INTRODUCTION

Abstract. These are lecture notes of a three-course series on the mathematical foundations of topological quantum computation.

Mathematics is the art of quantifying and shaping a man-made logic world[1]. Since the physical world is also logical, we find many matches between mathematical theories and physical models. The ultimate connection is exemplified by Newton, who is arguably the best mathematician and physicist in history.

On a practical level, many pure mathematicians seek out interesting patterns, then codify and classify them. One of the interesting patterns in quantum physics and quantum computation is the pattern of long-range entanglement. To elucidate the role of the long-range entanglement in quantum phases of matter and quantum algorithms calls for new mathematical tools. The mathematical foundations of long-range entanglement lie at the frontiers of current research in mathematics, physics, and computer science. One terrain that is ideal for vanguards to explore is topological quantum computation (TQC)—an interdisciplinary field at the triple juncture of mathematics, physics, and computer science.

1. Topological Quantum Computation

The goal of TQC is the construction of a large scale quantum computer based on braiding non-abelian anyons—the central part of a futuristic field *anyonics* broadly defined as the science and technology that cover the development, behavior, and application of anyonic devices. Two important mathematical structures underpinning TQC are topological quantum field theory (TQFT) and modular tensor category (MTC). TQFTs arise as low energy effective theories of topological phases of matter, whose elementary excitations in two spatial dimensions are anyons—quantum particles with statistics more general than bosons and fermions.

[1]On many occasions, I have been asked if pure mathematics is part of sciences. My answer is simply no. Mathematics and physics, for instance, aim at different worlds: the mathematical one consists of mind-created theories, while the physical one is all the stuff out there. There are other differences between the two subjects. The ultimate judgement for a mathematical theory is logical correctness-consistent, sound, and complete, while the acceptance of a physical model is determined by the valid experimental results predicted by the model. A physical attribute is defined by experiments whose results are given by a collection of numbers. Subsequently, physical quantities should be algorithmically computable, while mathematics is full of algorithmically uncomputable structures. Similar differences exist between mathematics and other sciences.

The algebraic models of anyons are unitary MTCs, which are also the algebraic data for unitary (2+1)-TQFTs.

Solid, liquid, and gas are all familiar states of matter (the words state and phase are used interchangeably). But by a more refined classification, each state consists of many different states of matter. For example, within the crystalline solid state, there are many different crystals distinguished by their different *lattice*[2] structures. All those states are classical in the sense they depend crucially on the temperature. More mysterious and challenging to understand are quantum states of matter: states of matter at zero temperature (in reality very close to zero). The modeling and classification of quantum states of matter is an exciting current research area in condensed matter physics and topological quantum computation. In recent years, much progress has been made in a particular subfield: topological phases of matter (TPMs). Besides their intrinsic scientific merits, another motivation comes from the potential realization of fault-tolerance quantum computation using non-abelian topological phases of matter.

To carry out quantum computation, we need quantum memories, quantum circuits, and protocols to write and read information to and from the quantum systems. In the anyonic quantum computing model, we first fix a non-abelian anyon type, say x. Then information is stored in the ground state manifold $V_{n,x}$ of n type x anyons (for simplicity, we ignore the boundary conditions.) As n goes to infinity, the dimension of $V_{n,x}$ goes asymptotically as d_x^n, where d_x is the quantum dimension of x. Since x is non-abelian, $d_x > 1$. It follows that when n is large enough, we can encode any number of qubits into some $V_{n,x}$. The ground state manifold $V_{n,x}$ is also a unitary representation of the n-strand braid group B_n; hence, unitary representation matrices serve as quantum circuits. An initial state of computation is given by creating anyons from the ground state and measurement is done by fusing anyons together to observe the possible outcomes. There are important subtleties regarding encoding qubits into $V_{n,x}$ because their dimensions are rarely powers of fixed integers. There is also the important question of whether the braiding matrices alone will give rise to a universal gate set.

2. Quantization and Categorification

A famous quote about quantization attributed to E. Nelson is: “first quantization is a mystery, but second quantization is a functor”. The first quantization in general will undoubtedly continue to generate deep insights into the quantization process, but the situation that we are interested in is very simple—the quantization of a finite set S. First quantization is the process to go from a classical system to a quantum system that modeled by a Hilbert space and a Hamiltonian operator. In the case of a finite set S as the classical configuration space, the quantization

[2]We mean lattices as in physics in the sense they are regular graphs, which are not lattices in the mathematical sense because they are not necessarily subgroups of $\mathbb{R}^n$ for some n.

is simply the linearization of S—the Hilbert space is just $\mathbb{C}[S]$ spanned by the elements of S. Second quantization is the process to go from a single particle Hilbert space to a multi-particle Hilbert space. For simplicity, if we consider a fermion with a single particle Hilbert space V of dimension= n, then the multi-particle Fock space is just the exterior algebra $\wedge^* V$ of dimension 2^n. The process of dequantization is measurement: when we measure a physical observable $\mathcal{O}$ at state $|\Psi\rangle$, we arrive at a normalized eigenvector e_i of $\mathcal{O}$ with probability $p_i = \langle e_i|\mathcal{O}|\Psi\rangle$.

According to M.-M. Kapranov and V.-A. Voevodsky, the main principle in category theory is: "in any category it is unnatural and undesirable to speak about equality of two objects". The general idea of categorification is to weaken an equality to some version of isomorphism. Naively, we want to replace a natural number n with a vector space of dimension n. Then the categorification of a finite set S of n elements should be $\mathbb{C}[S]$ of dimension n. It follows that the equality of two sets $S_i, i = 1, 2$ should be relaxed to an isomorphism of the two vector spaces $\mathbb{C}[S_i], i = 1, 2$. Isomorphisms between vector spaces can be functorial or not as the isomorphisms between V and V^{**} or V^* demonstrated: while the first one is functorial, the second one is not.

We are interested in finding procedures for quantization and categorification that are as functorial as possible. Categorifications of monoids and rings lead to monoidal categories and algebroids—an active research area in quantum algebra and quantum topology.

3. Quantum Mathematics

Our long term goal is to study quantum mathematics and their application to quantum physics and quantum computing. Quantum mathematics will be a catch-all for mathematics needed to model quantum systems and mathematics based on quantum logic. A large part of classical mathematics is based on set theory and classical logic. Quantum logic is not mature enough that would enable us to pursue a new layer of mathematics. As a relevant step, we propose that higher algebroid theory as part of the quantum mathematics. We use *algebroid* to mean $\mathbb{C}$-linear category. We are interested in representations of categories with additional structures such as the monoidal categories of n manifolds and tangles in $\mathbb{R}^3$ into the algebroid of finite dimensional vector spaces.

Representation is a powerful method to study new structures through familiar ones, e.g. we gain deep understanding of groups via their representations in linear transformations. One of the most important concept in TQC is an anyon—a quantum particle whose statistics can be more general than boson and fermion. The mathematical model of an anyon is a simple object in a unitary MTC. To understand an object x in a unitary MTC $\mathcal{C}$, we consider the functor from $\mathcal{C}$ to **Hil**: $x \to \text{Hom}(x, y)$, where **Hil** is the category of finite-dimensional Hilbert spaces. The Hilbert space $\text{Hom}(x, y)$ models the processes between the two states x, y. In classical physics, the excited states of a particle at a point $x \in M$ form its tangent space T_xM. Analogously, we may treat the MTC as the quantum tangent space of an anyonic point. More speculatively, we could imagine the quantum phase space as some lattice (or triangulation) limit of the tangent bundle by replacing a vertex with an anyonic point and the fiber over the vertex with the MTC. Algebroids with additional structures vs categories is the same as manifolds vs topological spaces. Therefore, we are interested in algebroids with certain structures that could be thought as quantum manifolds. In a sense, a MTC is just a quantum finite set.

It has been often complained that category theory is just a language for organizing things. But higher order language is important for the progress of mathematics, and deep mathematical insights are just some kinds of powerful languages. For the opposite direction, if we adopt the Bourbaki logical formalism, then we need $4,523,659,424,929$ many symbols together with $1,179,618,517,981$ links between certain of those symbols in order to express the structuralist definition of the number 1. How big the number 1 would be in this language if each symbol and link has a legible physical size?

4. Plan

As we continue to gain deeper understanding of $(2+1)$-dimension, we will also expand into $(3+1)$-dimension, and symmetry-enriched TPMs. In the real world, $3D$ materials are more common and TPMs are always coupled to conventional degrees of freedom, therefore conventional symmetry such as time-reversal can also be present. It is important to understand how to combine conventional symmetry with topological order—symmetry enriched topological order (SET). Time-reversal symmetry combined with trivial topological order leads to the symmetry protected topological order (SPT) exhibited in topological insulators.

We plan to start the course with the Heisenberg model of a beautiful real quantum spin liquid—the mineral Herbertsmithite, and end the first quarter with the theorem that every Turaev-Viro type $(2+1)$-TQFT has a Hamiltonian realization via Kitaev or Levin-Wen models. Such Hamiltonian realizations of TQFTs are topological qubit liquids which generalize gapped quantum spin liquids. The second quarter will focus on Reshetikhin-Turaev type (or physically Witten-Chern-Simons type) $(2+1)$-TQFTs and their applications to the fractional quantum Hall effect and anyonic quantum computing. The last quarter will be on extensions of $2D$ theories to ones enriched with conventional group symmetries and in $3D$.