HIGHER ALGEBROID THEORY

A category can be thought as a categorification of a set. Interesting sets in mathematics include topological spaces—sets with additional structures. Among spaces, manifolds are most interesting. In this analogy, we would like to categorify spaces and eventually manifolds so we understand quantum manifolds and their geometry. Fusion categories are categories with many additional structures and could be thought as quantum finite sets.

1. Categories as Mathematical Structures

We will study categories as mathematical structures. In mathematics, there is a fruitful program called categorification, which is closely related to linearization and quantization. A large portion, if not all, of mathematics is based on set theory. We are interested in if category theory can enrich mathematical foundations. In linearization, sets will become bases of vector spaces, and the converse operation is projective measurement.

Quantum physics ushers in a new era in mathematics—quantum mathematics deriving inspiration from the hypothetical string theory to mundane quantum phases of matter. We will use category as our framework and take category seriously.

1.1. Algebroids as Quantum Linear Spaces.

1.2. **Representation of Higher Categories.** A function on a space is a one-dimensional representation of the space, and a functor from a space to Vec is like a vector bundle. But how to model superposition of points? We are interested in the algebraization of manifolds and computational processes. Two manifolds are related to Frobenius algebras and three manifolds to Hopf algebras. How about 4-manifolds?

Representation of categories is important and ubiquitous in mathematics. Representation of a finite group can be regarded as a representation of a groupoid. Categorifically speaking, TQFT, TPM, and anyonic quantum computation are all reps in Vec. TQFTs are reps of the space-time bordism categories, TPMs are reps of quantum processes, and circuit models are reps of computational steps, represented by a tangle category in 2D. In computational complexity, we quantify the computing steps by keeping track of how long a computation is.

2. Fusion Categories

3. Ribbon Fusion Categories

4. Drinfeld Center and Witt Group

4.1. Quantum Double and Annualization.

4.2. Mueger's Theorem.

5. Bicategories

5.1. The Bicategory CAT.

5.2. Bimodules.

5.3. $\pi_{\leq 2}(X)$ of a Space X.

5.4. 2-Groups.

6. Tri-category

7. Tetra-category