m \) become bigger and bigger. It follows that \( \text{inf } S = 0 + 0 - 1 = -1 \).

2) \( \text{max } S \) does not exist. Assume that it exists. Then \( \text{max } S = \text{sup } S = 2 \). But \( \text{max } S = \frac{1}{n} + \frac{1}{m} - \frac{1}{k} \) for some \( n, m \) and \( k \). It follows that \( 2 = \text{max } S \leq 1 + 1 - \frac{1}{k} < 2 \). This is a contradiction.

3) \( \text{min } S \) does not exist. Assume that it exists. Then \( \text{min } S = \text{inf } S = -1 \). But \( \text{min } S = \frac{1}{n} + \frac{1}{m} - \frac{1}{k} \) for some \( n, m \) and \( k \). Hence \( \text{min } S \geq \frac{1}{n} + \frac{1}{m} - 1 > -1 \). This is a contradiction. \( \blacksquare \)