ATIYAH-SEGAL TYPE FUNCTORIAL TQFTS

The subject of topological quantum field theories (TQFTs) is so diverse that we do not have a standard definition of TQFT yet. It is much like the early days of homology theory, and then category theory was invented to compare all the different definitions of homology groups. The most convenient formulation is to follow Atiyah who modeled the definition of TQFT on Segal's definition of conformal filed theory (CFT). In this formulation, a TQFT is a symmetric monoidal functor from a monoidal category of manifolds with certain structures to the category of finitely dimensional vector spaces. All kinds of variations and extensions are possible, and we will consider several later. In this definition, the detail is in the definition of the bordism category of manifolds[1] and the definition of a monoidal functor.

Mathematically, TQFTs are amenable to classification. Physically, we study which TQFTs can be realized by real physical systems, which will serve as the hardware for TQC. For computer science, we investigate how to compute using TQFTs. Each perspective leads to its own set of questions, and a real TQFT that is BQP complete will be a universal quantum computer—the ultimate goal of TQC.

1. Bordism Categories

1.1. **Manifold Topology Recalled.** A topological space is a set X with a class $\mathcal{O} = \{O_\alpha\}$ of subsets of X that satisfy the following axioms:

(1) The empty set $\emptyset$ and X are in $\mathcal{O}$
(2) The intersection of any two sets in $\mathcal{O}$ is in $\mathcal{O}$
(3) The union of any collection of sets in $\mathcal{O}$ is in $\mathcal{O}$.

In other words, a topology on X is collection of subsets of X that always contains the empty subset $\emptyset$ and X, and is closed under finite intersections and arbitrary unions. Sets in $\mathcal{O}$ are called open sets of X, and their complements are called closed sets. A topology defines a notion of closeness of points of X, which is a far-reach generalization of the notion of closeness given by a metric. Not every

[1]Bordism here is often referred to as cobordism. Our choice of bordism is based on two considerations: first historically co- in cobordism is to emphasize the symmetric roles of the two boundary manifolds. In TQFT for manifolds with orientations and other structures, we actually need to distinguish the in- and out- parts of the bordism. Secondly, bordism is historically used for generalized homology theory based on manifolds, while cobordism for generalized cohomology based on manifolds. TQFT is a functor, so bordism is more appropriate. It is also convenient to have the term cobordism for a slightly different manifold category.

topology of a space can be defined using a metric, but for manifolds that we are interested in, their topologies always come from metrics, i.e., they are topological spaces that are metrizable.

A topological n-manifold M for a fixed integer $n \geq 0$ is a topological space M with the extra property[2] that every point $x \in M$ has a neighborhood that is homeomorphic to $\mathbb{R}^n$ or $\mathbb{R}^n_+ = \{(x_1, \cdots, x_n), x_n \geq 0\}$. Points with neighborhoods that are homeomorphic to $\mathbb{R}^n_+$ are called boundary points. The boundary points of an n-manifold M form a subset of M, denoted as ∂M, which is an $(n-1)$-manifold sub-manifold of M. We assume that a manifold M is always Hausdorff and can be covered by countably many coordinate charts. While a topological manifold M being orientable is an extra property, an orientation M is an extra structure. Similarly, a topological manifold M has a smooth structure or not is an extra property, while a smooth manifold M is a topological manifold with an extra structure.

1.2. **Bord**$(n+1)$**.** Our manifolds will be smooth, oriented, and compact unless stated otherwise. In order to have a *small* category of manifolds, we will assume that all our manifolds are subsets of some Euclidean space $\mathbb{R}^n$. Though we will not identify manifolds by arbitrary diffeomorphisms, we consider a manifold M and all its translations as the same manifold. In particular, $M \times \alpha$ is identified with M when α is an index such as $0, 1, 2, \cdots$ ... or a point. All vector spaces are over the complex numbers $\mathbb{C}$ and finite dimensional unless stated otherwise.

Definition 1.1. (1) *An $(n+1)$-bordism X between two oriented smooth n-manifolds $\partial_- X$ and $\partial_+ X$ is an oriented $(n+1)$-manifold X whose boundary is equal to $-\partial_- X \sqcup \partial_+ X$, i.e., $\partial X = -\partial_- X \sqcup \partial_+ X$, not just diffeomorphism, where the $-$ sign in front of $\partial_- X$ means $\partial_- X$ with the opposite orientation.*

For the definition to make sense, we need to fix a convention for the induced orientation. We will use the polite convention: out (outsider) first, i.e., the outward normal vector of the boundary is the first vector for the induced orientation. In this convention, the standard circle in the plane has a counterclockwise induced orientation from the unit disk.

The separation of the boundary ∂X into two parts $\partial_- X$ and $\partial_+ X$ is part of the bordism. Strictly speaking, a bordism is a pair (X, p), where $p : \partial X \to \{-, +\}$ is a continuous map such that $\partial_\mp X = p^{-1}(\mp)$. Therefore, $X = Y \times I$ is not a bordism unless we divide its boundary $\partial X = Y \sqcup Y$ (here ∂X divided as unoriented manifolds) into two parts. We decided to divide ∂X using the induced orientation as follows: the $\partial_- X$ part is the oriented manifold whose orientation is opposite to the induced orientation,

[2]It is important in mathematics to distinguish extra property and extra structure on a mathematical structure. An abelian group is a group with the extra property $ab = ba$, but a Lie group is a group with an extra structure—a topology that is also compatible with the group multiplication.

while $\partial_+ X$ is the part whose orientation agrees with the induced one. If Y is oriented, there are many possible divisions such as $\partial_- X = Y, \partial_+ X = Y$, or $\partial_- X = -Y, \partial_+ X = -Y$, or $\partial_- X = Y \sqcup -Y, \partial_+ X = \emptyset$, or $\partial_- X = \emptyset, \partial_+ X = Y \sqcup -Y$, which lead to many inequivalent bordisms.

(2) *Two $(n+1)$-bordisms X, X' between the same two n-manifolds Y_- and Y_+ are equivalent if there is an orientation preserving diffeomorphism $f : X \to X'$ such that f is the identity on the boundary.*

(3) *Two important conventions: the empty set $\emptyset$ is a manifold of each dimension $n \geq 0$, and any diffeomorphism $f : Y \to Y$ is regarded as a bordism from Y to Y by forming the symmetric mapping cylinder $M_f = Y \times [0, \frac{1}{2}] \cup_f Y \times [\frac{1}{2}, 1]$. Note that M_f and M_g are equivalent bordisms if f and g are isotopic or pseudo-isotopy.*

Remark 1.2. In section 1 of Milnor's book "lectures on the h-cobordism theorem". Bordism here is called a smooth manifold triad. A cobordism from Y_1 to Y_2 there is defined as a 5-tuple $(X; \partial_1 X, \partial_2 X; h_1, h_2)$, where $(X; \partial_1 X, \partial_2 X)$ is a smooth triad and $h_i : \partial_i X \to Y_i, i = 1, 2$ are diffeomorphisms. This definition leads to a category of manifolds, which are often called the cobordism category. While convenient for studying cobordism theory, this formulation is inconvenient for the definition of a TQFT because the gluing map for two smooth manifold triads often induces a non-trivial linear isomorphism. The insertion of such non-trivial isomorphisms into the definition of a TQFT makes the assignments cease to be a functor. Furthermore, deep information of a TQFT is inside the representation of the mapping class groups, which can be obscured by forgetting such inserted isomorphisms.

An alternative way to define a manifold category convenient for TQFTs is to use cobordism with parameterized boundaries. Fix n and for each diffeomorphism class of closed connected n-manifold, choose one that will be called the model manifold, denoted as Σ. We will say that a manifold Y diffeomorphic to Σ is a manifold of type Σ.

Definition 1.3. *Given two closed n-manifolds ∂X_1 and ∂X_2, a cobordism from ∂X_1 to ∂X_2 is a 5-tuple $(X; \partial_1 X, \partial_2 X; p_1, p_2)$, where $(X; \partial_1 X, \partial_2 X)$ is a smooth triad and $p_i : \Sigma_i \to \partial_i X, i = 1, 2$ are parameterizations of the boundaries by model manifolds Σ_i. Two cobordisms $(X_1; \partial_1 X, \partial_2 X; p_1, p_2)$ and $(X_2; \partial_2 X, \partial_3 X; h_1, h_2)$ are glued only by the diffeomorphism g that commutes with the parameterizations p_2 and h_1, ie $g = p_2 \cdot h_1^{-1}$. It can be derived later that such a tautology diffeomorphism is the identity for gluing cobordisms.*

2. $(n+1)$-TQFTs

We require a TQFT to be defined for all space and space-time manifolds, which captures the idea of locality in the sense that a physical theory is defined completely by local interactions[3]. There are theories that are not defined for all manifolds such as the first TQFT—Witten-Donaldson theory. Such partial TQFTs are defined only for certain special space and space-time manifolds. We will also consider space manifolds with boundary later—extended TQFTs. They are important as they correspond to physical systems with excitations.

The most convenient way to organize the TQFT axioms is to first define a bordism category $\mathbf{Bord}(n+1)$ of manifolds.

Definition 2.1. *An object Y of the category $\mathbf{Bord}(n+1)$ is a closed oriented smooth n-manifold Y. The Hom-set $Hom(Y_1, Y_2)$ of two smooth oriented closed n-manifolds Y_1, Y_2 is the set of $(n+1)$-bordism classes X between Y_1 and Y_2.*[4]

Then an $(n+1)$-TQFT is a symmetric monoidal functor from $\mathbf{Bord}(n+1)$ to the category of finitely dimensional vector spaces $\mathbf{Vec}$. Since we have not defined symmetric monoidal functors yet, we will just list all axioms of TQFTs explicitly below.

Following the physical jargon, we will refer to the n-manifolds of the $\mathbf{Bord}(n+1)$ category as space manifolds and the bordism $(n+1)$-manifolds as space-time manifolds.

2.1. Axioms.

Definition 2.2. *An $(n+1)$-TQFT is a pair $F = (V, Z)$ of assignments such that V assigns to each smooth oriented closed n-manifold Y a finite dimensional vector space $V(Y)$ and Z assigns to each $(n+1)$-bordism X between two smooth oriented closed Y_- and Y_+ a linear map $Z(X) : V(Y_-) \to V(Y_+)$, which satisfy the following axioms. Note that each diffeomorphism $f : Y \to Y$ is assigned a linear map $Z(f) = Z(M_f) : V(Y) \to V(Y)$, and the vertical isomorphisms in the left side of the commutative diagrams below are induced by the obvious diffeomorphisms.*

(i): Functoriality

1. $Z(X) = Z(X') : V(Y_-) \to V(Y_+)$ *if X and X' are equivalent, i.e., the map Z depends only on the bordism class.*
2. $Z(Y \times I) = \mathrm{Id}_{V(Y)} : V(Y) \to V(Y)$.
3. $Z(X_2 \cup X_1) = Z(X_2) \cdot Z(X_1) : V(Y_1) \to V(Y_2) \to V(Y_3)$.

(ii): Monoidality

There are canonical isomorphisms

[3]For physical systems such as the herbertsmithite, the lattice for the quantum system is real and they exist only on certain manifolds. Therefore, partial TQFTs are also important for the understanding of the physical world. Similar to the restriction of topology by lattice, symmetry can also put restriction on manifolds

[4]The definition justifies the notation of $(2+1)$- or $(3+1)$-TQFTs rather than 3- or 4- TQFTs as the manifolds as objects are not diffeomorphism classes while the bordisms are.

(1) $V(\emptyset) \cong \mathbb{C}$
(2) $V(Y_1 \sqcup Y_2) \cong V(Y_1) \otimes V(Y_2)$

such that

$$\begin{array}{ccc} V((Y_1 \sqcup Y_2) \sqcup Y_3) & \cong & (V(Y_1) \otimes V(Y_2)) \otimes V(Y_3) \\ \downarrow & & \downarrow \\ V(Y_1 \sqcup (Y_2 \sqcup Y_3)) & \cong & V(Y_1) \otimes (V(Y_2) \otimes V(Y_3)) \end{array}$$

and

$$\begin{array}{ccc} V(\emptyset \sqcup Y) & \cong & \mathbb{C} \otimes V(Y) \\ \downarrow & & \downarrow \\ V(Y) & = & V(Y) \end{array}$$

(iii): Symmetry

There is a canonical isomorphism $V(Y_1 \sqcup Y_2) \cong V(Y_2 \sqcup Y_1)$ *such that*

$$\begin{array}{ccc} V(Y_1 \sqcup Y_2) & \cong & V(Y_1) \otimes V(Y_2) \\ \downarrow & & \downarrow \\ V(Y_2 \sqcup Y_1) & \cong & V(Y_2) \otimes V(Y_1) \end{array}$$

The map Z is called the (generalized) partition function. Inside $\mathbf{Bord}(n+1)$, there is a subcategory with the same objects, but only bordisms from diffeomorphisms. The map V restricted to this subcategory is called an n-dimensional topological modular functor. The word *canonical* in the axioms is subtle as what it really means is functorial.

2.1.1. *Manifold Invariants and Representation of MCGs.* One of the major goals in topology is to find topological invariants of manifolds. Each TQFT provides a topological invariant of manifolds. One of the widely open questions is how those quantum invariants are related to classical topological invariants such as homology and homotopy groups. The mapping class group representations also lead to topological invariants of mapping tori. Another open question is how to decide when a rep of the MCG is reducible and what are the closed images? This is intimately related to the BQP completeness of anyonic quantum computers. In general, it is also not known what is the computational complexity of computing exactly or approximating the quantum invariants.

Theorem 2.3. *Given an* $(n+1)$*-TQFT* $F = (V, Z)$*, then*

(1) $Z(M)$ *is a smooth topological invariant for closed* $(n+1)$*-manifolds* M*.*
(2) $V(Y)$ *is a representation of the mapping class group of* Y*.*
(3) *Direct sum and tensor product of two TQFTs is also a TQFT.*

2.1.2. *Some General Properties.*

Proposition 2.4. (1) *That* $V(Y)$ *is finite dimensional follows from the axioms.*

(2) *The vector space* $V(-Y)$ *is canonically isomorphic to* $V(Y)^*$.
(3) *Let* T_f *be the mapping torus of an orientation preserving diffeomorphism* $f : Y \to Y$, *then* $Z(T_f) = Tr(Z(f))$. *In particular,* $Z(Y \times S^1) = dim(V(Y))$
(4) *If* X *is the union of two handlebodies, then* $Z(X)$ *is the pairing of the two vectors.*

Proof. Let us consider the different bordisms from $X = Y \times I$. As in the definition 1.1, we have 2 different bordisms that lead to 2 maps as below.

Birth: As a bordism from $\emptyset$ to $-Y \sqcup Y$, we have a map: $b : \mathbb{C} \to V(Y) \otimes V(-Y)$.

Death: As a bordism from $-Y \sqcup Y$ to $\emptyset$, we have a map: $d : V(-Y) \otimes V(Y) \to \mathbb{C}$.

Let $b(1) = \sum_{ij} c_{ij} e_i \otimes \epsilon^j$ which is a finite sum.

Consider the following two S-bordisms which are obvious diffeomorphic to the identity bordisms. If we slice the bordism into the composition of three bordisms, we see the identity map from $V(\pm Y)$ to $V(\pm Y)$ are also the compositions:

$$\mathrm{Id}_{V(Y)} : V(Y) \to V(Y) \otimes \mathbb{C} \to V(Y) \otimes V(-Y) \otimes V(Y) \to \mathbb{C} \otimes V(Y) \to V(Y).$$

Similarly,

$$\mathrm{Id}_{V(-Y)} : V(-Y) \to V(-Y) \otimes \mathbb{C} \to V(-Y) \otimes V(Y) \otimes V(-Y) \to \mathbb{C} \otimes V(-Y) \to V(-Y).$$

Written in indices, for vectors $v \in V(Y), w \in V(-Y)$, then

$$\sum_{ij} c_{ij} d(\epsilon^j, v) e_i = v.$$

$$\sum_{ij} c_{ij} d(w, e_i) \epsilon^j = w.$$

Therefore, $V(\pm Y)$ are generated by $\{e_i\}$ and $\{\epsilon^j\}$, respectively, hence both are finite dimensional.

Furthermore, if v or w in the kernel of the pairing, then $v = 0$ or $w = 0$ because $d(\epsilon^j, v)$ and $d(w, e_i)$ will be 0 for all i, j. Therefore, the paring d is non-degenerate.

Now we assume that e_i, ϵ^j are dual basis of $V(Y)$ and $V(-Y)$. Let $v = v^k e_k$, from the S-identity 2.1.2, we have

$\sum_{ijk} c_{kj} d(\epsilon^j, e_i) v^i e_k = \sum_k v^k e_k$. It follows $\sum_i c_{ki} v^i = v^k$. Since this is true for all v and the matrix (c_{ij}) is the idenity matrix. So $b(1) = \sum_i \epsilon^i \otimes e_i$.

□

2.1.3. *(0+1)-TQFTs.* To see the general structure of TQFTs, we start with the simplest theories—$(0 + 1)$-TQFTs.

The only connected 0-manifold is a point pt. We will follow the standard convention that a pt is orientable with two orientations. Hence we can have a positive

$+pt$ or a negative $-pt$. One justification is that an orientation is choice of a component of $\wedge^n V$ for an n-dimensional real vector space V. Since $\wedge^0 V \cong \mathbb{R}$, so we have two components.

Given a $(0+1)$-TQFT $F = (V, Z)$, then by the paring Prop. 2.4, $V(-)$ is the dual vector $V(+)^*$. A sequence of signed points will be assigned the tensor product of $V(+)$ and $V(+)^*$. The morphisms are tensor of the identity, b, d, and $\dim(V) \cdot \mathrm{Id}$. The partition function of the closed circle is $Z(S^1) = \dim(V)$.

2.2. **Some Picture TQFTs.** We are not ready to construct general TQFTs yet because we need more topological and algebraic preparation. But we are going to study three important $(2+1)$-TQFTs for TQC: the toric code, the double Ising theory, and the double Fibonacci theory. They represent three kinds of TQFTs[5] and are closely related to real physical systems: herbertsmithite, fractional quantum Hall liquids at $\nu = \frac{5}{2}$, and engineered materials that generalzied Majorana zero modes. They are all TQFTs that are independent of orientations, so really are TQFTs for unoriented manifolds including non-orientable ones. We will leave the proof of the invariance for the general partition functions to later sections.

2.2.1. *Picture and Local Relations.* All three TQFTs belong to the class of TQFTs that will be called picture TQFTs. The vector space $V(Y)$ for a closed surface Y consists of linear combination of pictures in Y modulo some local relations—relations among pictures restricted to a disk. If a picture means an embeded unoriented 1-manifold, then in order to obtain a non-trivial TQFT, the local relations are highly constrained, and essentially the famous Jones-Wenzl projectors from subfactor theory, where quantum topology started. When pictures are generalized to string-nets with orientations and colors, then all Turaev-Viro $(2+1)$-TQFTs can be realized as picture TQFTs. Generalization is still actively pursued from many directions.

Definition 2.5.
(1) *Let Y be a surface, a multi-curve S in Y is an unoriented embeded 1-manifold. The empty set is always a multi-curve and S is not necessarily connected.*
(2) *Let n be a positive integer n, and fix $2n$ points on the boundary of the unit disk B^2. A Temperley-Lieb (TL) diagram D is a disjoint union of n arcs connecting the $2n$ points up to relative isotopy. There are exactly Catalan number many diagrams for each n, denoted as $D_i, i = 1, ..., C_n$ and $C_n = \frac{1}{n+1}\binom{2n}{n}$. Then a local relation is a formal sum of TL diagrams*

$$\sum_{i=1}^{C_n} c_i D_i = 0$$

for some complex numbers c_i.

[5] abelian, non-abelain but not braiding universal, and brading universal

The second Jones-Wenzl projector p_2 gives rise to the surgery local relation: Picture!!!

The third Jones-Wenzle for projector p_3 gives rise to the 5-term local relation: Picture !!!

Remark 2.6. In two dimensions, every multi-curve is tame in the sense it has a neighborhood that is homeomorphic to an annulus. But the Alexander horned sphere show that in three dimension and higher, we have to be careful. This is one reason that we consider smooth manifolds. Another reason is that the yet unrealized hope to study 4 manifolds using TQFTs is only interesting for the smooth category because topological 4 manifolds are well understood for a large class of fundamental groups, in particular simply connected ones.

2.2.2. *Picture Vector Space.* Given a local relation R, and a non-zero number d, we can define a vector space for each surface Y, not necessarily orientable. Let $\tilde{V}(Y)$ be the vector space spanned by finite linear combinations of multi-curves in Y. This is a vector space of uncountable dimension. Now we form a quotient space $V(Y)$ by modulo the following subspace generated by three type of relations:

(1) If S_1 and S_2 are isotopic, then $S_1 - S_2 = 0$.
(2) Two multi-curves S_1 and S_2 differ by m bounding simple closed curves (bscs), then $S_1 = d^m S_2$ if D_2 has m many more bscs than S_1. When m is negative, it is understood S_1 has $-m$ more bscs than S_2.
(3) If C_n multi-curves S_i are the same outside a topological disk B^2 in Y, and S_i restricted to B^2 is topologically the same as the D_i in the local relation R, then $\sum_{i=1}^{C_n} c_i S_i = 0$.

Almost every local relation will lead to either 0 or infinite dimensional quotient vector spaces $V(Y)$. The relations for multi-curves that can lead to finite dimensional non-zero vector space $V(Y)$ for all surfaces are essentially the Jones-Wenzl projectors p_r for TL diagrams.

2.2.3. *Toric Code and Double Ising* $(2+1)$*-TQFTs.*

Theorem 2.7. (1) *The second Jones-Wenzl projector gives rise to a TQFT, which is called the toric code TQFT.*
(2) *The third Jones-Wenzl projector gives rise to a TQFT, which is called the double Ising TQFT.*

2.2.4. *DFib* $(2+1)$*-TQFT.* The local relation for double Fibonacci is from the F-matrix of the Fibonacci TQFT.

A string-net S in a surface Y is an unoriented trivalent graph including multi-curves as special string-nets. A tadpole is ...PICTURE!!!

We can repeat the same construction above by replacing a multi-curve with a string-net. Now let $\tilde{V}(Y)$ be the vector space spanned by finite linear combinations of string-nets in Y. Then we form a quotient space $V(Y)$ by modulo the subspace generated by the following four type of relations:

(1) If two string-nets S_1 and S_2 are isotopic, then $S_1 - S_2 = 0$.
(2) Two string-nets S_1 and S_2 differ by m bscs, then $S_1 = \phi^m S_2$ if D_2 has m many more bscs than S_1, where ϕ is the golden ratio.
(3) If a string-net has a tadpole, then $S = 0$.
(4) If three string-nets S_i are the same outside a topological disk B^2 in Y, and $S_i, i = 1, 2, 3$, restricted to B^2, are topologically the same as the three string-nets, then $\sum_{i=1}^{3} c_i S_i = 0$.

Theorem 2.8. *The vector spaces $V(Y)$ and the state-sum partition functions below form a TQFT, which is called the double Fibonacci TQFT.*

Proposition 2.9. *Let Γ be a trivalent graph in Y whose complements are all simply connected, then the admissible colorings of Γ form a basis of the DFib vector space.*

Push the string-net into the graph.

To complete the above construction into TQFTs, we need to construct the generalized partition functions. The standard construction is by state-sum on a triangulation of a bordism. It will be very interesting to give a direct picture construction of the generalized partition functions. They should be vector spaces in bordisms spanned by relative surfaces or simple polyhedra.

Exercise 2.10.
(1) Find a basis for $V(T^2)$ for the toric code and DFib. Both are 4 dimensional. Color the embeded theta graph in T^2 to have 5 string-nets: 4 multi-curves plus a theta. They have one linear relation, which is given by the plaquette term of the Levin-Wen Hamiltonian. It is also a version of the 4th Jones-Wenzl projector.
(2) Fnd the representation of the MCG of the torus for the toric code and DFib, and show the image of DFib rep for the torus is $A_5 \times A_5$. For the torus code, it is probably S_3.

2.3. **Physical Interpretation of Pictures and Local Relations.** One way to understand the picture TQFTs is to regard them as continuous limits of lattice TQFTs. Each picture TQFT above has a realization as rigorously solvable lattice model with a LCP Hamiltonian that generalizes the toric code.

Given a surface Y, a mutli-curve can be thought as a quantum state in quantum magnetism such as the domain walls between spin-ups and spin-downs. The total Hilbert space is given by $\tilde{V}(Y)$, which will decompose into its energy eigenspaces $\tilde{V}_i(Y)$. Then one meaning of the local relation for $\tilde{V}_i(Y)$ is that the local relation specifies a local term of the Hamiltonian that will enforce this relation for the ground states. The quotient $V(Y)$ is a version of the ground state manifold. Here we should regard quotienting $\tilde{V}(Y)$ by the local relation as integrating out the higher energy states and the integration result in the low energy physics is encoded by the local relations.

2.4. **Mathematical Origin of the Local Relations.** The local relations are Jones-Wenzl projectors from subfactor theory. To understand them, we first define the Temperlieb-Lieb algebras which appeared first in Onsager's solution of the Ising model implicitly.

Fix a non-zero complex number d and an integer $n \geq 1$. Consider the square with n points at the bottom and top edges, respectively. Then the linear combinations of the TL diagrams form a vector space of dimension C_n. A multiplication of a TL diagram D_1 with D_2 can be defined by stacking D_2 on top of D_1 and then rescale the two squares back to one. When there is a loop, we delete the loop and multiply the remaining diagram by d. Extending this multiplication bilinearly, we obtain the Temperley-Lieb algebra $TL_n(d)$. It is not hard to see that $TL_n(d)$ is generated by the following $n-1$ diagrams U_i as an algebra:

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Theorem 2.11. *The TL ajgebra $TL_n(d)$ has the following presentation by the generators U_i:*

(1) $U_i^2 = dU_i$.
(2) $U_i U_{i\pm1} U_i = U_{i\pm1} U_i U_{i\pm1}$.

By drawing TL diagrams, it is easy to check the above relations hold. To see that any other relation follows, we show any isotopy can be realized by the two types.

Theorem 2.12. *There is unique no-zero element p_n in $TL_n(d)$ such that*

(1) $p_n^2 = p_n$
(2) *For any $x = a+B$, where B is monomial in its generators U_i, $p_n x = x p_n = a p_n$.*

Proof. Let us $p_n = c + U$, where U is a monomial in U_i's. Then $p_n^2 = p_n(c+U) = cp_n$, so $c = 1$. Let $p_n' = 1 + U'$ be another such element, then

$$p_n p_n' = (1+U)p_n' = p_n' = p_n(1+U') = p_n.$$

To prove existence, we construct p_n inductively.

□

There is the famous Markov trace on $TL_n(d)$ by connecting the corresponding points of the top and bottom edges of the square with n outside disjoint arcs. Given a TL diagram D, we dfine $Tr(D) = d^m$ if the closure of D has m loops. Denote $\Delta_n = Tr(p_n)$, then $\Delta_0 = 1, \Delta_1 = d$ and $\Delta_{n+1} = d\Delta_n - \Delta_{n-1}$—the Chebyshev polynomials of second kind. If d is not a root of Δ_n, then TL_n is a semi-simple algebra. If d is a root of Δ_n, then the kernel of the inner product induced by

Markov trace is generated by p_n and the quotient of TL by the ideal generated by p_n is semi-simple.

Easy calculation shows $\Delta_3 = d(d^2-2)$ and $\Delta_4 = (d^2-d-1)(d^2+d-1)$. Hence the values of d for the toric code, double Ising, and DFib are just roots of Δ. We will construct the TL category later and then p_i's will be simple objects of the category and serve as models for anyons.

2.5. **State Sum.** To construct the partition functions of the picture TQFTs, we need Pachner moves to relate different triangulations of the same manifold. We will only give the state-sum definition of the partition functions and leave the proof of the desired properties to later chapters.

The method of state sum is a mathematical version of state sums in statistical mechanics. The most important quantity is the partition function Z. A good example is the classical Ising model on a lattice G in a surface Y. Each vertex v can have two possible states $\{-1,1\}$, which is interpreted as spin-down and spin-up respectively. Then there will be $2^{V(G)}$ many possible sates s for the lattice (or graph) G, where $V(G)$ is the set of vertices of G. Each state s is an assignment of $s(v)=\pm 1$ to each vertex. We will use $v(e)_\pm$ to denote the two vertices of an edge e. We assign each state s an energy $H(s) = -\sum_e s(v(e)_+)s(v(e)_-)$. The partition function is then $Z(G) = \sum_s e^{-\beta H(s)}$, where β is a positive number, physically $\beta = \frac{1}{kT}$. To solve the model means to find an explicit analytic formula for Z. The most famous solution is Onsager's formula for the square lattice in the plane.

The partition function Z is the normalization factor to assign a probability to each state s according to its energy: $p(s) = \frac{e^{-\beta H(s)}}{Z}$. It follows that the lower the energy of a state is, the higher its probability. The importance of the partition functions is that most thermodynamic quantities of the system can be derived from the partition function. For example, the free energy $F = -\frac{1}{\beta}\partial_\beta lnZ$.

Mathematically for state sums, we first need to have a combinatorial presentation of each manifold, and a set of moves that will connect any two presentations of the same manifold. Secondly, we define states on a combinatorial presentation and form a state sum like the partition function. Finally, we prove topological invariance of the state sum, ie the state sum is independent of all the choices, in particular invariance under the moves that connect any two presentations. Basically the mathematical insight would be that the moves are related to some algebraic structures and the state sum is designed based on properties of such algebraic structures that will lead to topological invariance of state sums.

There are no standard procedures to write down all state sums. In general, we give a manifold a triangulation and color vertices, edges, faces... or part of them such as edges with basis of algebras. We call a coloring a state and assign a "weight" to a state and form a sum over all admissible states. For topological applications, the key to topological invariance is Pachner theorem which provides a finite set of moves that is sufficient to relate any two triangulations of the same manifold. We are going to discuss this theorem later.

To define the state sum for the toric code and double Fibonacci TQFTs, we need only two colors as in the Ising model above. We will call the two colors $\{0, 1\}$. Let us first consider only closed manifolds. Let X be a closed 3-manifold with a triangulation Δ. We will denote its vertices, edges, faces (or triangles), and tetrahedra as $V(\Delta), E(\Delta), F(\Delta)$ and $T(\Delta)$, and V, E, F, T will be the cardinality of each set. A state s of the triangulation Δ is an assignment of 0 or 1 to each edge. A state s is admissible for the toric code (or the double Fibonacci) if the three edges of any face have either 0 or 2 edges colored by 1 (or 0 or 2 or 3 edges colored by 1). To give a weight to a state, we will first define weights for a colored vertex, edge, face, or tetrahedron. For the toric code, the weight of an admissible state s is $w(s) = 1$, so the state sum for X is

$$Z(X, \Delta) = \frac{1}{2^V} \sum_{s:\text{admissible}} 1.$$

The state sum for the double Fibonacci is more complicated. We will denote the states $\{0, 1\}$ as $\{1, \tau\}$ to conform with the Fibonacci theory notation as τ is the Fibonacci anyon. The weights of vertex, edge, face, and tetrahedron are as follows for admissible states:

(1) The weight of a vertex is $\phi + 2$, where ϕ is the golden ratio.
(2) The weight of an edge colored by 1 is 1, and the weight of an edge colored by τ is ϕ, where ϕ is the quantum dimension of the Fibonacci anyon.
(3) The weight of a face with all three edges colored by 1 is 1, the weight of a face with two edges colored by τ is ϕ, and the weight of a face with three edges colored by τ is $\phi^{\frac{3}{2}}$.
(4) The weight of a tetrahedron with all 6 edges colored by τ is $-\phi$, 5 edges colored by τ is $\phi^{\frac{3}{2}}$, and 4 edges colored by τ is ϕ.

Then we define the weight of an admissible state s as

$$w(s) = \frac{\prod_{e \in E(\Delta)} w(e) \prod_{t \in T(\Delta)} w(t)}{\prod_{v \in V(\Delta)} w(v) \prod_{f \in F(\Delta)} w(f)}.$$

Finally, the state sum of X is $Z(X, \Delta) = \sum_{s:\text{admissible}} w(s)$.

It is very mysterious at this stage why this is topological invariance and the right thing to study. The local weights of vertices, edges, faces, and tetrahedron all come from a spherical fusion category which is invented to guarantee the topological invariance. For now, we just take it as a magic state sum that works. They also have physical meanings in condensed matter physics and potentially 3D quantum gravity.

Newt we will extend to bordisms with boundary. Given a bordsim X from Y_1 to Y_2, we fix a triangulation Δ_i for Y_i. Then find a relative triangulation Δ of X

that extends the two triangulations on its boundary. A state of Y_i or X will be an assignment of $\{0,1\}$ to each edge of the triangulation. Fix a state on each Y_i, a relative state for X is a state that is the two given states on the boundary. Now we will assignment a weight to each relative state as follows:

This relative state sum defines a map from the vector space ...

It descends to a linear map on...

To see the equivalence of lattice model with the string-nets, we push the string-nets into the 1 skeleton of a triangulation and use a triangulation to realize all string-nets.

Exercise 2.13. (1) Show the invariant for the toric code is $Z(X) = 2^{b_1(X)-1}$ for a closed 3-manifold, where $b_1(X)$ is the first $\mathbb{Z}_2$-Betti number of X.

(2) Show that the invariant Z from DFib is a sum

$$Z(X) = (5+\sqrt{5})^{-\chi(X)} \sum_{S \subset P} (-1)^V (\phi)^{-\chi(S)-V},$$

where P is any special spine of X and the sum is over all special subpolyhedra of P.

2.6. **Where Do the State Sums Come From.** It seems to be magic that such a complicated sum turns out to be a 3-manifold invariant. An explanation can be given using the "regular representation" of a ribbon fusion category. In each ribbon fusion category with the set x_i of isomorphism classes of simple objects, we can form a formal sum $\Omega = \sum_i d_i x_i$, where d_i is the quantum dimension of x_i. Suppose an oriented 3-manifold is given as a handle-decomposition $X = H_g \cup_f H_g$, where H_g is the standard genus=g handlebody in $\mathbb{R}^3$ and $f : H_g \to H_g$ is the identification diffeomorphism, then $Z(X) =< v, Z(f)v >$, where v is the vector in $V(\partial H_g)$ when H_g is regarded as a bordism from the empty set to ∂H_g. The map f is determined by the images of the standard meridians of Σ_g, denoted as $\alpha_i, i = 1, 2, \cdots, g$ and their images under f as $\gamma_j, j = 1, 2, \cdots, g$. The curves γ_j on the standard handlebody with α_i form the Heegaard diagram of a 3-manifold. From a Heegaard diagram, we get a link by pushing γ_j slightly into the handlebody H_g. This is a link with $2g$ components. Now attaching an Ω to each component and expand formally into a linear sum with coefficients and colored links. If we know how to evaluate the colored link invariants, then we get a number which is the state sum invariant. This is the so-called chain-mail formulation of Turaev-Viro invariant.

Now suppose X is given a triangulation. Then there is a Heegaard splitting by taking the dual of the triangulation. Applying the chain-mail construction to this Heegaard splitting, we will get the state sum above. This also leads to a better way to evaluate the 3-manifold invariant.

3. (1+1)-TQFTs

The construction and classification of $(1+1)$-TQFTs is an excellent example for the general theory of TQFTs. There are many ways to present a surface. We choose to use pants decompositions. A pair of pants is a three punctures sphere. One reason that the topology of surface is so simple is that except the sphere and torus, every closed orientable surface has a decomposition into pairs of pants and any two decompositions are related by simple moves that we will call them F-move and S-move. Then to construct a $(1+1)$-TQFT, we can assign a vector space to a surface with a pants decomposition and verify that the assignment is independent of the chosen pants decomposition by checking invariance under the moves. The moves between pants decompositions correspond well to the properties of commutative Frobenius algebras. Therefore, the category of $(1+1)$-TQFTs is equivalent to the category of commutative Frobenius algebras.

3.1. **From TQFT to Frobenius Algebras.** We will show that each $(1+1)$-TQFT leads to commutative Frobenius algebra. There are many equivalent ways to define a Frobenius algebra. We will use any of the following as a definition.

Proposition 3.1. *Let A be a finite dimensional associative unital algebra. Then the following are equivalent:*

(1) *There is an A-module isomorphism $\theta : A \to A^*$.*
(2) *There is a non-degenerate linear map $\tau : A \to \mathbb{C}$ in the sense that the kernel has only 0 ideals.*
(3) *There is a non-degenerate pairing $\lambda : A \otimes A \to \mathbb{C}$ such that $\lambda(a, bc) = \lambda(ab, c)$.*
(4) *A is a bi-algebra such that the multiplication and co-multiplication satisfy the following I=H relation.*

Proof. From (1) to (2), set $\tau = \theta(1)$. From (2) to (3), set $\lambda(a, b) = \tau(ab)$. From (3) to (1), take the adjoint.

□

Example 3.2. (1) Every group algebra $\mathbb{C}[G]$ is a Frobenius algebra with $\theta : \sum_g c_g g \to \mathbb{C}$ by $\theta(x) = c_e$.
(2) For a closed simply connected 4-manifold M, $H^2(M : \mathbb{C})$ is a Frobenius algebra with the pairing given by $\lambda(\omega_1, \omega_2) = \int_M \omega_1 \wedge \omega_2$.
(3) Every finite dimensional Hopf algebra is a Frobenius algebra.

Given a $(1+1)$-TQFT $F = (V, Z)$ and let S^1 be the standard circle in the complex plane. Then $A = V(S^1)$ is a finite dimensional vector space. A can be canonically identified with A^* as follows. The vector space $V(-S^1)$ is canonically dual to $V(S^1)$, so canonically isomorphic to A^* as a vector space. We need a lemma.

Proposition 3.3. *The mapping class group of the circle S^1 is trivial, i.e., every orientation preserving diffeomorphism of the circle S^1 is isotopic to the identity.*

Therefore, there is a unique way to identify $-S^1$ with S^1 up to isotopy. We can just choose the complex conjugation r which is an orientation-reversing map of S^1 or orientation preserving map from S^1 to $-S^1$. Hence $Z(r) : V(S^1) \to V(-S^1)$ is a canonical isomorphism. Composing with the isomorphism from $V(-S^1)$ to A^*, we have a map:

$$\theta : A \to A^*.$$

The vector space A has both a multiplication and a co-multiplication by the following trees:

Theorem 3.4. *The vector space $V(S^1)$ is a commutative Frobenius algebra.*

To understand the partition functions, we use the classification theorem of orientable surface to see that any bordism from m circles to n circles are diffeomorphic to the standard model bordism as follows: tree-handles–tree.

Therefore, the linear map $Z(X) : A^{\otimes m} \to A^{\otimes n}$ is

$$\Delta_{n-1} \cdots \Delta_1 \cdot H^{\otimes g} \cdot \mu_1 \cdots \mu_{m-1}.$$

In particular, for a closed surface of genus$=g$, $Z(Y_g) = Tr(H^{\otimes g})$.

3.2. **From Frobenius Algerba to TQFT.** Our combinatorial presentation for a surface will be a pants decomposition. This is basically a Morse function on the surface. To keep our presentation elementary, we will outline the idea from Morse theory, but re-prove the necessary theorems using combinatorial method.

When a surface is given as a pants decomposition, we need to know the moves that will connect any two pants decomposition. In terms of Morse functions, we will consider a path of Morse functions that connects the two given Morse functions. There will be finitely many isolated moments on the path that the function is not Morse. There are two possibilities; either two critical points have the same critical value or a birth-death event occurs. Birth-death events correspond to the birth-death of handle pairs of adjacent indices. The same critical values are for two critical points of index 1 so are supported inside a 4-punctured sphere. Then the Morse modification is the two different ways to decompose a 4-puncture spheres into two pairs of pants. Analyzing the theory completely, we obtain that each surface can be decomposed into a composition of five building pieces and any two such decompositions are related by 5 moves.

3.2.1. *Pants Decompositions.*

Theorem 3.5. (1) *Each bordism can be decomposed into a composition of the following* 5 *pieces:*

(2) *Every two pants decompositions of the same surface Y are related by the following 5 moves:*

Given a commutative Frobenius algebra A, we will assign A to the standard circle S^1. For any bordism X as the standard model, we assign the linear map:

We need to show that for any other pants decomposition, we will have the same map.

For any other circle Y, we will assign also A to Y, and choose a diffeomorphism from $S^1 \to Y$, then a bordism from Y_1 to Y_2 will be turned into a bordism for the standard circle. We will assign the map ...

3.3. Classification.

The best way to state the equivalence of the classification of $(1+1)$-TQFTs is to formulate the equivalence as an equivalence of two categories.

Theorem 3.6. *The category of $(1+1)$-TQFTs is equivalent to the category of commutative Frobenius algebras.*

Exercise 3.7. Show that the eigenvalues of the handle operators for all Frobenius algebras from $H^2(M;\mathbb{C})$ of a 4-manifold is 0.

4. CAT

All categories together with functors between them and natural transformations between functors form a bicategory that will be denoted as CAT. We will treat general bicategories in detail later.

4.1. Category Recalled.

A category can be large in the sense that its objects form a class that is not a set. Categories whose objects do form sets are called *small* categories. On the other hand, the morphisms between any two objects in a category are required to form a set. Since we will often promote a category to a higher category by assuming that we have a single new object for the higher category and treat the objects of the original category to be morphisms of this new object, we require all our categories to be small. This does lead to some foundational questions when we form categories such as the category of all sets or the category of all topological spaces[6]. A topological space is a set with a topology. Any set can be regarded as a topological space endowed with the discrete topology, which is actually a 0-dimensional manifold. In order to have a small category, we assume that any set or topological space is cat-isomorphic[7] to a subset of the

[6]Considering isomorphism classes of objects is not a good solution to this problem for TQFTs and its applications because we often need to glue together manifolds, and consider physical processes in the space manifolds.

[7]We will consider many different categories of manifolds with extra structures: orientation, spin structure, smooth structure, PL structure, framing,.. and homeomorphisms between such manifolds that preserve the extra structures. We will call such homeomorphisms cat-isomorphisms or sometimes just isomorphisms instead of homeomorphisms, orientation preserving diffeomorphisms, etc.

Euclidean space $\mathbb{R}^n$ for some integer $n \geq 0$. We consider $\mathbb{R}^n$ as the subset $\mathbb{R}^n \times 0$ of $\mathbb{R}^{n+1}$. Then every set or topological space that we discuss is a subset of $\mathbb{R}^\infty$.[8]

Definition 4.1. *A small category $\mathcal{C}$ consists of a set of objects and an associated set* $\mathrm{Hom}(x, y)$ *of morphisms to each ordered pair of objects x, y endowed with a distinguished element called the identity and a composition of composable morphisms*

$$\mathrm{Hom}(y, z) \times \mathrm{Hom}(x, y) \to \mathrm{Hom}(x, z)$$

satisfying the following axioms:

(1) *(quadrilateral axiom)*
(2) *(eye-glass axiom)*

There are two standard ways to compose the morphisms: composition of functions or juxtaposition of arrows. We choose to use the composition of functions. We will denote the set of objects of a category $\mathcal{C}$ by $\mathcal{C}^0$ and the morphism set of two objects $x, y \in \mathcal{C}^0$ as $\mathrm{Hom}(x, y)$ or ${}_y\mathcal{C}^1_x$.

4.1.1. *Examples of Categories.*

(1) Set
(2) Vec
(3) Hil
(4) Group
(5) Algebra
(6) $\pi_{\leq 1}(X)$ of a space X

4.2. **Functors.**

4.2.1. *Examples of Functors.*

(1) Double dual
(2) Representation of groups
(3) First and Second quantization

4.3. **Natural Transformations.**

4.3.1. *Example of Natural Transformations.*

(1) Double dual

4.3.2. *Yoneda Lemma.*

4.3.3. *Nat(Id,Id).*

[8] The history of the notion of a manifold is an interesting lesson about how mathematics is making progress in a strange way. Topologists spent decades to abstract the definition of a manifold from subsets of the Euclidean spaces. Then Whitney proved his famous embedding theorem that actually all the abstract manifolds can be embeded into Euclidean spaces. It seems that all the efforts for the general definition are wasted. Yet the secret to manifold topology is uncovered by the proof of Whitney's embedding theorem—the Whitley trick.

4.4. Functor Categories.

5. Monoidal Category

Monoidal categories serve two purposes: one for organizing the axioms of a TQFT, and the other, when endowed with extra structures, for input data for the construction of TQFTs. While TQFTs are inspired by quantization, monoidal categories are inspired by categorification of monads. Endowed with extra structures such as fusion and modular, they are beautiful mathematical structures that can be classified.

5.1. **Pentagons and Triangle Axioms.** A monad is a set with a binary operation that is associative with a unit. A monoidal category is a categorification of a monad so it also has a binary operation called a tensor product. Since sets are categorified to categories, so the binary operation for a monoidal category is a bi-fucntor (a functor with two variables). The associativity will be categorified to a coherent functorial isomorphism: a natural transformation which is an isomorphism between two functors. The coherence is encoded into the famous pentagon and triangle axioms.

5.2. Monoidal Functors.

5.3. Braided Monoidal Categories.

5.4. Group Categories and Graded Vector Spaces.

5.5. **Tangle and Tangle Diagram Categories.** Their reps are generalizations of the Jones polynomial.

6. Bord$^{\mathrm{CAT}}(n+1)$ and CAT-ASTF TQFTs

The same axioms of TQFTs can be used to define a variety of CAT-ASTF TQFTs for manifolds with extra structures. These TQFTs will be referred to as CAT-ASTF TQFTs. So the toric code and DFib are TOP-ASTF TQFTs, where TOP means the nonoriented TOP category.

Definition 6.1. *Let Bord$^{CAT}(n+1)$ be the symmetric monoidal category of $(n+1)$-CAT-manifold category. Then an $(n+1)$-CAT-ASTF TQFT is a symmetric monoidal functor from $F=(V,Z)$: Bord$^{CAT}(n+1)$ to* **Vec**.

Unfortunately, the famous Witten-Chern-Simons or mathematically Reshetikhin-Turaev TQFTs do not fit into this definition. In order to cover those TQFTs, we need an extended $(3,2,1)$-TQFt with anomaly. We will encode the anomaly by a projective functor so WCS (or RT) TQFTs are projective 2-functors from the manifold category of $1,2,3$-manifolds to the 2-vector space of modular categories. Turaev-viro TQFTs are anomaly-free, so they can be further extended to points.

To formalize Witten-Donaldson TQFT, we need further relaxation which has not been coded yet. Important ingredients will be grading and symmetry.

6.1. Structures on Manifolds.

6.1.1. *Spin, Spin^c, Pin$(\pm)$ Structure.* The spin structure on a manifold for the Dirac equation can be considered as a minor anomaly. The electron is not completely local.

6.1.2. *Combing, Framing, 2-framing and Lagrangians.*

6.2. Category of $(n+1)$-TQFTs.

7. Dijkgraaf-Witten TQFTs

8. Manifold Pairings as Universal TQFTs

9. Open Questions and Exercises

9.1. **Frobenius Algebas.** Show $V(T^2)$ from a (2+1)-TQFT is always a Frobenius algebra. For a unitary theory, the "handle operator" is Hermitian, so it can be diagonalized.

The eigenvectors of the handle operator form a basis and the unit can be expressed in this basis. Let e_i be a basis of $V(T^2)$ such that the handle operator is diagonalized, and $\eta(1) = \sum_i a_i e_i$. Then for any vector $v = v^i e_i \in V(T^2)$, the scalars v^i/a_i are well-defined. In particular for a knot in any 3-manifold, we have a colored knot invariant.

9.2. **Combinatorial Presentation of a Spin Structure.** Find a combinatorial presentation of spin structures on a 3-manifold like the presentation of spin structure on a surface using a dimer matching.

9.3. **TQFTs with Same Z.** Show the toric code and the double semion theory have the same Z on all closed 3-manifolds.

Do the double Ising theory and the double $SU(2)_2$ have the same Z?

9.4. **Characterize Values of Z for DFib.** With the right normalization, the values of Z on closed 3 manifolds is in $\mathbb{Z}[\phi]$ and is dense in the positive $\mathbb{R}$. Is Z a homotopy invariant?

9.5. **Relation to Classical Invariants.** Are 3-mfd invariants from DIsing and DFib homotpy invariants?

Property F TQFTs and homotopy inv.

Abelian theories? Yes.

9.6. **Topological Lattice Model for $(1+1)$-TQFTs.** There are lattice models for $(1+1)$-TQFTs with associative algebras as input. The resulting theories are only those $(1+1)$-TQFTs whose spectrum of the handle operators are discrete. How did this quantization of spectrum occur?

As is well-known that there is not intrinsic topological phases of matter in 1D, so all $(1+1)$-TQFTs do not describe stable intrinsic TPM. Either they are SPY or they are unstable with respect to perturbations.

9.7. **Loop Operators of Lattice Models.** Half-braidings.

9.8. $(2+1)$**-TQFTs with** $dimV(S^2) > 1$**.** Are they SPTs?