1. Find all the critical points of the function

$$ z = \frac{1}{1 + x^2 + y^2} $$

and use the second derivative test to determine their type (i.e. relative minimum, relative maximum or saddle points).

**SOLUTION:** First we compute the gradient of $z$:

$$ \nabla z = \left(-\frac{2x}{(1 + x^2 + y^2)^2}, -\frac{2y}{(1 + x^2 + y^2)^2}\right). $$

The critical points are solutions of $\nabla z = (0, 0)$. The only such point is the point $(0, 0)$. To determine the type, we must calculate the second derivatives:

$$ \frac{\partial^2 z}{\partial x^2}(0, 0) = -2 $$

$$ \frac{\partial^2 z}{\partial x \partial y}(0, 0) = 0 $$

$$ \frac{\partial^2 z}{\partial y^2}(0, 0) = -4 $$

Thus $D = z_{xx}z_{yy} - z_{xy}^2 = 4 > 0$ and $z_{xx} = -2 < 0$, so the critical point is a maximum.
2. Find all the critical points of the function
\[ z = e^y \sin x \]
and use the second derivative test to determine their type.

SOLUTION: First, the gradient of \( z \) is
\[ \nabla z = \left( y \cos x e^{y \sin x}, \sin x e^{y \sin x} \right). \]
\( \nabla z = 0 \) if and only if \( \sin x = 0 \) and \( y \cos x = 0 \). For \( \sin x = 0 \), we must have \( x = n\pi \) for some integer \( n \). Then \( \cos x = (-1)^n \), and so we must have \( y = 0 \). That is, the critical points are the points \( (n\pi, 0) \), for \( n \) any integer.

Next we must compute the second derivatives:
\[ \frac{\partial^2 z}{\partial x^2} = -y \sin x e^{y \sin x} + y^2 \cos^2 x e^{y \sin x} \]
\[ \frac{\partial^2 z}{\partial x \partial y} = \cos x e^{y \sin x} + y \cos x \sin x e^{y \sin x} \]
\[ \frac{\partial^2 z}{\partial y^2} = \sin^2 x e^{y \sin x} \]

So, at the critical point \( (n\pi, 0) \), \( z_{xx} = 0 \), \( z_{xy} = \cos(n\pi) = (-1)^n \), and \( z_{yy} = 0 \). Thus \( D = z_{xx}z_{yy} - z_{xy}^2 = -1 \leq 0 \), so every critical point is a saddle point.
3. Find the following double integral:

\[ \int \int_{D} xy^2 \, dx \, dy, \]

where \( D \) is the region bounded by the parabola \( y = x^2 \) and the parabola \( x = y^2 \).

SOLUTION: The two parabolas intersect at the points \((0, 0)\) and \((1, 1)\), and the parabola \( x = y^2 \) lying above \( y = x^2 \). Thus we can write the double integral as an iterated integral as:

\[ \int_{0}^{1} \int_{x^2}^{\sqrt{x}} xy^2 \, dy \, dx. \]

We then calculate

\[
\int_{0}^{1} \int_{x^2}^{\sqrt{x}} xy^2 \, dy \, dx = \int_{0}^{1} \left. \left( \frac{1}{3}xy^3 \right) \right|_{y=x^2}^{\sqrt{x}} \, dx \\
= \int_{0}^{1} \left( \frac{1}{3}x^{5/2} - x^7 \right) \, dx \\
= \left( \frac{2}{21}x^{7/2} - \frac{1}{24}x^8 \right) \bigg|_{0}^{1} \\
= \frac{2}{21} - \frac{1}{24} \\
= \frac{3}{56}
\]
4. Find the divergence of

\[ \mathbf{F}(x, y, z) = xy \mathbf{i} + yz \mathbf{j} + xz^2 \mathbf{k} \]

(2) Find the curl of the vector field in (1).

SOLUTION: (1) The divergence of \( \mathbf{F} \) is

\[ (\partial_x, \partial_y, \partial_z) \cdot (xy, yz, xz^2) = y + z + z^2. \]

(2) The curl of \( \mathbf{F} \) is

\[
\begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\partial_x & \partial_y & \partial_z \\
x y & y z & x z^2 \\
\end{vmatrix} = (-y, -z^2, -x).
\]
5. Find 
\[ \int_C \mathbf{F} \cdot d\mathbf{r}, \]
where \( \mathbf{F} \) is the vector field given by 
\[ \mathbf{F}(x, y) = (xy + 1, xy - 1) \]
and \( C \) is the curve 
\[ r(t) = (t^2 + 1, t^2 - 1), 0 \leq t \leq 1. \]

SOLUTION: By definition,
\[ \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \mathbf{F}(r(t)) \cdot r'(t) \, dt. \]
We calculate 
\[ \mathbf{F}(r(t)) = (t^4, t^4 - 2) \]
and 
\[ r'(t) = (2t, 2t). \]
Thus 
\[ \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 4t^5 - 4t \, dt 
\quad = \frac{2}{3} - 2 
\quad = -\frac{4}{3} \]
6. Use Green’s Theorem to evaluate the line integral
\[ \int_C (x^2y^2 + e^{x^2}) \, dx + (xy^2 - \sin(y^3)) \, dy, \]
where \( C \) is the boundary of the rectangle \([0, 1] \times [0, 1]\) oriented clockwise.

**SOLUTION:** Recall that Green’s Theorem says that, if \( C \) is a counterclockwise oriented curve, then
\[
\int_C P \, dx + Q \, dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dx \, dy.
\]
In our problem, we have that
\[
P = x^2y^2 + e^{x^2},
\]
\[
Q = xy^2 - \sin(y^3),
\]
and so, by Green’s Theorem
\[
\int_C (x^2y^2 + e^{x^2}) \, dx + (xy^2 - \sin(y^3)) \, dy = -\int_0^1 \int_0^1 (Q_x - P_y) \, dx \, dy
\]
\[
= -\int_0^1 \int_0^1 (y^2 - 2xy^2) \, dx \, dy
\]
\[
= -\int_0^1 (y^2 - y^2) \, dy
\]
\[
= 0.
\]