Midterm Exam
Math 124B
Summer 2014
Prof. R. Ye

Your Name:
Your Signature:
Your Perm Number:

Scores:
1.
2.
3.
4.
Total: (out of 100)

Please present detailed steps of your solutions.

Write letters in larger size, and press harder when writing with pencils!

1. (30 points) Solve the equation $u_{xx} + u_{yy} = x^2 + y^2$ in $r < 2$ with $u = 1$ on $r = 2$. Hint: Use the polar coordinates.

Solution In polar coordinates, the equation $u_{xx} + u_{yy} = x^2 + y^2$ becomes

$$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = r^2. \tag{0.1}$$

We look for a solution $u = u(r, \theta) = u(r)$ which is independent of θ. Then this equation becomes

$$\frac{1}{r}\frac{d}{dr}(r\frac{du}{dr}) = r^2. \tag{0.2}$$

We deduce

$$\frac{d}{dr}(r\frac{du}{dr}) = r^3. \tag{0.3}$$

Integrating yields

$$r\frac{du}{dr} = \frac{1}{4}r^4 + C_1. \tag{0.4}$$

Hence

$$\frac{du}{dr} = \frac{1}{4}r^3 + \frac{C_1}{r}. \tag{0.5}$$

It follows that

$$u = \frac{1}{16}r^4 + C_1 \ln r + C_2. \tag{0.6}$$

Since $\ln r$ is singular at the origin, we discard it. Setting $r = 2$ we then infer $u = 1 + C_2$. But $u = 1$ for $r = 1$, hence $C_2 = 0$. It follows that $u = \frac{1}{16}r^4$. ■

2. (25 points) Solve $u_{xx} + u_{yy} = 0$ in the disk $r < 3$ with the boundary condition $u = 3 + 2xy + x^2 - y^2$. Hint: $\cos(2\theta) = \cos^2\theta - \sin^2\theta$.

Solution In polar coordinates we have $x = r\cos\theta, y = r\sin\theta$. Hence the boundary condition can be written as follows:

$$\begin{aligned} u &= 3 + 2r^2(\cos\theta)(\sin\theta) - r^2(\cos^2\theta - \sin^2\theta) \\ &= 3 + 3^2\sin(2\theta) + 3^2\cos(2\theta). \end{aligned} \tag{0.7}$$

The general solution is given by

$$u = \frac{a_0}{2} + \sum_{n\ ge1} r^n(a_n\cos(n\theta) + b_n\sin(n\theta)). \tag{0.8}$$

We deduce that the desired (unique) solution is given by

$$u = 3 + r^2(\sin(2\theta) + \cos(2\theta)). \tag{0.9}$$

3. (30 points) 1) Let u be a harmonic function in the disk $r < a$ with the boundary condition $u = h$. Write down the Poisson formula for $u = u(r, \theta)$.
2) Show that the function

$$P(r,\theta) = \frac{a^2 - r^2}{a^2 - 2ar\cos\theta + r^2} \tag{0.10}$$

in the Poisson formula is harmonic.

Solution 1) The Poisson formula is as follows

$$u(r,\theta) = \frac{a^2 - r^2}{2\pi} \int_0^{2\pi} \frac{h(\phi)}{a^2 - 2ar\cos(\theta - \phi) + r^2} d\phi \tag{0.11}$$

2) **Approach 1** Let $z = re^{i\theta}$. We calculate

$$\mathbf{Re}(1 + 2\frac{z}{a - z}) = 1 + \frac{z}{a - z} + \frac{\bar{z}}{a - \bar{z}}, \tag{0.12}$$

where **Re** denotes the real part and $\bar{z} = re^{-i\theta}$ is the complex conjugate of z. It follows that

$$\begin{aligned} \mathbf{Re}(1 + 2\frac{z}{a - z}) &= 1 + \frac{z(a - \bar{z}) + \bar{z}(a - z)}{(a - z)(a - \bar{z})} \\ &= 1 + \frac{az - 2z\bar{z} + a\bar{z}}{a^2 - az - a\bar{z} + z\bar{z}} \\ &= 1 + \frac{are^{i\theta} - 2r^2 + are^{-i\theta}}{a^2 - are^{i\theta} - are^{-i\theta} + r^2}. \end{aligned} \tag{0.13}$$

Applying the Euler formula $e^{i\theta} = \cos\theta + i\sin\theta$ we then infer

$$\begin{aligned} \mathbf{Re}(1 + 2\frac{z}{a - z}) &= 1 + \frac{2ar\cos\theta - 2r^2}{a^2 - 2ar\cos\theta + r^2} \\ &= \frac{a^2 - r^2}{a^2 - 2ar\cos\theta + r^2} = P(r,\theta). \end{aligned} \tag{0.14}$$

The function $1 + 2\frac{z}{a-z}$ is complex analytic (for $z \neq a$), and hence its real part is harmonic. It follows that $P(r,\theta)$ is harmonic.

Approach 2 Let $\mathbf{x} = (x, y) = (r\cos\theta, r\sin\theta)$. Then $|\mathbf{x}| = r$, $|\mathbf{x}|^2 = x^2 + y^2$ and $|\mathbf{x} - (a, 0)|^2 = (x - a)^2 + y^2$. We calculate

$$\begin{aligned} |\mathbf{x} - (a,0)|^2 &= |\mathbf{x}|^2 + a^2 - 2\mathbf{x} \cdot (a, 0) \\ &= r^2 - 2ar\cos\theta + a^2. \end{aligned} \tag{0.15}$$

It follows that

$$\begin{aligned} P(r,\theta) &= \frac{1}{2\pi} \frac{a^2 - |\mathbf{x}|^2}{|\mathbf{x} - (a,0)|^2} \\ &= \frac{1}{2\pi} \frac{a^2 - x^2 - y^2}{(x - a)^2 + y^2}. \end{aligned} \tag{0.16}$$

Consequently, we have

$$\begin{aligned}\frac{\partial}{\partial x}P(r,\theta) &= \frac{1}{2\pi}\cdot\frac{\partial_x(a^2-x^2-y^2)((x-a)^2+y^2)-\partial_x((x-a)^2+y^2)(a^2-x^2-y^2)}{((x-a)^2+y^2)^2} \\ &= \frac{1}{2\pi}\cdot\frac{-2x((x-a)^2+y^2)-2(x-a)(a^2-x^2-y^2)}{((x-a)^2+y^2)^2} \\ &= \frac{a}{\pi}\cdot\frac{(x-a)^2-y^2}{((x-a)^2+y^2)^2}. \qquad (0.17)\end{aligned}$$

We then deduce

$$\begin{aligned}\frac{\partial^2}{\partial x^2}P(r,\theta) &= \frac{a}{\pi}\cdot\frac{(\partial_x((x-a)^2-y^2)((x-a)^2+y^2)^2-\partial_x((x-a)^2+y^2)^2((x-a)^2-y^2)}{((x-a)^2+y^2)^4} \\ &= \frac{a}{\pi}\cdot\frac{2(x-a)((x-a)^2+y^2)^2-4((x-a)^2+y^2)(x-a)((x-a)^2-y^2)}{((x-a)^2+y^2)^4} \\ &= \frac{2a}{\pi}(x-a)((x-a)^2+y^2)\frac{3y^2-(x-a)^2}{((x-a)^2+y^2)^4}. \qquad (0.18)\end{aligned}$$

Similarly, we have

$$\begin{aligned}\frac{\partial}{\partial y}P(r,\theta) &= \frac{1}{2\pi}\cdot\frac{\partial_y(a^2-x^2-y^2)((x-a)^2+y^2)-\partial_y((x-a)^2+y^2)(a^2-x^2-y^2)}{((x-a)^2+y^2)^2} \\ &= \frac{1}{2\pi}\cdot\frac{-2y((x-a)^2+y^2)-2y(a^2-x^2-y^2)}{((x-a)^2+y^2)^2} \\ &= \frac{2a}{\pi}\cdot\frac{y(x-a)}{((x-a)^2+y^2)^2} \qquad (0.19)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2}{\partial y^2}P(r,\theta) &= \frac{2a}{\pi}\cdot\frac{\partial_y[y(x-a)]\cdot((x-a)^2+y^2)-\partial_y((x-a)^2+y^2)^2\cdot y(x-a)}{((x-a)^2+y^2)^4} \\ &= \frac{2a}{\pi}\cdot\frac{(x-a)((x-a)^2+y^2)^2-2((x-a)^2+y^2)2y\cdot y(x-a)}{((x-a)^2+y^2)^4} \\ &= \frac{2a}{\pi}(x-a)((x-a)^2+y^2)\frac{(x-a)^2-3y^2}{((x-a)^2+y^2)^4}. \qquad (0.20)\end{aligned}$$

Combining the above results we then arrive at $\Delta P(r,\theta)=0$. ■

Approach 3 This approach is to employ polar coordinates. The calculation here is similar to the one in Approach 2. Here is the first step of the calculation. Further steps are omitted here, as the calculation in Approach 2 has been presented in details.

There holds

$$\begin{aligned}\partial_r P(r,\theta) &= \frac{(\partial_r(a^2-r^2)(a^2-2ar\cos\theta+r^2)-(a^2-r^2)\partial_r(a^2-2ar\cos\theta+r^2)}{(a^2-2ar\cos\theta+r^2)^2} \\ &= \frac{(-2r(a^2-2ar\cos\theta+r^2)-(a^2-r^2)(-2a\cos\theta+2r)}{(a^2-2ar\cos\theta+r^2)^2} \\ &= 2a\frac{-2ar+(a^2+r^2)\cos\theta}{(a^2-2ar\cos\theta+r^2)^2}. \qquad (0.21)\end{aligned}$$

4. (15 points) Solve the equation $u_{xx} + u_{yy} = 0$ in the quarter disk $\{x^2 + y^2 < a^2, x > 0, y > 0\}$ with the boundary condition $u = 0$ on $x = 0$, $u = 0$ on $y = 0$, and $\frac{\partial u}{\partial r} = 1$ on $r = a$.

Solution The quarter disk is a wedge with angle $\beta = \frac{\pi}{2}$. Hence we have

$$\begin{aligned} u(r,\theta) &= \sum_{n\ ge1} A_n r^{n\pi/\beta} \sin\frac{n\pi\theta}{\beta} \\ &= \sum_{n\ ge1} A_n r^{2n} \sin(2n\theta), \end{aligned} \tag{0.22}$$

and

$$\begin{aligned} A_n &= \frac{2a^{1-n\pi/\beta}}{n\pi} \int_0^{\pi/2} 1 \cdot \sin\frac{n\pi\theta}{\beta} d\theta \\ &= \frac{2a^{1-2n}}{n\pi} \int_0^{\pi/2} \sin(2n\theta) d\theta \\ &= \frac{2a^{1-2n}}{n\pi} \frac{1}{2n} (1 - \cos(n\pi)) \\ &= \frac{a^{2n-1}}{n^2\pi} (1 - (-1)^n). \end{aligned} \tag{0.23}$$

We arrive at

$$u(r,\theta) = \sum_{n\geq 1} \frac{a^{1-2n}}{n^2\pi} (1 - (-1)^n) r^{2n} \sin(2n\theta). \tag{0.24}$$