FROM HIGHER ALGEBROIDS TO TQFTS

We will call $(2+1)$-TQFTs Turaev-Viro (TV) type TQFTs, and $(3+1)$-TQFTs Crane-Yetter (CY) type TQFTs.

1. Picture Modular Functors

1.1. **Colored Curve Diagrams and Local Relations.**

2. Triangulation and PL Manifolds

2.0.1. *Simplicial and Combinatorial Triangulation.*

Definition 2.1.
(1) *Simplicial complex...*
(2) *Star or link*
(3) *Joint*
(4) *PL and combinatorial n-manifolds*

Definition 2.2. *Triangulation*

Example 2.3. The 5-manifold $E_8 \times S^1$

3. State Sum and Pachner Theorem

The construction of TQFTs by state-sum has two ingredients: a presentation of manifolds by some combinatorial objects with moves that connect any two presentations, and some algebraic data. The algebraic data is used to write down a state-sum for the generalized partition function, which is a finite version of the path integral. The difficulty lies in the choice of algebraic data because they need to possess properties to render the state sum invariance under all the moves. The magic is basically the algebraitization of the moves into defining equations of the algebraic data. The solutions of such equations will lead to invariance. A good example is the Kauffman bracket definition of the Jones polynomial of links. Here an oriented link is presented as a link diagram in the plane, and any two link diagrams are related by the three types of Reidemeister moves. The algebraic input is two variables A, B and a state of a diagram is an assignment of A or B to a crossing. The invariance of the Kauffman bracket under Reidemeister move II implies that $B = A^{-1}$ and a contractible loop is $d = -A^2 - A^{-2}$. Then the state-sum is automatically invariant under the third Reidemeister move. A normalization that incorporate Reidemeister one change will lead to the Jones polynomial.

In the following, we will present manifolds by their triangulations. Then any two triangulations are related by Pachner moves.

3.1. Pachner Theorem.

3.2. State Sum.

4. Turaev-Viro Type TQFTs

4.0.1. *3D Pachner Moves.*

5. Crane-Yetter Type TQFTs

5.0.2. *4D Pachner Moves.*

6. From State Sum to Hamiltonian

6.1. Hamiltonian Realization of TV Type TQFTs.

6.1.1. *Kitaev Model.*

6.1.2. *Levin-Wen Model.*

6.2. Hamiltonian Realization of CY Type TQFTs.

6.2.1. *Lattice Model.*

7. Unitary Higher Algebroids of Elementary Excitations

7.1. Loop Operators.

8. Fault Tolerance of TQFTs

8.1. Disk Axiom and Error Correction Code.