QUANTUM MATTERS

What is the matter? Einstein's general relativity unveils the origin of ordinary matter, and Landau's symmetry breaking and phase transition theory reveals the secret of classical phases of matter. How about quantum matter and quantum phases of matter?[1]

Quantum mechanics is a set of rules that predict the responses of the microscopic world to our measuring devices. The most salient feature is the superposition of different states. With the advent of quantum information science, another quantum correlation comes to center stage: entanglement–"the spooky action at a distance" according to Einstein.

The conventional formalism of quantum theory consists of four principles: superposition, linear evolution, entanglement, and probabilistic outcome of a measurement. The mathematical model to embody superposition and linear evolution is through Hilbert spaces and unitary transformations. We arrive at such a counter-intuitive model through a long period of trial-and-error in theory and experiment. The most controversial principle is measurement, though it models well the uncontrollable disturbance of the fragile quantum state by our measuring devices. With so much energy injected from our measuring apparatus into the quantum system, what we obtain is like a snapped string from an overwhelming plucking.

Mathematical model of quantum mechanics assigns a Hilbert space to each quantum system. Two operations on Hilbert spaces that are used to describe superposition and entanglement, respectively, are the direct sum and the tensor product. While interference is arguably more fundamental, a deeper understanding of quantum mechanics would come from the interplay of the two. The role for entanglement is more pronounced for many-body quantum systems such as systems of 10^{11} electrons in condensed matter physics. Advancing our understanding of the role of entanglement for both quantum computing and condensed matter physics lies at the frontiers of current research.

Section 1 serves mainly as motivation for our mathematical pursuit, and Section 2 introduces mathematical quantum systems through the Hamiltonian formalism. Logically, Section 1 is independent of the pure mathematics that we begin from Section 2. But some general knowledge of the materials will make many later topics more natural and is essential for the formulation of a mathematical theory of quantum phases of matter.

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[1]Calculate how hot you are and your angular frequency using $E = mc^2 = kT = \hbar\omega$. Do these numbers make sense?

1. Topological Qubit Liquids

This section is about the new quantum phases of matter that we are interested in. We do not have a mathematical theory yet, so we will just outline the ways how physicists are approaching the subject. Our goal is not to make physicists' common practice rigorous, rather to use physics as inspiration to develop new quantum mathematics. There are many questions that we cannot answer mathematically yet. We will proceed by studying many examples first and then try to axiomatize what we are interested in mathematically. Hopefully in the end, what we do as pure mathematics will be helpful for our understanding of the physical world.

1.1. **Spin Liquids.** Two fascinating macroscopic quantum phenomena are the high temperature superconductivity and the fractional quantum Hall effect (FQHE). Both classes of quantum matters are related to topological qubit liquids[2]. While the quantum Hall liquids are universally accepted examples of topological phases of matter, the role of topology in high temperature superconductors is still under debate. But the new state of matter "quantum spin liquid" suggested for high temperature superconductors by P. Anderson is beautifully realized by a mineral: herbertsmithite.

1.1.1. *Herbertsmithite.* In 1972, Adib and Ottemann's effort to make "anarakite" into a new mineral species failed. Over the years better experimental results eventually established "anarakite" being of species rank, and the mineral is renamed herbertsmithite in honor of Dr. G. F. Herbert Smith. Recently, single crystal sample of herbersmithite was grown in MIT labs and experiments provide strong evidence that herbertsmithite is in a quantum spin liquid state—an exotic form of quantum magnetism. Various experiments find no conventional order for herbertsmithite down to the low temperature of 50 mk, and neutron scattering is consistent with theoretically predicted spinon excitations. Experimental data so far also point to a gapless spin liquid, but disorder and other factors can easily hide a small gap.

Herbertsmithite is a mineral with the chemical formula $Cu_3Zn(OH)_6Cl_2$. The unit cell has three layers of copper ions Cu^{2+} which form three perfect kagome lattices in three parallel planes. The copper ion planes are separated by zinc and chloride planes. Since the Cu^{2+} planes are weakly coupled and Cu^{2+} has a $S = \frac{1}{2}$ magnetic moment, herbertsmithite can be modeled as a perfect $S = \frac{1}{2}$ kagome antiferromagnet with perturbations.

[2]The common term for them in physics is "spin liquids", as we will call them some time too. But the word "spin" implicitly implies an SU(2) symmetry, which is not present in general. Therefore, we prefer "qubit liquids" or really "qudit liquids", and use "spin liquids" only for those with an SU(2) symmetry.

1.1.2. *Fundamental Hamiltonian vs Model Hamiltonian.* Theoretically, we can write down the *fundamental*[3] Hamiltonian for a herbertsmithite crystal and try to find its ground state wave function. Then we would deduce its physical properties from the ground state wave function. Unfortunately, this fundamental approach from first principle physics cannot be implemented in almost all realistic systems in condensed matter physics. Therefore, what we do instead in condensed matter physics is to have an educated guess: to write down a model Hamiltonian and derive physical properties of the state of matter from this model system. This emergent approach has been extremely successful in our understanding of the fractional quantum Hall effect.

The model Hamiltonian for herbertsmithite is the Heisenberg $S = \frac{1}{2}$ kagome antiferromagnet with perturbations:

$$H = J \sum_{\langle i,j \rangle} S_i \cdot S_j + H_{\text{pert}},$$

where J is the exchange energy, and S_i are the spin operators (we will discuss them more carefully later). Experiments show that $J = 170K \sim 190K$. The indices i, j refer to the vertices of the lattice (also called sites in physics), and $\langle i, j \rangle$ means that the sum is over all pairs of vertices that are nearest neighborhood of each other.

1.1.3. *Competing Models.* There are many perturbations (small terms) to the Heisenberg spin exchange Hamiltonian H. Three prominent ones are the anisotropy of the spin exchange action, the next nearest neighbor interaction, and the disorder from impurities in the zinc plane. Different perturbations lead to different potential spin liquid states: numerical simulation using DMRG shows that next nearest neighborhood exchange will stabilize a gapped spin liquid, while other models lead to gapless spin liquids. These competing models can be organized into a phase diagram, which represents the rich physics of the different phases when parameters of the model Hamiltonian H change.

1.1.4. *Toric Code on the Honeycomb Lattice.* The Heisenberg model of herbertsmithite with next nearest neighborhood interaction is found to be in a topological spin liquid state in the same universal class of the toric code, or the $\mathbb{Z}_2$-gauge theory. This is a numerical result so we will it a *physical theorem.*

[3]By fundamental here, we mean the Hamiltonian comes from first physical principles such as Coulomb's law. Of course they are also model Hamiltonians. By model here, we mean that we describe the system with some effective degrees of freedom, and keep only the most relevant part of the interaction and treat everything else as small perturbations.

Physical Theorem:

For $J_1 > 0, 0.05 < \frac{J_2}{J_1} < 0.15,$, the spin $S = \frac{1}{2}$ antiferromagnetic $J_1 - J_2$ Heisenberg model $H_{J_1,J_2} = J_1 \sum_{\langle i,j \rangle} S_i \cdot S_j + J_2 \sum_{\langle\langle i,j \rangle\rangle} S_i \cdot S_j$, where $\langle\langle i,j \rangle\rangle$ means a sum over next nearest neighborhood, represents a topological phase in the same universal class as the toric code.

The spin=$\frac{1}{2}$ moments in the kagome Heisenberg model sit on the sites (or vertices) of the kagome lattice, while the qubits of the toric code are on the edges of the underlying graph. But the kagome lattice is the the medial graph of the honeycomb lattice. Therefore, numerical simulation proves that the kagome Heisenberg model on the kagome lattice with next nearest neighborhood interaction is in the same universal class as the toric code on the honeycomb lattice. The toric code is one of the most beautiful rigorously solvable lattice models in physics, and an understanding of this relation as part of the phase diagram for herbertsmithite will be very illuminating. More speculative would be the possible appearance of DFib phase in the phase diagram of herbertsmithite.

1.2. **Physical Quantization.** What is a quantum system? What is a quantum phase of matter? How to characterize and classify them? Where do Hilbert spaces come from? Where do the model Hamiltonians and observables come from? What are the principles to follow? What is a phase diagram? What is a phase transition? What does it mean that a model is exactly solvable or rigorously solvable?

When we have answers to all these questions, our subject will be so mature and cease to be exciting. Fortunately, Nature would never fail to surprise us with her new tricks and present us with new questions.

1.2.1. *The Statehood of a Quantum State.* By treating classical theories as limits of quantum theories, we consider the classical world as part of the quantum world. By examining things around us, we can easily grasp the idea of a state of a classical object. In quantum world, classical states are eigenstates of the Hamiltonian operator. But without the aid of eyes and senses, it is hard to believe that quantum systems have definite states. After all, the most well-known feature of quantum theory is the uncertainty principle. Then is the wave function or the state vector *real*? If so, how can we establish the statehood of a quantum state? The existence of a quantum state can be derived from its properties and its relations with other states. The two-tier structure of a category implements this philosophy perfectly: an object x in a category is something that we are interested in, but hard to get the feel for. Instead of giving up, we delve into its relations with other objects y which are embodied in the collection of sets Hom(x,y) for all y. This is analogous to the important idea in mathematics: representation theory. Following this idea, the map Hom(x,—) from x to Hom(x,y) is a representation of the object x

into something that we are more familiar with—sets. The object x is completely characterized by the sets Hom(x,y).

1.2.2. *Unitarity and Locality.* In the sense above, all physical systems are quantum systems. Their theoretical models are ultimately established by a trial-and-error process from experiments. Theoretically, most of quantum systems come from a quantization process. Two basic principles for quantum systems are Unitarity and Locality, i.e., our quantum system's Hamiltonian is Hermitian and a sum of local terms. Since every physical quantum system is subject to un-controlled perturbations and we want our Hamiltonian to represent a stable phase of matter, and ideally to be good for information processing, stability and universality for BQP are among our desiderata.

1.2.3. *To Quantize is to Wear a Hat.* Physicists have a powerful conceptual method to quantize a classical observable: put a hat on the classical quantity. For example, to quantize a classical particle with position x and momentum p, we define two Hermitian operators $\hat{x}$ and $\hat{p}$, which satisfy the famous commutator $[\hat{x}, \hat{p}] = i\hbar$. The Hilbert space of states are spanned by eigenstates of $\hat{x}$ denoted as $|x\rangle$, where x is the corresponding eigenvalue of the Hermition operator $\hat{x}$. Similarly, we can use the momentum p, but not both due to the uncertainty principle. The two quantizations using x or p are related by the Fourier transform. Therefore, physical quantization is easy: wear a hat. In general, a wave function $\Psi(x)$ is really a spectral function for the operator $\Psi(\hat{x})$ at state $|\Psi(x)\rangle$: $\Psi(\hat{x})|\Psi(x)\rangle = \Psi(x)|\Psi(x)\rangle$.

1.3. **Particle and Phase of Matter.** Particles are just special quantum states, which are localized exponentially in space. They are fuzzy spots in space modeled by Gaussian wave packets. While they are not points and occupy space of non-zero size, it is hard to define any classical shape, size, or position.

1.3.1. *Ribbon Trajectory of Particles.* Classical particles are perceived as points, therefore their configuration spaces are topological spaces, which are generally manifolds. A defining feature of a quantum particle such as the electron is its quantum spin. A deep quantum principle is the spin-statistics connection. It is common to picture the spin of a particle through a 2π rotation as follows:

Insert two pictures.

When we pull tight the string, it becomes the straight line. Hence, it seems that spinning a particle has no observable consequence. A remedy for this misleading picture is to use a ribbon. Then after we pull tight the ribbon, we see a full twist of one side around the other side. So indeed spinning a particle makes its trajectory different from a straight line. One justification of the ribbon idea is as follows. A quantum particle is an elementary excitation in a quantum system, so represented by a non-zero vector in a Hilbert space. Even after we normalize the state vector, there is still a phase ambiguity $e^{i\theta}, \theta \in [0, 2\pi)$, which parameterizes the standard circle in the complex plane. So a semi-classical picture of the particle sitting at the origin of the plane would be an arrow from the origin to a certain angle θ (this arrow is really in the tangent space of the origin). Hence one picture model of a quantum particle could be a small arrow. Visualizing a quantum particle as an infinitesimal arrow leads to the ribbon picture of the worldline of the quantum particle. When we twist the particle by 2π, the resulted full twist in the ribbon encodes the spin-statistics connection.

1.3.2. *Picture Calculus.* A powerful tool to study higher algebroid theory is graphical calculus—a far reaching generalization of spin network. One important graphical tool is the tensor network states, which represent an important class of quantum states including the hugely successful matrix product states for DMRG.

While we will use category to organize our structures, we will emphasize the role of pictures for logical deductions. It is of paramount importance in this book to understand all algebraic formulas or proofs through pictures. It takes efforts to be able to make such arguments rigorous and bring life to the algebraic formulas as a rich world of particle hops and interactions.

1.3.3. *Phase of Matter and Phase Diagram.* Topological phases of matter (TPMs) in nature include quantum Hall states–integral and fractional—and the recently discovered topological insulators.

Roughly, a phase of matter is an equivalence class of quantum systems that certain properties are the same within the equivalence class. The devil is in the definition of the equivalence relation. Homotopy is a good example for this investigation. Classical phases of matter illustrate the idea well, e.g., water within the temperature between 0 and 100 under normal conditions are all in the same liquid state. Less familiar is that liquid and gas are also in the same state as we can continuously change a liquid to a gas without passing a border in the phase diagram. All this information is presented by the so-called phase diagram: a diagram represents all possible phases and some domain walls indicating the phases transitions. Quantum systems in the same domain are the same, and the domain walls are where certain physical quantities such as the ground state energy per particle become singular.

A general theory for classical phases of matter and their phases transitions is formulated by L. Landau. In Landau's theory, phases of matter are characterized by their symmetry groups, and phases transitions are characterized by symmetry breaking. It follows that group theory becomes an indispensable tool in condensed matter physics. While classical phases of matter depend crucially on the temperature, quantum phases of matter are all phases of matter at zero temperature (in reality very close to zero). TPMs do not fit into the Landau paradigm, and intensive effort in physics right now is to develop a post-Landau paradigm to classify quantum phases of matter.

TPMs are phases of matter whose low energy physics and universal properties can be modeled well by TQFTs and their enrichments. To characterize TPMs, we focus either on the ground states or their first excited states. The ground state dependence on the topology of spaces where the quantum system resides is organized into a TQFT, and the algebraic models of elementary excitations of the TPMs are unitary modular categories.

1.4. **Tensor Network Representation of a State.** Quantum field theories are effective theories so our theories are valid up to certain energy scales. The particles at such an energy scale will be referred to as effective constituent particles (ECPs). From ECPs, new beautiful physics at larger scale or lower energy emerges following the renormalization flow. We will take our qubits as ECPs, and quantum states will be represented by layers of graphical states—tensor networks.

2. Mathematical Quantum Systems

We will start the mathematical foundations of topological quantum computation from this section. We will be careful to distinguish mathematical statements from physical ones. Definitions and theorems are all mathematical ones. Physical theorems should be treated as mathematical conjectures and they would follow if mathematical foundations are set up for certain physical principles. Physical quantum field theories that are made mathematically rigorous include conformal field theories (CFTs), topological quantum field theories (TQFTs), and certain non-linear sigma models.

Unless stated otherwise, all our Hilbert spaces are finitely dimensional ones, hence are isomorphic to $\mathbb{C}^d$ for some integer d. We will refer to $\mathbb{C}^d$ as a *qudit* following the quantum information science language. We will also refer to $\mathbb{C}^d$ as a spin $s = \frac{d-1}{2}$ following the physical jargon, then the quantum system should have an $\mathrm{SU}(d)$ symmetry.

2.1. **Linear Algebra in Quantum Theory Language.** To a first approximation, the subject of mathematical quantum systems is complex linear algebra in quantum theory language. In practice, it is a linear algebra problem that even the most powerful classical computer cannot handle. A major constraint is memory. State-of-the-art computational physicist can handle Hilbert spaces of dimension $\approx 2^{72}$. But for a real quantum system, this dimension is very small: in quantum Hall physics, there are about 10^{11} electrons per cm^2. So we will have thousands of electrons per square micron. If we consider the Hilbert space for the electron spins, we will deal with a Hilbert space of dimension $\approx 2^{1000}$. Therefore, it is almost an impossible task to solve such a problem exactly. To gain understanding of such problems, we have to rely on ingenious approximations or extrapolations from a small number of electrons.

One interesting open question is to understand thermodynamical properties of quantum systems, which are quantum systems with Hilbert spaces as large as possible. A good approximation for arbitrarily large Hilbert spaces are infinite dimensional Hilbert spaces and their von Neumann algebras. Presumably, von Neumann algebras are important for our understanding of quantum systems. The subtlety is how to topologize oberservable algebras to define the appropriate limits.

Definition 2.1. (1) *A mathematical quantum system (MQS) is a triple* $(\mathcal{L}, b, H)$, *where* $\mathcal{L}$ *is a Hilbert space with a distinguished orthonormal basis*[4] $b = \{e_i\}$, *and* H *an Hermitian matrix regarded as an Hermitian operator on* $\mathcal{L}$ *using* b. *The Hermitian operator* H *is called the Hamiltonian of the quantum system, and its eigenvalues are the energy levels of the system. The distinguished basis elements* $\{e_i\}$ *are classical states or configurations.*

A non-zero vector $|\Psi\rangle$ *is called a state vector or a wave function of the quantum system. So the zero vector is the only vector that is not a quantum state, and thus not physical. A state* $|\Psi\rangle$ *is normalized if its norm is* 1.

At each moment, the quantum system is in a state given by a wave function $|\Psi\rangle$. *An evolution of the quantum system from one state* $|\Psi_0\rangle$ *to another* $|\Psi_1\rangle = U|\Psi_0\rangle$ *is through a unitary operator (matrix)* U, *which generalizes the time evolution operator* $e^{i\Delta t H}$ *for some time interval* Δt.

(2) *An observable or a measurement* $\mathcal{O}$ *of a quantum system* $(\mathcal{L}, b, H)$ *is an Hermitian operator (matrix) on* $\mathcal{L}$, *in particular* H *is an observable.*

If the observable $\mathcal{O}$ *is measured when the quantum system is in a normalized state* $|\Psi\rangle$, *then the state of the system after measurement is in a normalized eigenstate* $|e_i\rangle$ *of* $\mathcal{O}$ *with probability* $p_i = |\langle e_i|\mathcal{O}|\Psi\rangle|^2$.

It follows that experiments always result in some eigenstates of the observable. Conceptually, mathematical quantum mechanics is like a square root of probability theory because amplitudes are square roots of probabilities.

(3) *A quantum system on a graph* $\Gamma = (V, E)$ *with local degrees of freedom* $\mathbb{C}^d$ *is a mathematical quantum system whose Hilbert space* $\mathcal{L} = \otimes_{e \in E} \mathbb{C}^d$ *with the tensor orthonormal bases from* $\{e_i\}$ *of* $\mathbb{C}^d$, *where* E *are the edges (bonds or links) of* Γ.

Using the physical jargon, we will say there is a spin $s = \frac{d-1}{2}$ *on each edge as the Hilbert space for spin* s *is of dimension* $d = 2s + 1$. *More precisely, we will say there is a qudit on each edge following quantum information terminology.*

A quantum system on a graph is a composite quantum system consisting of identical subsystems of local spins or qudits. We will use the Dirac notation to represent the standard basis of $\mathbb{C}^d$ *by* $e_i = |i - 1\rangle, i = 1, \cdots, d$. *When* $d = 2$, *the basis elements of* $\mathcal{L}$ *are in one-one correspondence with bit-strings or* $\mathbb{Z}_2$*-chains of* Γ.

[4]The preferred basis is very important for later discussion of entanglement because it allows locality to be defined. The role of basis is also essential for superposition to be meaningful.

Since we are interested in quantum states of matter, we are not focusing on a single quantum system. Rather we are interested in a collection of quantum systems and their properties in some limit which is related to the physical idea of thermodynamic or long-wave length/low energy limit. In most cases, our graphs are the 1-skeleton of some triangulation of a manifold.

The most important Hermitian matrices are the Pauli matrices, which are spin=$\frac{1}{2}$ operators. Pauli matrices wear two hats because they are also unitary.

Example 2.2. Let Γ be a kagome lattice on the torus, i.e., a kagome lattice in the plane with periodic boundary condition. The Heisenberg Hamiltonian is

$$H = \sum_{<i,j>} \sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y + \sigma_i^z \sigma_j^z,$$

where the Pauli matrices $\sigma_k^\alpha, \alpha = x, y, z, k = i, j$ acts on the k-qubit in the sense σ_k^α is the Pauli matrix σ^α on the k tensor factor extended by Id to other factors.

Definition 2.3. *Given a mathematical quantum system* $(\mathcal{L}, b, H)$*. Let* $\{\lambda_i, i = 0, 1, \cdots\}$ *be the eigenvalues of* H *ordered in an increasing order and* $\mathcal{L}_{\lambda_i}$ *the corresponding eigenspace.*

(1) L_{λ_0} *is called the ground state manifold*[5] *and any state in* L_{λ_0} *is called a ground state. Any state in its complement* $\oplus_{i>0}\mathcal{L}_{\lambda_i}$ *is called an excited state. Usually we are only interested in the first excited states in* $\mathcal{L}_{\lambda_1}$*, but sometimes also states in* $\mathcal{L}_{\lambda_2}$*. Bases states in* $\mathcal{L}_{\lambda_i}$ *are called minimal excited states.*
(2) *The partition function of a quantum system is* $Z = Tr(e^{-\beta H})$*.*
(3) *A quantum system is rigorously solvable if we can find bases of the ground state manifold and minimal excitations for a few low excited state manifolds, and its partition function as an explicit analytic function of* β*. A quantum system is exactly solvable if the exact answers as in rigorously solvable models are given as physical theorems.*

Minimal excitations vs elementary excitations vs particles

Example 2.4. Ising chain with open boundary condition: elementary excitation

The linear algebra problems appeared in quantum physics involve Hilbert spaces $\mathcal{L}$ whose dimensions are exponentially large and it is too costly to run any standard linear algebra packages. Moreover, most physical quantities are only approximations so exact solutions are not necessarily better. But exact solutions are important at least for two reasons: one is to establish physical concepts without any doubt. On such example is the famous toric code below which established the quantum spin liquid concept theoretically. Secondly it can be used as a measuring stick for approximating methods.

[5]Physicists use the word manifold to mean multi-fold, but L_{λ_0} indeed is a simple manifold.

Entanglement is a property of a composite quantum system. If we have a quantum system with several subsystems given by a collection of Hilbert spaces $\{L_i\}, i \in I$, then the Hilbert space for the composite system is their tensor product $\mathcal{L} = \bigotimes_{i\in I} L_i$.

Definition 2.5. *Given a composite quantum system with Hilbert space $\mathcal{L} = \bigotimes_{i\in I} L_i$, then a state is entangled with respect to this tensor decomposition into subsystems if it is not a direct product of the form $\otimes_{i\in I}|\psi_i\rangle$ for some $|\psi_i\rangle \in L_i$.*

Definition 2.6. *A unitary U is a symmetry of H if $UHU^\dagger = H$. An Hermitian operator K is a symmetry of H if $[K, H] = 0$, and then e^{itK} is a unitary symmetry for any t. When U is a symmetry, then each energy eigenspace $\mathcal{L}_{\lambda_i}$ is further decomposed into eiegnspaces of U. Those eigenvalues of U are called good quantum numbers.*

Definition 2.7.
(1) *Correlation function: $\langle 0|\mathcal{O}(r - r')|0\rangle$.*
(2) *Expectation value $\langle 0|\mathcal{O}|0\rangle$ is the expectation value for measuring $\mathcal{O}$.*

Definition 2.8.
(1) *A MQS is on a control space Y if $\mathcal{L} = \bigotimes_{i\in I} L_i$ and there is a map $p : i \to Y$.*
(2) *A MQS is local if $H = \sum H_i$ such that each H_i is of the form $A \otimes Id$ and the support of A is bounded.*
(3) *A Hamiltonian is a sum of commuting local projectors (CLP) if $H = \sum H_i$ such that each H_i is a local projector and $[H_i, H_j] = 0$ for all i, j.*

2.2. **Gap, Stability, and CLP Himiltonians.** When we specify a Hamiltonian, we are giving instructions on how to write down a class of mathematical quantum systems. We will call such instructions for defining quantum systems as a Hamiltonian schema. In this book, Hamiltonian schemas are given for triangulations of or lattices in the space manifolds of a fixed dimension. Usually, we also need additional structures on the manifolds or lattices.

Definition 2.9. *A Hamiltonian schema (HS) is gapped or has a spectral gap if the difference of energies $\lambda_1 - \lambda_0$ is bounded below by a non-zero constant for all the quantum systems resulted from this HS. The spectral gap is difficult to establish analytically and is crucial for application to quantum computing.*

How to establish a gap is in a technique vacuum. One obvious case is for Hamiltonians which are CLP.

Proposition 2.10. *If H is a CLP then it has an energy gap and Ψ is a ground state iff $H_i\Psi = 0$ for each i.*

2.3. **Toric Code.** Kitaev's toric code is arguably the most important model Hamiltonian in TQC. It is rigorously solvable and represents a topological phase of matter whose low energy physics is given by a TQFT based on the quantum double of $\mathbb{Z}_2$.

2.4. **DFib.** While the toric code is the paradigm example for TQC, the elementary excitations are bosons and fermions, whose braiding statistics are very simple. The DFib is a rigorously solvable model whose elementary excitations are anyons whose braiding statistics is universal for quantum computation.

2.5. **Characterization of TPMs.**

2.5.1. *TQFT as Low Energy Effective Theory.*

2.5.2. *UMC as Model of Elementary Excitations.*

2.5.3. *Quantum Circuits.* Preserve locality and connected to the identity.

2.5.4. *Long-range Entanglement.*

2.5.5. *Error Correction Code.* An error-correcting code is an embedding of $(\mathbb{C}^2)^{\otimes n}$ into $(\mathbb{C}^2)^{\otimes m}$ such that information in the image of $(\mathbb{C}^2)^{\otimes n}$ is protected from local errors on $(\mathbb{C}^2)^{\otimes m}$. We call the encoded qubits the logical qubits and the raw qubits $(\mathbb{C}^2)^{\otimes n}$ the constituent qubits. Let V, W be logical and constituent qubit spaces.

Theorem 2.11. *The pair (V, W) is an error-correcting code if there exists an integer $k \geq 0$ such that the composition*

$$V \xhookrightarrow{i} W \xrightarrow{O_k} W \xrightarrow{\pi} V$$

is $\lambda \cdot id_V$ for any k-local operator O_k on W, where i is inclusion and π projection.

When $\lambda \neq 0$, O_k does not degrade the logical qubits. But when $\lambda = 0$, it rotates logical qubits out of the code subspace, introducing errors. But it always rotates a state to an orthogonal state, so errors are detectable and correctable.

Example 2.12. 1-qubit in 5-qubits

This error-correcting code is generated by the frustration-free Hamiltonian $H = \sum_{i=1}^{4} H_i$, i.e. $[H_i, H_j] = 0$, on $(\mathbb{C}^2)^{\otimes 5}$, where

$$H_1 = \sigma_x \otimes \sigma_z \otimes \sigma_z \otimes \sigma_x \otimes \sigma_0, \qquad H_2 = \sigma_0 \otimes \sigma_x \otimes \sigma_z \otimes \sigma_z \otimes \sigma_x,$$
$$H_3 = \sigma_x \otimes \sigma_0 \otimes \sigma_x \otimes \sigma_z \otimes \sigma_z, \qquad H_4 = \sigma_z \otimes \sigma_x \otimes \sigma_0 \otimes \sigma_x \otimes \sigma_z,$$

where σ_x, σ_z are Pauli matrices and $\sigma_0 = id$. The ground state space is isomorphic to $\mathbb{C}^2$. The unitary matrices $X = \sigma_x^{\otimes 5}$, $Z = \sigma_z^{\otimes 5}$ are symmetries of the Hamiltonian, hence act on the ground states. Therefore X and Z can be used to process encoded information. They are called logical gates. An error basis can be detected using measurements and then corrected

3. Exercises

3.1. **Graph codes.** Shor code and $\mathbb{R}P^2$, Reed code and the Peterson graph

3.2. **Matrix Product State.**

4. Questions and Research Topics

4.1. Definition of Quantum Phases of Matter.

4.2. Phase Diagram of Herbetsmithite. Can DFib appear in the phase diagram of herbertsmithite? Anyonic versions?

4.3. Rigourously Solvable Anyonic Chains.

4.4. Controlled Linear Algebra.

4.5. Quantum Logic.

4.6. Algebraic Structure for 4-Manifolds.

4.7. How to Topologize Higher Categories.

4.8. Gauge Theory Dual of Manifolds.