Controlled Linear Algebra

Michael H. Freedman
Zhenghan Wang

1 Introduction

These notes derive from lectures given at UCSD in 1990. The purpose of those lectures was to elucidate the two main theorems of controlled linear algebra: the vanishing theorem for Whitehead and K_0-type obstructions when only "simply connected directions" are left uncontrolled and the squeezing principle—that once a threshold level of geometric control is obtained any finer amount of control is also available. These notes present in detail (and with perhaps some new estimates on the optimal relations between ϵ, δ and dimension) the vanishing theorem of Frank Quinn's Section 8 [Q1] in the context of Whitehead torsion. Quinn's Section 8 is brilliant and difficult. In our efforts to understand the details we developed—with copious advice from Frank—the argument presented here. We have neglected all the important applications to savor the intricacy of the central argument.

The idea that controlled linear algebra might exist and be useful to topology goes back to Connell and Hollingsworth [CH] in the late 1960s. By 1977, Ferry [F1] and Chapman [C1] had obtained by geometric arguments theorems which strongly suggested the algebraic unification reached in Quinn's papers [Q1][Q2]. These effectively realized Connell and Hollingsworth's program. Since that time controlled algebraic topology and its cousin bounded algebraic topology have thrived (see e.g. [FP],[RY] and [FHP]). The theorem of Bryant-Ferry-Mio-Weinberger [BFMW] on the existence of "non-euclidean manifolds"—perhaps the most striking result at the wild end of topology in a decade—is evidence of the subject's vigor. Controlled linear algebra also has found applications outside of topology. The interested reader may consult [FrP] for an application to signal processing.

Although these notes do not cover the squeezing principle, the basic techniques of pulling back modules and truncating "flanges" by running the "standard deformation" of $A \oplus A^{-1}$ to id can be applied there as well. The reader who wishes to follow the approach of these notes to reach the squeezing principle may appreciate a hint: given any constant $c > 1$, the identity map on any manifold control space is homotopic to a composition f_c of folding maps which stretch all sufficiently small distances by a factor $\geq c$. For ϵ sufficiently small, the maps f_c can be used (in the place of handle-wise torus tricks) to deform an ϵ-isomorphism to an arbitrarily small ϵ'-isomorphism ($0 < \epsilon' < \epsilon$).

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2 Controlled linear algebra

Suppose X is a Riemannian manifold with the metric d, and π is a fixed group. Let $Z[\pi]$ be the group ring of π. A *controlled module* M over X is defined to be a free Z[π]-module with a preferred basis $S_M = \{e_i\}_{i=1}^{+\infty}$ and a proper map of the basis $\varphi : S_M \longrightarrow X$. We require that S_M is countable and endowed with the discrete topology. X is called the *control space* of M and φ the *control map* of M. The controlled module is denoted by a triple (M, X, φ). But in the following, only one or two symbols are given if the others can be understood without confusion. From the definition, for each compact subset of X, there are only finitely many basis elements mapping to it. It is convenient to identify the basis elements with their images in the control space. We will do this whenever no confusion will result.

Suppose two controlled modules (M, X, φ), (N, X, ϕ) are given with preferred bases $S_M = \{e_i\}_{i=1}^{+\infty}$ and $S_N = \{f_j\}_{j=1}^{+\infty}$, respectively.

Definition 2.1 Let R be a relation between S_M and S_N, then the spread of R at the basis element e_i is defined to be $r_i = sup\{d(\varphi(e_i), \phi(f_j)) \mid f_j \in R(e_i)\}$. The **spread** of R is the supremum of r_i for all i. The spread might be infinite if X is not compact. Given $\delta : X \longrightarrow (0, \infty)$, the spread of R is less than δ if for each i, $d(\varphi(e_i), \phi(f_j)) < \delta(\varphi(e_i))$ for every $f_j \in R(e_i)$.

Let $\alpha : M \longrightarrow N$ be a homomorphism, the *underlying set relation* $\underline{\alpha}$ of α between S_M and S_N is defined to be $\underline{\alpha}(e_i) = \{f_j \mid f_j$ has a nonzero coefficient in $\alpha(e_i) = \sum_j \alpha_{ij} f_j\}$. For a subset E of $S_M = \{e_i\}$, $\underline{\alpha}$ (E) $= \bigcup_{e_i \in E} \underline{\alpha}(e_i)$. A homomorphism $\alpha : M \longrightarrow N$ is a *δ-homomorphism* for $\delta : X \longrightarrow (0, \infty)$ if the spread of $\underline{\alpha}$ is less than δ. If there is also a δ-homomorphism β : N $\longrightarrow$ M such that $\alpha\beta = \mathrm{id}$, $\beta\alpha = \mathrm{id}$, then α is called a *δ-isomorphism*. A *geometric map* from M to N is defined to be the map induced by a bijection of the bases of M and N.

Given two positive functions δ_1, δ_2 on X, the nonabelian composition $\delta_2 \circ \delta_1$ of δ_1 and δ_2 is defined by:

$$\delta_2 \circ \delta_1(x) = sup\{\delta_1(x) + \delta_2(x') \mid x' \in B_{\delta_1(x)}(x)\}.$$

The composition of a δ_1-homomorphism followed by a δ_2-homomorphism is a $\delta_2 \circ \delta_1$-homomorphism.

Suppose K is a subset of X, a submodule $M \mid_K$ of M with the control space K and the control map $\varphi \mid_K$ is defined as follows: $M \mid_K$ is the submodule of M generated by all basis elements e_i satisfying $\varphi(e_i) \in$ K and $\varphi \mid_K$ is the restriction

of φ to the basis of $M\mid_K$. The submodule $M\mid_K$ is called the *restriction* of M to K. Given a homomorphism $\alpha : M \longrightarrow N$, then $\alpha\mid_K$ is the map $p_K \cdot \alpha \cdot i_K$, where i_K is the inclusion $M\mid_K \xrightarrow{i_K} M$ and p_K is the projection $N \xrightarrow{p_K} N\mid_K$. A δ-homomorphism $\alpha : M \longrightarrow N$ is called a *δ-isomorphism* over K if there is a δ-homomorphism $\beta : N \longrightarrow M$ such that $(\alpha\beta)\mid_K$ and $(\beta\alpha)\mid_K$ are both the identity.

Since M and N both have fixed bases, a homomorphism from M to N is one-one correspondent to an infinite matrix with only finitely many nonzero elements in each row. In our convention, a vector is written as a column. The corresponding matrix of α is denoted as $A(\alpha)$. The spread of α can also be described as follows: let $A(\alpha) = (a_{ij})$, the spread of α at e_i is $sup_{a_{ij}\neq 0}\{d(\varphi(e_i), \phi(f_j))\}$, where i is fixed and j varies through integers. The spread of α is the supremum of its spread at each basis element. In the following, a map is always thought as a matrix with a fixed indexing of the basis in mind.

In the following examples, the control space will be the Euclidean space E^1 with the standard metric, π will be the trivial group. So we have free abelian groups Z^n with the i-th basis element e_i mapped to i of E^1.

Example 2.1 Every $n \times n$ matrix represents a homomorphism of spread $\leq$ n. In particular, the spread of the identity matrix is 0.

Example 2.2 Let $A : Z^n \longrightarrow Z^n$ be the following $n \times n$ matrix:

$$A = \begin{pmatrix} 1 & -1 & & & & \\ & 1 & -1 & & & \\ & & 1 & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & 1 & -1 \\ & & & & & 1 \end{pmatrix}$$

It has spread 1 which does not depend on n. But its inverse

$$A^{-1} = \begin{pmatrix} 1 & 1 & 1 & \cdot & \cdot & 1 \\ & 1 & 1 & \cdot & \cdot & 1 \\ & & 1 & \cdot & \cdot & 1 \\ & & & \cdot & \cdot & 1 \\ & & & & \cdot & 1 \\ & & & & & 1 \end{pmatrix}$$

has spread n.

Example 2.3 Let $A = (a_{ij}) : Z^{+\infty} \longrightarrow Z^{+\infty}$ be a matrix. If there exists an integer $p > 0$ such that $a_{ij} = 0$ when $\mid i - j \mid > p$, A is called a 2p-band matrix. Its spread is 2p, so A is a 2p-isomorphism. Then the main theorem (Theorem 4.1) says, as a special case, that A can be diagonalized by row operations and stabilizations described below within some larger band about the diagonal.

Now we describe the *elementary operations* that will be used to deform a δ-isomorphism $\alpha : M \longrightarrow N$ to a geometric map. There are two types of them:

Type I. *Stabilizations.* Given a controlled module (M, X, φ), M can be stabilized by adding countably many new basis elements to S_M and specifying an extension of the control map φ. For a δ-homomorphism $\alpha : M \longrightarrow N$, we stabilize it as follows: stabilize M and N by adding new basis elements e to S_M and f to S_N, extend α to $\alpha' : Z[\pi]e \bigoplus M \longrightarrow Z[\pi]f \bigoplus N$ by sending e to f and the other basis elements the same as α. To extend the control maps, choose any two points p_1, p_2 in X with distance $< \delta(p_1)$, extend φ and ϕ by sending e to p_1 and f to p_2. This procedure is called an *elementary stabilization*. In particular, if $M = N$, f is chosen to be the same as e. This procedure can be repeated as many times as necessary. A *stabilization* of α is a locally finite composition of such elementary stabilizations, where locally finite means only finitely many basis elements can be mapped to any compact subset of X. After a stabilization, α is changed to a map $\alpha \mid_{stabilized}: M \mid_{stabilized} \longrightarrow N \mid_{stabilized}$ which has the same spread of α. If $M = N$, then in matrix notation $A(\alpha)$ is stabilized to $I \bigoplus A(\alpha)$.

Type II. *Row Operations.* A map $\beta : M \longrightarrow M$ is called an *elementary map* if it satisfies the following: suppose $\beta(e_i) = \sum \beta_{ij} e_j$, then a) $\beta_{ii} \in \pi$ or $-\beta_{ii} \in \pi$ for all i and b) $\beta_{ij} = 0$ for all pairs (i, j) with $i \neq j$ except possibly one pair. Given a homomorphism $\alpha : M \longrightarrow N$, an elementary row operation of α is the composition of α from left with an elementary map. Algebraically, the matrix $A(\alpha)$ is multiplied from left by the following matrix:

$$\begin{pmatrix} \beta_{11} & & & & & \\ & \beta_{22} & & & & \\ & & \cdot & a_{ij} & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & \beta_{nn} & \\ & & & & & I \end{pmatrix}.$$

for some nonnegative integer n, where $\beta_{ij} \in \pi$ or $-\beta_{ij} \in \pi$, and $a_{ij} \in Z[\pi]$. A *row operation* is a locally finite composition of such elementary row operations. Locally finite means that each basis element can be only involved in finitely many elementary row operations.

Definition 2.2 Given a homomorphism $\alpha : (M, X, \varphi) \longrightarrow (N, X, \phi)$. A **deformation** H of α is a locally finite composable string of possibly infinitely many elementary operations. A deformation H can always be written as a composition of a stabilization H_1 followed by a row operation H_2 of the stabilized map $\alpha \mid_{stabilized} = H_1 \cdot \alpha$. So $H_2 = \cdots \beta_2 \cdot \beta_1 : M \mid_{stabilized} \longrightarrow N \mid_{stabilized}$ is a row operation of $\alpha \mid_{stabilized}$, where β_i is an elementary row operation. Given an $\epsilon : X \longrightarrow (0, \infty)$, H is an ϵ-deformation if the following relation R between the

basis S'_M of $M\mid_{stabilized}$ and itself has spread less than ϵ: for each $e_i \in S'_M$, $R(e_i) = \{e_j \mid e_j \in \cdots \underline{\beta_2} \cdot \underline{\beta_1}(e_i)$ *or* $e_j \in \underline{\beta_1^{-1}} \cdot \underline{\beta_2^{-1}} \cdots (e_i)\}$. Here the composition makes sense because of the local finiteness.

If $K \subset X$, then the spread of H restricted to K is the spread of the relation $R\mid_K$, where $R\mid_K$ is the relation between $S\mid_K = \{e_i \mid \varphi(e_i) \in K, e_i \in S'_M\}$ and itself defined by $R\mid_K (e_i) = R(e_i) \bigcap S\mid_K$.

As exercises, if H_1 is a δ_1-deformation and H_2 is a δ_2-deformation, then $H_2 \cdot H_1$ is a $\delta_2 \circ \delta_1$-deformation. If we write H_1, H_2 as matrices A_1 and A_2, then the spread of the composition of H_1 followed by H_2 is $\geq$ the spread of $A_2 \cdot A_1$ in general.

Definition 2.3 Let G be a controlled module over R^m and $h : G \longrightarrow G$ be an isomorphism of G. Let $B_{r_1}(0)$ and $B_{r_2}(0)$ be balls of radii r_1 and r_2 with centers 0, $0 < r_1 < r_2$. An isomorphism $h' : G \longrightarrow G$ is called a **truncation** of h to the ball $B_{r_2}(0)$ if

(i). On $G\mid_{B_{r_1}(0)}$, h' is the same as h up to stabilization.

(ii). On $G\mid_{R^m \setminus B_{r_2}(0)}$, h' is the identity.

After a truncation, h' may be viewed as a map of the finite module G' obtained from G by discarding the basis elements outside the ball $B_{r_2}(0)$.

Whitehead group is well studied in K-theory. The definition is recalled for convenience. Identify each $M \in GL(n, Z[\pi])$ with $\begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix} \in GL(n+1, Z[\pi])$, then $GL(1, Z[\pi]) = U(Z[\pi]) \subset GL(2, Z[\pi]) \subset \cdots$. The union $GL(Z[\pi])$ is called the *infinite general linear group*. A matrix is elementary if it is the same as the identity matrix except for one off-diagonal entry. Let E($Z[\pi]$) be the subgroup generated by elementary matrices, then $E(Z[\pi])$ is a normal subgroup of $GL(Z[\pi])$. Define $K_1(Z[\pi]) = GL(Z[\pi])/E(Z[\pi])$ and $\overline{K_1}(Z[\pi]) = K_1(Z[\pi])/\{\pm I\}$, note that $\pi \subset GL(1, Z[\pi])$, the *Whitehead group* of π is defined to be $\overline{K_1}(Z[\pi])/image(\pi)$. From the definition, it is clear that to deform a δ-isomorphism to the identity is the controlled analogue of the isomorphism being zero in the Whitehead group.

3 Marriage theorem

Given two sets G and B with a relation R between them, say, G is a set of girls and B is a set of boys. We say a girl $g \in G$ knows a boy $b \in B$ if g has the relation R with b. A *marriage* from G to B is an injective map which preserves the relation. So every girl marries a boy that she knew.

Theorem 3.1 *Given a finite set G of girls and a set B of boys with a relation R between them. Let P be a subset of B, boys in P will be called handsome boys. Assume every subset of girls know at least that many boys and every subset of*

handsome boys are known by at least that many girls. Then there is a marriage from G to B such that every handsome boy gets married.

Proof. The set of girls G is *forced* if there is a subset S of G knowing exactly that many boys or if there is a subset T of P known by exactly that many girls. S and T are called forced subsets of G and P, respectively. Otherwise, G is *unforced.*

Theorem 3.1 will be proved by induction on the number of girls in G. If G has only one girl, it is obvious. Suppose the theorem is true for $\mid G \mid < n$, either G is forced or unforced. Then for $\mid G \mid = n$, there are two cases:

(i) G is unforced. Pick a handsome boy b in P and a girl g knowing him. Consider $G' = G - \{g\}$ and $B' = B - \{b\}$, then the hypothesis of the theorem is satisfied for G' and B'. By the inductive hypothesis, there is a marriage from G' to B'. Let g marry b, we get a marriage from G to B.

(ii) G is forced. Let S be a forced subset of G with the smallest number of girls, and T be a forced subset of P with the smallest number of boys, and let $k = min\{|S|, |T|\}$. Subcase a): $k < n$. If there is a subset S of girls knowing exactly $k = |S|$ boys, consider $G' = G - S$ and $B' = B - RS$, where RS is the subset of all boys known by at least one girl in S. For any subset Q of G', since $Q \cup S$ know at least $|Q| + |S|$ boys in B and RS have only $|S|$ boys, therefore, Q know at least $|Q|$ boys in B'. For any subset of $P - RS$, they are known by girls that none is in S. By the inductive hypothesis, there is a marriage from G' to B'. It is easy to check that the inductive hypothesis holds for S and RS. Combine the marriage from G' to B' with one from S to RS, we get a marriage from G to B. If it happens there is a subset T of P known by exactly $k = |T|$ girls $R^{-1}T$. Consider $G - R^{-1}T$ and $B - T$. Any subset of $G - R^{-1}$T know none in T, so they know enough boys, and for any subset of $P - T$, none is known by a girl in $R^{-1}T$. As before, we have a marriage from G to B by combining two marriages. Subcase b): $k = n$, then the n girls know exactly n handsome boys and any proper subset is unforced. The argument is similar to the unforced case.

4 Main theorem

In Section 8 [Q1], Quinn proved the ϵ-δ cancellation theorem. In this section, we prove the following version of the ϵ-δ cancellation theorem which makes explicit the relation between ϵ and δ.

Theorem 4.1 *Given a Riemannian manifold X of dimension n, and a group π. Suppose for each i, $0 \leq i \leq n$, $Wh(\pi \times Z^i)$=0. Then there exists an $\bar{\epsilon}: X \longrightarrow (0, \infty)$ such that if $\epsilon : X \longrightarrow (0, \infty)$ and $\delta : X \longrightarrow (0, \infty)$ satisfy $\epsilon < \bar{\epsilon}$ and $\sqrt{n} \cdot 6^{n^2+1} \cdot \delta < \epsilon$, then for any δ-isomorphism α of controlled modules M and N over X, there exists an ϵ-deformation H of α such that $H\alpha : M_{stabilized} \longrightarrow N_{stabilized}$ is a geometric map.*

The geometry of the control space determines the threshold $\bar{\epsilon}$, but for ϵ small, the asymptotic relation between ϵ and δ does not depend on the geometry of X, only on the dimension of X. When ϵ is small enough, the relation between ϵ and δ is linear. The constant for the linear relation between ϵ and δ depends only on the dimension n, and it is $\sqrt{n} \cdot 6^{n^2+1}$. However, if the control space X^n is a subset of Euclidean space E^n, then the constant for the linear relation between ϵ and δ is only $\sqrt{n} \cdot 6^{n+1}$ (see the following Lemma 4.2). A handle decomposition of a Riemannian manifold can be made arbitrarily fine so that each handle is (1+0.001)-quasi-isometric to some Euclidean subset, although the shapes of these handles may vary. These handles are subdivided into what we call *small handles* and our modification of α proceeds over these small handles. This two-stage process leads to the n^2 in the exponent.

Some notations:

C^m is the standard m-cube $\{x \mid x \in R^m, \mid x_i \mid \leq 1, i = 1, 2, \cdots, m\}$.

$T^{m-1} \times I$ is the annulus, where T^{m-1} is the $m-1$ *flat* torus and $I = [-1, +1]$ is the 1-cube.

$B_r(x)$ is the ball of radius r and center x.

Suppose X is a metric space, $K \subset X$, $\epsilon > 0$, then $K^{+\epsilon}$ is the ϵ-neighborhood of K, and $K^{-\epsilon} = X - (X - K)^{+\epsilon}$.

Suppose $K \subset R^m$ has a distinguished point x_0, for any $\lambda > 0$, λK denotes the set $\{\lambda x \mid x \in K\}$ when we choose x_0 as the origin of R^m. By *a half k-handle* we mean $C^k \times \frac{1}{2}C^{m-k}$, where the distinguished point for the second factor is the center.

We are constantly deforming maps in the proof of Theorem 4.1 without renaming them to keep the notation simple.

Observation I: If the diameter of the control space X is less than a constant ϵ, then any deformation over X is an ϵ-deformation.

Observation II: Given a family of self homomorphisms α_i of controlled modules M_i, i belongs to an index set I. If the control spaces of M_i are disjoint subsets of a space X, and H_i is a deformation of α_i with spread ϵ_i for each i, then $H = \bigoplus H_i$ is a deformation of $\alpha = \bigoplus \alpha_i : \bigoplus M_i \longrightarrow \bigoplus M_i$, and the spread of H is the supremum of $\{\epsilon_i\}$.

Lemma 4.1 (local cancellation of inverses) *If A: $G \longrightarrow G$ is a δ-isomorphism of a controlled module G of finite rank, then*

$$\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} : G \bigoplus G \longrightarrow G \bigoplus G$$

is a 3δ-deformation from the identity to $\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix}$ *or a 3δ-deformation from* $\begin{pmatrix} A^{-1} & 0 \\ 0 & A \end{pmatrix}$ *to the identity.*

Proof. If $A : G \longrightarrow G$ is a δ-homomorphism, it follows from the following factorization that $\begin{pmatrix} I & \pm A \\ 0 & I \end{pmatrix}$ is a δ-deformation of any map from $G \bigoplus G$ to itself:

$$\begin{pmatrix} I & \pm A \\ 0 & I \end{pmatrix} = \prod_{i,j=1}^{d} \{ \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \pm a_{ij} \begin{pmatrix} I & E_{ij} \\ 0 & I \end{pmatrix} \},$$

where E_{ij} is the matrix with all entries 0 except 1 at the (i,j) position. Similarly, $\begin{pmatrix} I & 0 \\ \pm A & I \end{pmatrix}$ is a δ-deformation of $G \bigoplus G$. We interpret the following matrix identity on $G \bigoplus G$ as a composition of deformations from the identity to $\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix}$ or from $\begin{pmatrix} A^{-1} & 0 \\ 0 & A \end{pmatrix}$ to the identity. The proof is completed by observing that $\begin{pmatrix} I & 0 \\ \pm I & I \end{pmatrix}$ is a 0-deformation of $G \bigoplus G$.

$$\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} = \begin{pmatrix} I & -A \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ A^{-1} & I \end{pmatrix} \begin{pmatrix} I & -A \\ 0 & I \end{pmatrix}$$
$$\begin{pmatrix} I & I \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ -I & I \end{pmatrix} \begin{pmatrix} I & I \\ 0 & I \end{pmatrix}$$

Addendum. In the following, this lemma will be used several times. But each time, it is a relativized form we use. The modification is easy. For any $K \subset X$, let P=$A\mid_K$, and Q=$A^{-1}\mid_K$, then

$$\begin{pmatrix} I & -P \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ Q & I \end{pmatrix} \begin{pmatrix} I & -P \\ 0 & I \end{pmatrix} \begin{pmatrix} I & I \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ -I & I \end{pmatrix} \begin{pmatrix} I & I \\ 0 & I \end{pmatrix}$$

is a 3δ-deformation from

$$\begin{pmatrix} (I-PQ)P+P & I-PQ \\ I-QP & Q \end{pmatrix}$$

to the identity. For example, if restricted to the subset $K^{-3\delta} \subset K$, $PQ\mid_{K^{-3\delta}} =$ *identity*. Then, there is a 3δ-deformation from

$$\begin{pmatrix} A\mid_{K^{-3\delta}} & 0 \\ 0 & A^{-1}\mid_{K^{-3\delta}} \end{pmatrix}$$

to the identity.

Lemma 4.2 *(Estimate for the Euclidean case) Suppose Y is a compact subset of E^n, and M is a controlled module over $Y^{+6^n\delta} \subset E^n$, and $\gamma : M \longrightarrow M$ is a δ-homomorphism. If γ is a δ-isomorphism over Y, then there is a $\sqrt{n} \cdot 6^{n+1} \cdot \delta$-deformation H such that $H\gamma : M\mid_{stabilized} \longrightarrow M\mid_{stabilized}$ is the identity map over Y.*

The proof of this lemma is the first induction in the proof of Theorem 4.1.

Proof. (small induction) The standard lattice structure of R^n gives a cubing of R^n. This cubing can be thought as a handle decomposition of R^n whose handles are cubes of the same size (see Fig. 1 for the 2-dimensional case).

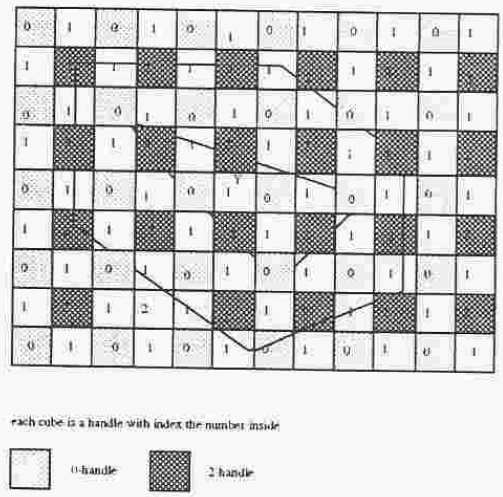

Figure 1.

By similarity, there is such a handle decomposition of R^n with largest side length of the cubes (handles) so that if a k-handle meets Y, then all adjacent handles of lower indices are inside $Y^{+6^n\delta}$. The handles in this decomposition will be called *small handles* or just handles. Over each small handle of index less than n, the argument uses Kirby's torus trick.

Step 1: Work on small 0-handles.

Splitting Procedure. Each small 0-handle is a standard n-cube C^n. Let $j : T^{n-1} \times I \longrightarrow C^n$ be an embedding such that $\frac{1}{2}C^n \subset j(T^{n-1} \times I)$, and the distance between the boundary of $j(T^{n-1} \times I)$ and the boundary of the half 0-handle is large enough(Fig. 2). The precise estimate is given in Step 4.

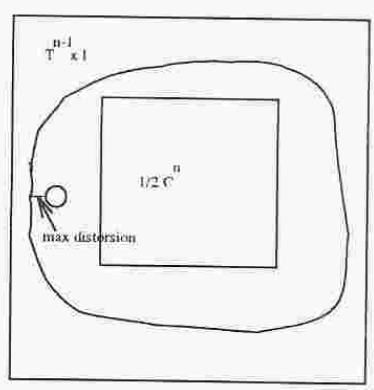

Figure 2.

Pull back γ and γ^{-1} to $T^{n-1} \times I$, and denote them by A and A^{-1} with a fixed indexing of the basis of M. Notice that A^{-1} is not exactly the inverse of A near the boundary of the annulus $T^{n-1} \times I$. Glue the two copies of $T^{n-1} \times I$ with A and A^{-1} over them together along collars of the boundary components of $T^{n-1} \times I$. The two identified collars will be called *flanges*. The cross section looks like Figure 3. Note that we do not reverse the radial orientations of the annuli, and what we get is a torus with two flanges. Suppose each flange has sufficiently large radial length, say 9δ. Again the precise estimate is given in Step 4. Over each flange, the map is $A \bigoplus A^{-1}$ from $M \bigoplus M \mid_{flange}$ to itself.

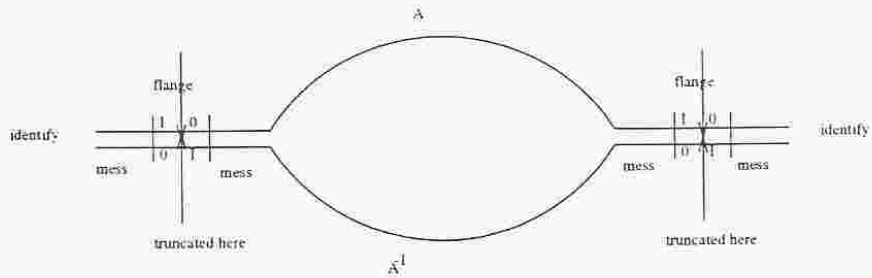

Figure 3.

Apply the 3δ-deformation in Lemma 4.1 to the map over the middle 3δ segment of each flange. After this deformation, $A \bigoplus A^{-1}$ is deformed to the identity over the middle 3δ segment of each flange. Truncate along the middle, we have a map $U = A \bigoplus A^{-1} \mid_{truncated}$ of the module over the torus with two half flanges. Since $\gamma \bigoplus \gamma^{-1}$ is invertible away from the boundary of the torus with two flanges and U is the restriction of $\gamma \bigoplus \gamma^{-1}$ to a region away from the boundary after a 3δ-deformation, so U is an invertible matrix.

Finally, we project U over the torus with two half flanges to a map over the torus T^n. Since the distances among basis elements decrease, the map over the torus T^n, denoted still by U, is an automorphism of the module $G \mid_{T^n}$ over the torus T^n.

In next step, we will work in a general setting for a map of a module G over the flat torus T^m.

Truncation Procedure. Given a δ-automorphism U of a controlled module G over the torus T^m, let $p : R^m \longrightarrow T^m$ be the universal covering map of T^m. Since δ is very small compared to the injectivity radius of T^m, we can pull back U uniquely to an automorphism $\hat{U} : \hat{G} \longrightarrow \hat{G}$, where the infinitely generated free $Z[\pi]$-module $\hat{G}$ over R^m is the lifting of $G \mid_{T^m}$ by p. The control map extends naturally. It is well known that the covering transformations give $\hat{G}$ a structure of a finitely generated free $Z[\pi \times Z^m]$-module over R^m. So, $\hat{U}$ can be viewed as an automorphism of the free $Z[\pi \times Z^m]$-module $\hat{G}$. As an element in $Wh(\pi \times Z^m)$, $\hat{U}$ is zero. Hence, after a stabilization by direct sum with an identity matrix, $\hat{U}$ is the product of elementary matrices. By factorizing these elementary matrices, it is sufficient to consider the following two different kinds

of basic matrices:

$$\textit{Type I.} \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & g & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix},$$

where $g \in \pi \times Z^m$. It shifts basis elements by g.

$$\textit{Type II.} \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & \dots & h & & \\ & & & \ddots & \vdots & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix},$$

where $h \in Z[\pi \times Z^m]$.

Further factorizing type I and II matrices, we may assume that for type I matrices, $g \in Z^m$ (Type I_1) or $\pm g \in \pi$ (Type I_2) and for type II matrices, $h \in Z[\pi]$. Now fix a factorization of $\hat{U}$ as products of such matrices $\hat{U} = E_1 \cdot E_2 \cdots E_t$, where E_i is either a matrix of type I_1, I_2 or type II and is essentially an elementary deformation. But these E_i's are not directly useful because they are maps of a module over R^n, and not δ controlled. So first we will truncate these E_i's to get a map over a large ball. In next step, we will use a retraction to regain δ control.

The spread of $\hat{U}$ over a ball $B_r(0) \subset R^m$ is increasing as r increases. But when r is bigger than some constant $r_0 = r_0(\hat{U})$, the spread of $\hat{U}$ is a constant. The constant r_0 is determined by the number of the type II and type I_2 matrices, and the largest spread of the type I_1 matrices. Pick an $r_1 > r_0$ and fix two concentric balls of radii r_1 and $r_2 > r_1$ with centers 0. If r_1 is large enough, no basis elements of $\hat{G}$ can be mapped out of the ball $B_{r_1}(0)$ by $\hat{U}$. We may assume $B_{r_1}(0)$ contains many copies of the fundamental domains and the region between the two balls consists of many copies of the fundamental domains.

For the type I_2 and type II matrices, the truncation is easy because they leave every fundamental domain invariant. The truncated E_i, denoted by E_i', is defined as follows: inside the ball $B_{r_1}(0)$, it is the same as E_i, outside the larger ball $B_{r_2}(0)$, it is the identity, between the two balls, it is still the same as E_i.

For a type I_1 matrix, assume g is a generator of Z^m. In general, the argument is similar if several fundamental domains together are treated as a "fundamental domain". Then g shifts basis elements from one domain to another. If $m \geq 2$,

we use the trick "*ship in water*" to truncate the matrix. As before, choose two fixed concentric balls. We need a map which is a quasi-isometry among the basis elements inside the ball $B_{r_2}(0)$ and is the same as E_i inside the smaller ball $B_{r_1}(0)$. To get such a quasi-isometry, let the basis elements inside the ball $B_{r_1}(0)$ be shifted according to the matrix and the basis elements between the balls follow the water flow as if a ship (the ball $B_{r_1}(0)$) moves forwards in the lake (ball $B_{r_2}(0)$). Figure 4 indicates the directions of the basis elements to go. It is really a permutation outside the ball $B_{r_1}(0)$. If the spread of the truncated map is larger than that of the map before, we add more basis elements between the two balls to get the same spread as before.

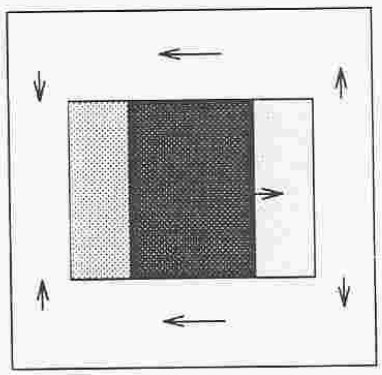

Figure 4. Ship in water.

If m=1, there is the flux problem because there is no extra direction to move basis elements backwards outside $B_{r_1}(0)$. The *flux* of a map is the difference between the number of the basis elements shifted by the map in two opposite directions.

For a type I_1 matrix with g being the generator t of $\pi_1(T^1) = Z$, the element t acts on one basis element e_i by shifting e_i to te_i and on the others by the identity. We index the domain and the basis element in it both by t^l for some integer l. To solve the flux problem, fix $r_1 = r, r_2 = r+1$ and two balls of radii r_1 and r_2 with centers at 0. The obvious solution is to add a new basis element $t^l e_i'$ to the t^l-th domain for each l. Extend the map by sending $t^{r+1}e_i$ to $t^{r+1}e_i'$, $t^l e_i'$ to $t^{l-1}e_i'$, $-r \leq l \leq r$ and $t^{-r-1}e_i'$ to $t^{-r-1}e_i$. But this map is not in general a δ-map.

To get a δ-map, we introduce more basis elements in each domain. Suppose $\frac{\epsilon}{\delta} \cong k$, we add k new basis elements $e_i^j, j = 1, 2, \cdots, k$ to the fundamental domain and extend the map by:

$$\begin{cases} e_i^k & \longrightarrow & e_i^{k-1} \\ \vdots & & \\ e_i^2 & \longrightarrow & e_i^1 \\ e_i^1 & \longrightarrow & t^{-1}e_i^k \end{cases}$$

Repeat this procedure in each domain inside the interval except the two end domains. For the left end domain, we define the map as follows:

$$\begin{cases} t^{-r}e_i^1 & \longrightarrow & t^{-r-1}e_i^k \\ t^{-r-1}e_i^k & \longrightarrow & t^{-r-1}e_i^{k-1} \\ \vdots & & \\ t^{-r-1}e_i^2 & \longrightarrow & t^{-r-1}e_i^1 \\ t^{-r-1}e_i^1 & \longrightarrow & t^{-r-1}e_i \end{cases}$$

Similarly for the right end domain. The truncated matrix is different from the original one by a δ permutation matrix which is essentially a stabilization after a marriage of basis elements.

After the truncation, we obtain a module $\hat{G}'$ over $B_{r_2}(0)$ and an automorphism of this module $\hat{U}' = E_t' \cdots E_1'$ which can be thought as a deformation from the identity to $\hat{U}'$. The spread of the truncated E_i' is independent of both r_1 and r_2, neither is $\hat{U}' = E_t' \cdots E_1'$. Therefore, we have the freedom to choose r_1 and r_2.

Retraction Procedure. Apply the truncation procedure ($m = n$) to the module $G\mid_{T^n}$ and the map U obtained in the splitting procedure. We get a deformation over $B_{r_2}(0)$ from the identity to $\hat{U}' = E_1' \cdot E_2' \cdots E_t'$, where E_i' is the truncated matrix of E_i. As an elementary deformation, E_i' is not δ controlled. But they are all uniformly bounded over $B_{r_2}(0)$. Therefore, we can project them back to a 0-handle C^n to regain δ control. Inside each fundamental domain in $B_{r_1}(0)$, there is a copy of A^{-1} which is the same as γ^{-1} over a neighborhood of $\frac{1}{2}C^n$. Fix such a copy of A^{-1} inside $B_{r_1}(0)$ and identify the control space in $B_{r_1}(0)$ of A^{-1} with $\frac{1}{2}C^n$. In this way, $\frac{1}{2}C^n$ is treated as a subset of $B_{r_1}(0)$. Let $q : R^n \supset B_{r_2}(0) \longrightarrow C^n$ be an embedding which is the identity over $\frac{1}{2}C^n$ and has the following property:

(*) For any constant L, the diameter of the image of the ball $B_L(x)$ under q goes to 0 as x goes to infinity.

It is easy to see there exists such a function. Under q, the truncated module $\hat{G}'$ over $B_{r_2}(0)$ is mapped to a module $q_*\hat{G}'$ over C^n and $\hat{U}'$ is mapped to a deformation $q_*\hat{U}'$. The deformation $q_*\hat{U}'$ can be made into a 3δ-deformation over C^n as follows. The spread of $q_*\hat{U}'$ is bounded since the spread of $\hat{U}'$ over $B_{r_2}(0)$ is a constant and the projection q decreases the spread. But $q_*\hat{U}'$ might not be a 3δ-deformation. For each E_i', choose r_1 large enough, by the property (*) of q, the spread of q_*E_i' can be made as small as possible outside $q(B_{r_1}(0))$, say for each i, the spread of $q_*E_i{}'$ is $< \frac{\delta}{t}$, then the spread of $q_*\hat{U}'$ is $< t \cdot \frac{\delta}{t} = \delta$. Since $\hat{U}'$ is the same as U over $B_{r_1}(0)$, so $q_*\hat{U}'$ consists of many copies of $\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix}$ over $q_*(B_{r_1}(0))$. By Lemma 4.1, this is a 3δ-deformation. Therefore, if r_1 is large enough, using a tapering retraction we can control the spread of $q_*\hat{U}'$ over the region outside $\frac{1}{2}C^n$ within 3δ. The

result is a 3δ-deformation from the identity to a map with A^{-1} over $\frac{1}{2}C^n$. It is important to note that the spread of $q_*\hat{U}'$ does not depend on U and r_1, r_2.

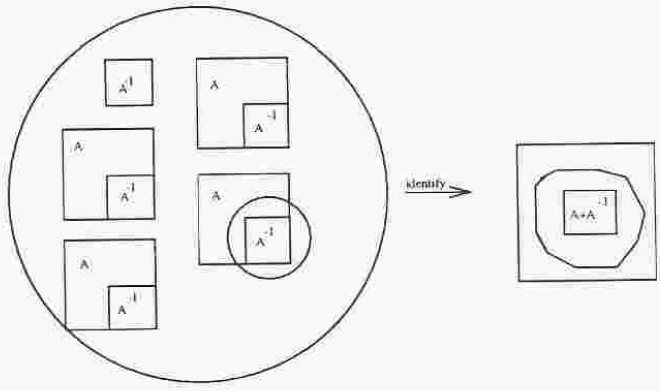

Figure 5.

Cancellation on small 0-handles. Over a small 0-handle, stabilize the module according to $q_*\hat{G}'$ above and run the 3δ-deformation $q_*\hat{U}'$. So we have $\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix}$ over the half 0-handle (Fig. 5). Although A^{-1} is not exactly the inverse of A over the 0-handle, it really is over some neighborhood of the half 0-handle. Applying Lemma 4.1, we run the 3δ-deformation to cancel A and A^{-1} over the half 0-handle. Do this simultaneously over all 0-handles, we deform γ to the identity over all half 0-handles. Notice that the deformation used is a 6δ-deformation as Lemma 4.1 was used twice. It is easy to check that the 3δ-deformation used in the splitting procedure does not contribute to the estimate. Therefore, the resulting map outside all half small 0-handles is a 6δ-isomorphism.

Step 2: Work on general small k-handles. Each small k-handle is a cube $C^k \times C^{n-k}$. Let $j : C^k \times T^{n-k-1} \times I \longrightarrow C^k \times C^{n-k}$ (T^0 is a point) be an embedding whose image contains $C^k \times \frac{1}{2}C^{n-k}$(half k-handle) and is the identity on the factor C^k (Fig. 6).

Assume the distance between the boundary of the image of j and the boundary of the half k-handle is large enough. For the precise estimate, see the following Step 4. Over a k-handle the map is a $6^k \cdot \delta$-isomorphism. Do the analogue of the splitting procedure to get a controlled module over $C^k \times T^{n-k}$. As $Wh(\pi \times Z^{n-k}) = 0$, factorize the matrix analogous to $\hat{U}$, do the truncation procedure for $m = n - k$ with the identity on the factor C^k. The situation is almost exactly the same as the 0-handle case. After the retraction procedure, we get a 3δ-deformation over each k-handle, and this is used to deform a stabilization over each k-handle to a 3δ map with the desired inverse over the half k-handle. After cancellation over the half k-handle using Lemma 4.1, γ has been deformed to the identity over $C^k \times \frac{1}{2}(C^{n-k})$ by a 6δ-deformation. Again we can

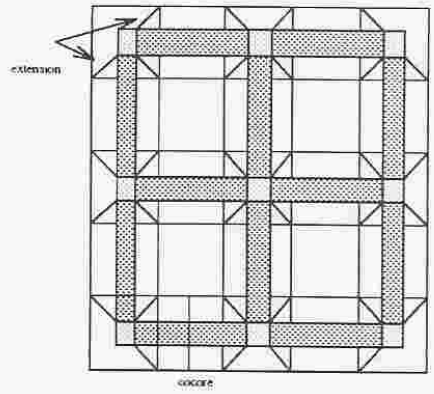

Figure 6.

do this simultaneously over all small k-handles. Continuing inductively when k reaches $n-1$, we find that a $6^n\delta$-deformation has deformed γ to the identity over all half small k-handles ($k \leq n-1$) (Fig. 6). The largest deformation happens over the region near the boundaries of half handles of index $\leq n-1$. Notice that after we deform γ to the identity over half k-handles ($0 \leq k \leq n-2$), it is necessary to extend the (k+1)-handles to the half k-handles left. Figure 6 illustrates the situation. Also note that there exists the flux problem over (n-1)-handles.

Step 3: Cancellation on small n-handles. Now the map is not the identity only over the disjoint n-handles. The n-handles here are the original n-handles enlarged a little. Since $Wh(\pi) = 0$, do a finite algorithm over all n-handles simultaneously, γ is deformed to the identity. Since the side length of each handle (cube) is $6^{n+1} \cdot \delta$ (see following step 4), the diameter of each enlarged n-handle is $(\sqrt{n} \cdot 6^{n+1}) \cdot \delta$. The n-handles are disjoint and the map over them has been already deformed by a $6^n\delta$-deformation, so the deformation used above is a $\sqrt{n} \cdot 6^{n+1} \cdot \delta + 6^n \cdot \delta$-deformation by observations I and II. The key property needed in the second big induction step is although the deformation has spread $\sqrt{n} \cdot 6^{n+1} \cdot \delta$ over $Y^{+6^n\delta}$, the map restricted outside Y has been deformed only by a $6^n\delta$-deformation.

Step 4: Estimate. Let $\bar{\delta}$ denote the side length of small handles (cubes). The side length $\bar{\delta}$ is determined by the radial length of the two flanges in the splitting procedure. We require that the two flanges have radial length 9δ. As over small k-handles γ is a $6^k\delta$-isomorphism, the cocore must be large enough for the torus trick to work. In particular, over a small (n-1)-handle is γ a $6^{n-1}\delta$-isomorphism, so we choose $\bar{\delta}$ to be $4 \cdot 9 \cdot 6^{n-1}\delta = 6^{n+1}\delta$. We will show that if $\bar{\delta} > 6^{n+1}\delta$, then there exists an embedding of $C^k \times T^{n-k-1} \times I$ into $C^k \times C^{n-k}$ so that the

flanges have radial length 9δ. In the following, $\bar{\delta}$ will be treated as 1. Let the embedding of $T^{n-k-1} \times I$ into R^{n-k} be as follows:

$$T^{n-k-1} = S^1_{2^{n-k-1}} \times S^1_{2^{n-k-2}} \times \cdots \times S^1_2,$$

where $S^1_{2^{n-k-i}}$ is the circle in the $x_i x_{i+1}$-plane with radius 2^{n-k-i} for $1 \leq i \leq n-k-1$. Thickening S^1_2 to a two-sided collar of thickness 1, we have an embedding of $T^{n-k-1} \times I$ into R^{n-k}. Choose a cube C^{n-k} with center $x_0 \in S^1_2$, then $\frac{1}{2}C^{n-k}$ is sitting inside $T^{n-k} \times I$. There is a retraction from R^{n-k} into C^{n-k} so that it retracts $T^{n-k} \times I$ to C^{n-k}, and is the identity on $\frac{1}{2}C^{n-k}$, and its maximum distortion is $2^{n-k-1}+2$. Cross with the identity on the factor C^k, we have a desired embedding of $C^k \times T^{n-k-1} \times I$ into $C^k \times C^{n-k}$. The part with maximum distortion is indicated in Fig. 2. The necessary inequality for $\bar{\delta}$ is $2 \cdot \frac{1}{2^{n-k-1}+2} \cdot \bar{\delta} \geq 18 \cdot 6^k \delta$. This is subsumed by $6^{n+1}\delta < \bar{\delta}$.

Remark. At the end of Step 2, the handle decomposition of X has been extended. This changes the shapes and sizes of the handles. Hence this extension contributes some constant to the final estimate. But note that instead of deforming the map to the identity over only half k-handles at all steps, we can deform the map to the identity over more, say 9/10, of all k-handles. Then the distortion due to the extension of handle decomposition could be neglected.

Proof of the theorem (big induction).

Step I: Fix a fine handle decomposition of X so that each handle is $(1+0.001)$-quasi-isometric to a Euclidean subset of E^n. The handles in this decomposition will be referred to as *big handles*. The mesh of this handle decomposition is the threshold $\bar{\epsilon}$ in the theorem. Given any $\epsilon < \bar{\epsilon}$, if the theorem holds for some $\delta > 0$, then it holds for any $0 < \delta' \leq \delta$. So it is sufficient to determine the largest δ for a given ϵ such that the theorem holds. Let us fix a very small δ now, and the relation of ϵ and δ will be determined in the end of the proof.

Step II: As an application of the marriage theorem, α will be changed to an automorphism γ of M. To do this, we prove there is a bijection between the bases S_M and S_N such that each $f_j \in S_N$ can be identified with an $e_i \in S_M$ and $d(\varphi(e_i), \phi(f_j)) < \delta(\varphi(e_i))$.

Fix a decomposition of X= $\bigcup_{i=1}^{+\infty} K_i$ such that each K_i is compact, $\overline{K_i} \subset intK_{i+1}$ and the distance between K_i and $X - K_{i+1}$ is $> max_{x \in K_i}\delta(x)$. Define a basis element $e_i \in S_M$ knows $f_j \in S_N$ if the coefficient of f_j in $\alpha(e_i)$ is not zero. Let $G_i = \varphi^{-1}(K_i)$ be the set of girls, $B_i = \phi^{-1}(K_i^{+\delta})$ be the set of boys, where $K_i^{+\delta} = \{x \mid d(x, K_i) \leq max_{x \in K_i}\delta(x)\}$ and define the handsome boys P_i of B_i as $\phi^{-1}(K_i^{-\delta})$, where $K_i^{-\delta} = \{x \mid d(x, X - intK_i) \geq mim_{x \in K_i}\delta(x)\}$, then $P_i \subset B_i$. To check that the hypothesis of the marriage theorem holds for G_i and B_i, note that $M\mid_{K_i}$ is the submodule of M generated by G_i, the conditions for G_i and B_i follow from the fact that $\alpha\mid_{K_i}$ is a δ-isomorphism over K_i. The hypothesis for handsome boys follows from the same fact for the inverse of α. By the marriage theorem, there are marriages $\{\rho_i\}$ from G_i to B_i such that $\rho_i(G_k) \subset B_k$ and

$\rho(G_i) \supset B_{i-1}$ for $k \leq i$. Since each B_i is compact (actually finite), so is the product $\times_{i=1}^{+\infty} B_i$. Each marriage ρ_i determines a closed subset F_i of $\times_{i=1}^{+\infty} B_i$ as follows: $F_i = \rho_i(G_1) \times \rho_i(G_2) \times \cdots \times \rho_i(G_i) \times B_{i+1} \times \cdots$. Since F_i has only finite possibilities for each i, there is an l large enough such that $F_i \supset F_{i+l}$. Pick a subsequence $i_1, i_2 \cdots$ such that $F_{i_1} \supset F_{i_2} \supset \cdots$. Then by the compactness of $\times_{i=1}^{+\infty} B_i$, $F = \cap_{j=1}^{+\infty} F_{i_j} \neq \emptyset$. Since $S_M = \bigcup G_i, S_N = \bigcup B_i = \bigcup P_i$, any element in F is a desired marriage.

Choose a marriage μ between S_N and S_M, if we identify f_j with its image e_i, then $\gamma = \alpha\mu : M \xrightarrow{\alpha} N \xrightarrow{\mu} M$ is a 2δ-automorphism of M. After this marriage, the matrix $A(\alpha)$ is changed by reordering of the columns and the spread of $A(\alpha)$ is changed by 2δ. We will still use $A(\alpha)$ to denote the matrix after the marriage to keep the notation simple.

Step III: Induction on big handles. As each big 0-handle is (1+0.001)-quasi-isometric to a Euclidean subset of E^n, so big 0-handles can be safely treated as Euclidean subsets. Applying Lemma 4.2 to each big 0-handle, we deform the map γ over all big 0-handles simultaneously to the identity. As observed in Step 3 of Lemma 4.2, the largest deformation so far, a $(\sqrt{n} \cdot 6^{n+1} + 6^n)\delta$-deformation, was only used to deform the map over all disjoint small n-handles to the identity. The module over these disjoint small n-handles never enters into later modifications. By observation II, they are subsumed by later deformations. The map outside big 0-handles has been deformed only by a $6^n\delta$-deformation. Hence, the resulting map γ is a $6^n\delta$-isomorphism over big 1-handles. Next we proceed to deform γ over all big 1-handles to the identity. As over a big 1-handle γ is a $6^n \cdot \delta$ isomorphism, when Lemma 4.2 is used, the decompositions of big 0-handles and big 1-handles into small handles over a collar of big 0-handles mismatch with different sizes. Now proceed as above, the only difference is that over a big k-handle γ is a $6^{kn} \cdot \delta$-isomorphism. When k reaches $n-1$, we need that the big n-handles are large enough so that the deformation over big k-handles ($k \leq n-1$) happens over disjoint pieces of the control space. The modification to deform the map over big (n-1)-handles to the identity proceeds over a $6^{n^2} \cdot \delta$ neighborhood of big (n-1)-handles, so the diameter of a big n-handle must be at least $2 \cdot 6^{n^2} \cdot \delta$. Therefore, ϵ and δ should satisfy the inequality: $\epsilon > 2 \cdot 6^{n^2}\delta$. Finally, the map is not the identity only over disjoint big n-handles. Since $Wh(\pi) = 0$, so we can do the usual linear algebra. In the end, the map γ has been deformed to the identity over the whole control space X. The deformation used over big n-handles is a $2 \cdot 6^{n^2} \cdot \delta + 6^{n^2}\delta < 6^{n^2+1}\delta$-deformation by observations I and II. However, the largest deformation happens when Lemma 4.2 is used to deform the map over big (n-1)-handles to the identity. As over a big (n-1)-handle the map is a $6^{(n-1)n}\delta$-isomorphism, so the deformation used is a $\sqrt{n} \cdot 6^{n+1} \cdot 6^{(n-1)n} \cdot \delta = \sqrt{n} \cdot 6^{n^2+1}\delta$-deformation. Composed with the inverse of the marriage, the map α is the same as being deformed to a geometric map. As the marriage used in step II is only a δ marriage, this adds to the spread of the deformation only a negligible 2δ. Thus the proof is completed.

Remarks. (1) If we are not seeking the best constant for the relation between ϵ and δ or consider only compact Riemannian manifolds, the second induction in the proof is not necessary. By the Nash isometric embedding theorem [N], any Riemannian n-manifold can be isometrically embedded into R^{cn^3} (or R^{cn^2} for the compact case) for some universal constant c. So it follows from Lemma 4.2 that the constant for the relation between ϵ and δ is exponential cn^3 (or cn^2 for the compact case)

(2) It follows from Whitney's triangulation method for manifolds [Wh] and the Nash isometric embedding theorem [N] that any compact Riemannian manifold of dimension n can be triangulated with arbitrarily small mesh and every top n-simplex is c_n-quasi-similar to the standard Euclidean n-simplex Δ^n. The constant c_n depends only on n. It seems possible to get a better estimate via this approach, but details have not been worked out.

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