

BOOK REVIEW OF INTRODUCTION TO TOPOLOGICAL QUANTUM COMPUTATION BY J. PACHOS

ZHENGHAN WANG

Shor's quantum factoring algorithm and the efficient simulation of quantum evolutions by quantum computers are compelling evidence that quantum physics will change the landscape of information science in the future. Despite great progress in the last two decades, the construction of a large scale quantum computer is still in its infancy, largely due to decoherence. A revolutionary idea that emerged during the late 1990s is to use topology to protect information, thus solving the problem of the fragility of the qubit.

Topology is a pure mathematical subject that studies properties about spaces that are invariant under small deformations. By their very nature, topological properties are non-local, yet they can manifest themselves through local geometric properties of spaces such as curvature. A famous example of this phenomenon is the beautiful Gauss-Bonnet theorem that the Euler characteristic of an oriented surface Y with a Riemannian metric is the integral of the Gaussian curvature K at each point: $\frac{1}{2\pi} \int_Y K dA = \chi(Y)$, where the Euler characteristic $\chi(Y) = 2 - 2g(Y)$, $g(Y)$ being the genus or the number of handles of Y . This is an excellent example of a local encoding of non-local information: the non-local Euler characteristic $\chi(Y)$ of a surface Y spreads into the local curvature (reminiscent of the storing of Lord Voldemort's soul in seven Horcruxes). The Gauss-Bonnet theorem can be used to give an explanation for why the Hall conductance in the integer quantum Hall effect is so precise. This connection between the Gauss-Bonnet formula and the Hall conductance in the integer quantum Hall effect is covered in Section 2.3 of the book under review.

Miraculously, there are exotic quantum phases of matter with topological degrees of freedom—topological phases of matter—that possess topological properties. Those are quantum phases of matter (phases of matter at zero temperature) whose low energy effective theories are topological quantum field theories. Then information that is encoded into topological degrees of freedom is automatically immune to local errors. Therefore protection of information from local interactions with the environment is conferred at a physical level with no active error correction needed—an idea due to A. Kitaev of Caltech around 1997 [K1]. Around the same time, M. Freedman of Microsoft Research was independently working on a computing model based on topological quantum field theories, which has been shown to be polynomially equivalent to the standard circuit model of quantum computing [F]. The synthesis of the two ideas ushered in topological quantum computation.

To characterize topological phases of matter, we study their elementary excitations, which are quasi-particles. Amazingly, those quasi-particles in two spacial dimensions do not necessarily obey the standard bose- or fermi- statistics. Instead, when two quasi-particles are exchanged in two spacial dimensions among many such quasi-particles, their wave function can acquire a phase $e^{i\theta}$ for any θ between 0 and 2π . Quasi-particles with $\theta \neq 0$ or π are called *anyons*. There are experimental data that strongly support the theory that anyons exist as elementary excitations in fractional quantum Hall liquids of filling factors such as $\nu = \frac{1}{3}$. A more exotic kind of anyons

are quasi-particles which possess a nearly degenerate multi-dimensional ground state manifold when many of them are well-separated with fixed locations. Such quasi-particles are called *non-abelian anyons*. Theoretical algebraic models of anyons are unitary modular categories. According to this model, an anyonic system first comes with a label set which consists of all the possible anyon types. Then there are fusion rules which describe possible outcomes when two anyons are brought together. For each anyon type, there is a number, called the quantum dimension of the anyon type, which controls the asymptotic rate of the degeneracy of any multi-anyon space. Using quantum dimension, we have another more fundamental characterization of non-abelian anyons. If the quantum dimension is 1, then the anyon is abelian. Otherwise, the quantum dimension is bigger than 1 and the anyon is non-abelian. Non-abelian anyons lead to multi-dimensional representations of the braid group, rather than simply a phase as for abelian anyons.

To carry out quantum computation, we need quantum memories, quantum circuits, and protocols to write and read information to and from the quantum systems. In the anyonic quantum computing model, we first fix a non-abelian anyon type, say x . Then information is stored in the ground state manifold $V_{n,x}$ of n type x anyons (for simplicity, we ignore the boundary conditions in this discussion.) As n goes to infinity, the dimension of $V_{n,x}$ goes asymptotically as d_x^n , where d_x is the quantum dimension of x . Since x is non-abelian, $d_x > 1$. It follows that when n is large enough, we can encode any number of qubits into some $V_{n,x}$. The ground state manifold $V_{n,x}$ is also a unitary representation of the n -strand braid group B_n ; hence, unitary representation matrices serve as quantum circuits. An initial state of computation is given by creating anyons from the ground state and measurement is done by fusing anyons together to observe the possible outcomes. There are important subtleties regarding encoding qubits into $V_{n,x}$ because their dimensions are rarely powers of fixed integers. There is also the important question of whether the braiding matrices alone will give rise to a universal gate set.

In anyonic quantum systems, the topological degrees of freedom of anyons in $V_{n,x}$ are believed to be error-correction codes. This conjecture can be proved in a large class of spin models—the Levin-Wen model, which is a dual generalization of the quantum double model including the celebrated toric code [K1]. Therefore, an anyonic quantum computing model is inherently fault-tolerant. Since the effective Hamiltonian on the topological degrees of freedom is 0, no continuous evolutions are possible on $V_{n,x}$; hence, $V_{n,x}$ is a stable quantum memory.

There has been important progress in the last several years toward finding or engineering non-abelian anyons, especially for the Ising anyonic system. This is an anyonic system with a label set $L = \{1, \sigma, \psi\}$, where 1 denotes the ground state, σ a non-abelian anyon of quantum dimension $d = \sqrt{2}$, and ψ a Majorana fermion. Experimental signature of the Ising anyon σ and the Majorana fermion ψ is reported in fractional quantum Hall liquids of filling fractions $\nu = \frac{5}{2}$. The Majorana fermion ψ is abelian, but it is responsible for the ground state degeneracy of well-separated multi-Ising anyons with fixed locations, which is a representation of the braid group. The same braid group representation can be realized projectively by Majorana zero modes in nanowires, and exciting experimental progress has been reported recently. But strictly speaking, neither the non-abelian Ising anyon σ nor the Majorana fermion ψ exists in nanowires. What is realized is the projective non-abelian braid group representation on the Majorana zero mode space.

Topological quantum computing is an interdisciplinary field that requires a good understanding of wide-ranging subjects in mathematics, physics, and computer science. As such, it has been difficult to understand the basic ideas and concepts; therefore, an introduction to the field

is very valuable. The book under review fills this need. There are three parts in the book. Part I is an introduction to the theory of anyons and their application to quantum computation. Since anyons arise as elementary excitations of topological phases of matter, in Part II, the author covers three kinds of theoretical models of anyonic systems which harbor non-abelian anyons. The level of exact solvability decreases for the three kinds of models. In the last part, two topics are selected: quantum algorithms for approximating link invariants, and a characterization of topological order using entanglement entropy.

Part I consists of four chapters. Chapter 1 is an introduction to some basic concepts in topological quantum computation: particle statistics, anyons, topological systems, abelian and non-abelian anyonic statistics, and how to use anyons to perform quantum computation. In Chapter 2, the author explains anyon statistics as geometric Berry phases. The title “geometric and topological phases” confused the reviewer. While there is no definition of topological phase in the book, the reviewer’s interpretation is that the author used it to mean anyonic statistics as topological phases. This use of the word “phase” clashes with another use of “phase” in topological phases of matter, which is a synonym of “state” in states of matter. The example of integer quantum Hall effect in Section 2.3 is not logically at the best place, as there are no anyons in such systems. Chapter 3 summarizes the quantum circuit model for quantum computation and a few other computing models. Chapter 4 is an excellent introduction to the algebraic theory of anyons. Essentially it is an introduction to the theory of unitary modular categories using $6j$ symbols. There is one subtle sign which is missing sometimes in Section 4.1.6—spin and statistics. In Equation (4.22) on page 65, when the anyons q and $\bar{q}$ are of the same type, then $R_{qq}^1 = \nu_q \cdot e^{2\pi i s}$, where ν_q is the Frobenius-Schur indicator of anyon q and s its topological spin (usually there is a minus sign in the exponent of the right-hand side due to conventions for braidings and twists.)

Part II is the main content of the book. The author picks three classes of models that support anyons. Chapter 5 deals with the quantum double models in [K1]. These are lattice spin models based on any finite group G . The realized anyonic system is given by the unitary modular category which is the representation category of the Drinfeld double of the group algebra $\mathbb{C}[G]$. The connection between those models and error correction codes is emphasized. Most of the statements in this chapter can be made into mathematical theorems. Chapter 6 covers the honeycomb spin model that contains a realization of the Ising anyonic system in [K2]. This is the first exactly solvable spin model that harbors non-abelian anyons in a chiral theory. The author has done extensive numerical and analytic research on this model. This chapter contains valuable insight into this beautiful spin model. The title of Section 6.3—Ising anyons as Majorana fermions—is very confusing, to say the least. Ising anyons are non-abelian, and Majorana fermions are fermions, therefore abelian when regarded as anyons. There are important differences between Majorana zero modes and Majorana fermions. Majorana zero modes shared among Ising anyons are responsible for the degenerate anyonic Hilbert space for multi-Ising anyons. Such Majorana zero modes can be described using the Majorana (fermion) operators, but they are not the same as Majorana fermions. Majorana fermions are quasi-particle with non-zero energy. How and when the shared Majorana zero modes materialize as Majorana fermions when Ising anyons are brought together in real systems is an interesting question. In Chapter 7, the author introduces Witten-Chern-Simons theories. Those topological quantum field theories are conjectured to be effective theories for fractional quantum Hall liquids.

The author chooses two special topics to cover in Part III. In Chapter 8, the author provides all the basic materials to understand the Jones polynomial and Jones representations of the

braid group. The additive approximation of evaluations of the Jones polynomial is sketched. Chapter 9 is on the characterization of topological order using entanglement entropy. This new approach to topological order leads to better understanding of symmetry protected topological phases such as the topological insulators.

The author has done a good job of introducing the most important concepts and ideas in topological quantum computation and choosing the best examples to illustrate them. Readers with a physical background will benefit the most from the book. Mathematically inclined readers might have some trouble with some physical arguments, but persistent readers will be rewarded with some physical intuition and a deeper understanding of the subject. Because the style and level of rigor are appropriate for physically inclined readers, there are no formal definitions or proofs as in the mathematical literature. For mathematically oriented readers, a monograph of the reviewer, not intended as an introduction, summarizes many mathematical results in the subject [W]. More in-depth covering of the topics in the book can be found in J. Preskill's online notes [P], and in A. Kitaev's seminal papers [K1, K2]. Another excellent reference is the survey article of C. Nayak et al [NSSFD].

Overall, the book is a well-conceived introduction to the foundation of topological quantum computation, as well as a valuable addition to the literature. Chapters 4 and 6 are the reviewer's favorites.

REFERENCES

- [F] M. Freedman, *P/NP, and the quantum field computer*, Proc. Natl. Acad. Sci. USA **95**, 1 (1998), 98–101.
- [K1] A. Kitaev, *Fault-tolerant quantum computation by anyons*, Ann. Physics **303**, 1 (2003), 2–30, arXiv:quant-ph/9707021.
- [K2] A. Kitaev, *Anyons in an exactly solved model and beyond*, Ann. Physics **321**, 1 (2006), 2–111, arXiv:cond-mat/0506438.
- [NSSFD] C. Nayak, S. Simon, A. Stern, M. Freedman, and S. Das Sarma, *Non-abelian anyons and topological quantum computation*, Rev. Mod. Phys. **80**, 3 (2008), 1083–1159, arXiv:0707.1889.
- [P] J. Preskill, <http://www.theory.caltech.edu/~preskill/ph219/topological.pdf>.
- [W] Z. Wang, *Topological quantum computation*, CBMS monograph vol. 112, American Mathematical Society, 2010.