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The N-Eigenvalue Problem and Two Applications

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1 Introduction

Consider the following two questions.

(1) When does a compact Lie group $G \subset U(n)$ have an element $g \in G$ possessing exactly two eigenvalues?

(2) When does a compact Lie group $G \subset U(n)$ have a cocharacter $U(1) \to G$ such that the composition $U(1) \to U(n)$ is a representation of $U(1)$ with exactly two weights?

A solution to the second problem gives a family of solutions to the first, by choosing g to be almost any element of the image of $U(1)$. The converse is not true. For one thing, any noncentral element of order 2 in G has exactly two eigenvalues. To eliminate these essentially trivial solutions, we can insist that the ratio between the two eigenvalues is not -1. There remain interesting cases of finite groups G satisfying the first (but obviously not the second) condition, especially when the ratio of eigenvalues is a third or fourth root of unity (see [3, 18, 35] for classification results). On the other hand, when G is infinite modulo center, the solutions of the two problems are essentially the same, though the historical reasons for considering them were quite different. The first problem was recently solved in the infinite-mod-center case by Freedman, Larsen, and Wang [13] with an eye toward understanding representations of Hecke algebras. The second problem was solved by Serre [27] nearly thirty years ago in order to classify representations arising from Hodge-Tate modules of weight 1.

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This paper is primarily devoted to an effort to understand the analogue of the first problem (the "N-eigenvalue problem" of the title) when the number $N \geq 3$ of eigenvalues is fixed and G is infinite modulo its center. As a consequence, we also say something about the second problem. We are especially interested in the case $N = 3$, both because the results can be made quite explicit and because it is especially relevant to the applications we have in mind. To specify our problem more precisely, we make the following definitions.

A *pair* (G, V) consists of a compact Lie group G and a faithful irreducible complex representation $\rho : G \to GL(V)$. Let N be a positive integer. We say a pair (G, V) satisfies the N-*eigenvalue property* if there exists a *generating element*, that is, an element $g \in G$ such that the conjugacy class of g generates G topologically and the spectrum X of $\rho(g)$ has N elements and satisfies the *no-cycle property*: for all roots of unity ζ_n, $n \geq 2$, and all $u \in \mathbb{C}^\times$,

$$u\langle \zeta_n \rangle \not\subset X. \tag{1.1}$$

Our goal is to classify pairs satisfying the N-eigenvalue property.

From the perspective of [13], the most obvious reason to consider the N-eigenvalue property is that certain naturally occurring representations of the Artin braid groups $\mathcal{B}_n$ satisfy this condition. The braid generators (half-twists) in the braid group form a generating conjugacy class, and given any braided tensor category $\mathcal{C}$ and any object $x \in \mathcal{C}$, we get a representation of $\rho_{n,x} : \mathcal{B}_n \to GL(V_{n,x})$. When $\rho_{n,x}$ is unitary with respect to a Hermitian form on $V_{n,x}$, the closure of $\rho_{n,x}$ is a compact Lie group endowed with a natural faithful representation and a generating conjugacy class. It is often possible to control the eigenvalues of half-twists, to guarantee the N-eigenvalue condition, and to guarantee irreducibility. In the case when the braided tensor category $\mathcal{C}$ is modular, we obtain in addition representations of the mapping class groups $\mathcal{M}(\Sigma_g)$ of closed oriented surfaces Σ_g for each genus g. It is well known that $\mathcal{M}(\Sigma_g)$ is generated by the (mutually conjugate) Dehn twists D_c on $3g - 1$ nonseparating simple closed curves c on Σ_g (see [14]). If $\mathcal{C}$ has m simple object types, then each D_c has at most m distinct eigenvalues as the eigenvalues of D_c consist of twists θ_i of the simple objects. When the values θ_i satisfy the no-cycle condition, it follows that each irreducible constituent of the representation of $\mathcal{M}(\Sigma_g)$ arising from $\mathcal{C}$ defines a pair satisfying the N-eigenvalue property for some $N \leq m$.

The original motivation for the work of [13] was for applications to quantum computing. In [12], topological models of quantum computing based on unitary topological quantum field theories (TQFTs) are proposed. Given a topological model of quantum computing, an important issue is whether or not this topological model is capable of simulating the universal circuit model of quantum computing [23]. This question actually depends on the specific design of the topological quantum computer. But for the models based on braiding anyons in [12], the universality question is translated into a question about the closures of the braid group representations. Quantum computing is the processing of information encoded in quantum state vectors in certain Hilbert spaces V_n by unitary transformations. Universality is the ability to efficiently move any state vector $v \in V_n$ sufficiently close to any other state vector in V_n. A theorem of Kitaev-Solovay (see [23]) guarantees efficiency if the available unitary transformations in $U(V_n)$ form a dense subset of $SU(V_n)$. Therefore, universality of topological models in [12] is equivalent to the density of braid group representations.

The unitary Witten-Reshetikhin-Turaev Chern-Simons TQFTs based on the gauge groups $SU(N)$ and $SO(N)$ are of particular interest due to their relevance to braid statistics in condensed matter physics. It was discovered in the 1980s that in dimension 2, there are quasiparticles which are neither fermions nor bosons [41]. The most interesting of these *anyons* are nonabelian: when two such quasiparticles are exchanged, their wave function is changed by a unitary matrix, rather than a complex number, which depends on the exchanging paths (braiding). It is predicted by physicists that the braid statistics of quasiparticles in certain fractional quantum Hall liquids are described by Jones' unitary braid groups representations or equivalently the braid representations coming from the $SU(2)$ TQFTs. Physicists have also proposed models of braid statistics based on the $SO(3)$ [9] and $SO(5)$ [37] TQFTs. Therefore, it may well be the case that both the Jones and the BMW braid group representations describe braid statistics of quasiparticles in nature. Experiments are proposed to confirm those predictions [7].

Problem (2) is significant partly because of its relation to problem (1), but in addition, there are number-theoretic applications, in the spirit of [27]. We mention a global one: assuming the Fontaine-Mazur conjecture, we can prove that if K is a number field, $\bar{K}$ is an algebraic closure of K, $G_K = \mathrm{Gal}(\bar{K}/K)$, and X is a nonsingular projective variety over K, then E_8 does not occur as a factor of the identity component of the Zariski-closure of G_K in the second étale cohomology group of $\bar{X}$.

The paper is organized as follows. Sections 2 and 3 treat the infinite-mod-center case of the N-eigenvalue problem. Section 2 gives the general shape of the solution for all N, and Section 3 gives an actual list for $N = 3$. Section 4 shows that a fairly weak hypothesis on the actual eigenvalues is enough to guarantee that G is infinite modulo its center. Section 5 gives applications to number theory, and Section 6 gives applications to braid group representations. We conclude with a discussion of possible future applications to topology and quantum computing.

2 Infinite groups

In this section, we consider the general N-eigenvalue problem for infinite compact groups. Our methods come directly from [13, 19].

Lemma 2.1. Let $V = V_1 \oplus \cdots \oplus V_k$ be a complex vector space and $T : V \to V$ a linear transformation permuting the summands V_i nontrivially. Then the spectrum of V does not satisfy (1.1). □

Proof. Renumbering if necessary, we may assume V permutes $V_1, V_2, \ldots, V_r$ cyclically, where $r \geq 2$. Let $W = V_1 \oplus \cdots \oplus V_r$, let $\zeta_r = e^{2\pi i/r}$, and let $S : W \to W$ act as the scalar ζ_r^i on V_i. Then

$$ST|_W S^{-1} = \zeta_p T|_W, \tag{2.1}$$

so the spectrum of $T|_W$ is invariant under multiplication by ζ_p. It is therefore a union of $\langle\zeta_p\rangle$-cosets. ■

Lemma 2.2. Let (G_1, V_1) and (G_2, V_2) be pairs and let G denote the image of $G_1 \times G_2$ in $\mathrm{GL}(V_1 \otimes V_2)$. If G satisfies the N-eigenvalue property, then there exist integers N_1 and N_2 such that $N_1 + N_2 - 1 \leq N$, and subgroups $G_1' < G_1$ and $G_2' < G_2$ such that (G_i', V_i) satisfies the N_i-eigenvalue property and $G_i' Z(G_i) = G_i$ for $i = 1, 2$. □

Proof. Let $g \in G$ be a generating element, and let $(g_1, g_2) \in G_1 \times G_2$ map to G. Let G_i' denote the subgroup of G_i generated by the conjugacy class of g_i. As the conjugacy class of g generates G, $(\ker G_i \to G)G_i' = G_i$. By construction, $\ker G_i \to G \subset Z(G_i)$.

The spectrum of $\rho(g)$ is the product of the spectra of $\rho_i(g_i)$. So the lemma reduces to the following claim: if X_1 and X_2 are finite subgroups of an abelian group A such that $X_1 + X_2$ does not contain a coset of a nontrivial subgroup of A, then $|X_1 + X_2| \geq |X_1| + |X_2| - 1$. This is well known (see, e.g., [17]). ■

If (G, V) arises in this way, we say it is *decomposable*; otherwise, it is *indecomposable*. Note that the tensor product of pairs which satisfy the N_1 and N_2-eigenvalue conditions need not satisfy the $N_1 + N_2 - 1$-eigenvalue condition. For one thing, the product of sets of cardinality N_1 and N_2 could be as large as $N_1 N_2$. For another, the product of sets satisfying the no-cycle property may itself fail to satisfy the no-cycle property.

Proposition 2.3. Let (G, V) be an indecomposable pair. If G is infinite modulo its center and (G, V) satisfies the N-eigenvalue property for some N, then $G = G^\circ Z(G)$. □

Proof. Let $g \in G$ be a generating element. Then $g \in G^\circ$ implies $G = G^\circ$, in which case there is nothing to prove. If $V|_{G^\circ}$ is not isotypic, then g acts nontrivially on the isotypic factors, and by **Lemma 2.1**, the spectrum of g fails to satisfy property (1.1). If $V|_{G^\circ} = W^n = W \otimes U$, where G° acts trivially on U and irreducibly on W, then the span of $\rho(G^\circ)$ is $\mathrm{End}(W) \otimes \mathrm{Id}_U \subset \mathrm{End}(V)$, so $\rho(G)$ lies in the normalizer of $\mathrm{End}(W) \otimes \mathrm{Id}_U$, which is $\mathrm{End}(W)\,\mathrm{End}(U)$. Thus ρ maps G to $(\mathrm{GL}(W) \times \mathrm{GL}(U))/\mathbb{C}^\times$. Let $\widetilde{G}$ denote the Cartesian square

$$\begin{tikzcd} \widetilde{G} \arrow[r, "\widetilde{\rho}"] \arrow[d, "\pi"'] & \mathrm{GL}(W) \times \mathrm{GL}(U) \arrow[d] \\ G \arrow[r, "\rho"] & \big(\mathrm{GL}(W) \times \mathrm{GL}(U)\big)/\mathbb{C}^\times \end{tikzcd} \tag{2.2}$$

If $\widetilde{g} \in \pi^{-1}(g)$, then the projections of $\widetilde{\rho}(\widetilde{g})$ to $\mathrm{GL}(W)$ and $\mathrm{GL}(U)$ have spectra satisfying the no-cycle property, since the product of these spectra is the spectrum of $\widetilde{\rho}(\widetilde{g})$. If $\dim W$ and $\dim U$ are both ≥ 2, then (G, V) is decomposable, contrary to hypothesis. As G° is not in the center of G, $\dim W \geq 2$. It follows that $\dim U = 1$, that is, the restriction of V to G° is irreducible. Thus every element of G which commutes with G° lies in $Z(G)$.

It follows that for every $g \notin G^\circ Z(G)$, conjugation by g induces an automorphism of G° which is not inner. By [**29**, Theorem 7.5], this implies that there exists a maximal torus T of G° such that $gTg^{-1} = T$ but conjugation by g induces a nontrivial automorphism of T. The characters of T appearing in $V|_T$ span $X^*(T) \otimes \mathbb{R}$ since V is a faithful representation. Therefore, a nontrivial automorphism of T must permute the weights of V nontrivially. By **Lemma 2.1**, this implies that the spectrum of g violates the no-cycle property, contrary to hypothesis. Thus $g \in G^\circ Z(G)$, and since the conjugacy class of g generates G, it follows that $G = G^\circ Z(G)$. ■

Proposition 2.4. Let (G, V) be as in **Proposition 2.3**. Then G is the product of the derived group D of G° and a group of scalar matrices in V. The group D is simple modulo its center, and the restriction of V to D is irreducible. If the highest weight λ of $V|_D$ is written as a linear combination $\sum_i a_i \varpi_i$, where ϖ_i are the fundamental weights, then $\sum_i a_i b_i \leq N - 1$, where the b_i are positive integers determined by the root system of D. □

Proof. As G° is connected, $G^\circ = DZ(G^\circ)$. As $V|_{G^\circ}$ is irreducible, $Z(G^\circ)$ contains only scalars, as does $Z(G)$. Thus $G = DZ(G^\circ)Z(G)$, and the product $Z(G^\circ)Z(G)$ is scalar in $\mathrm{GL}(V)$. The centralizer of D in $\mathrm{GL}(V)$ equals the centralizer of $G^\circ = DZ(G^\circ)$ since $Z(G^\circ)$

is scalar. It follows that $V|_D$ is irreducible. Any tensor decomposition of $V|_D$ extends to G since scalars respect any tensor decomposition; it follows that $V|_D$ is tensor indecomposable and therefore that D is simple modulo its center. Let λ denote its highest weight.

Let g be a generating element, and let $t \in D$ be such that $g^{-1}t$ is a scalar. Let T be a maximal torus of D containing t, R the root system of D with respect to T, and $(\cdot,\cdot)$ the Killing form on $X^*(T) \otimes \mathbb{R}$. Let

$$\langle \beta, \alpha \rangle = \frac{2(\beta,\alpha)}{(\alpha,\alpha)}, \tag{2.3}$$

and fixing a Weyl chamber, let γ denote the root dual to the highest root in R. Thus γ is the highest short root. By [6, Chapter VIII, Section 7, Proposition 3(i)], the maximal arithmetic progression of the form $\lambda, \lambda-\gamma, \lambda-2\gamma, \ldots$ contained in the set of weights of V has length

$$1 + \langle \lambda, \gamma \rangle = 1 + \sum_i a_i b_i, \tag{2.4}$$

where the positive integers b_i are the coefficients in the representation of the highest root in R in terms of the simple roots. If this sum exceeds N, then the geometric progression of values

$$\lambda(t), (\lambda-\gamma)(t), (\lambda-2\gamma)(t), \ldots \tag{2.5}$$

must either take $\geq N+1$ distinct values, or fail (1.1), or be constant. The first two possibilities are ruled out by hypothesis, and it follows that $\gamma(t) = 1$. If w belongs to the Weyl group, the same considerations apply to the weight sequence $w(\lambda), w(\lambda) - w(\gamma), w(\lambda) - 2w(\gamma), \ldots$, so $w(\lambda)(t) = 1$. On the other hand, the short weights in a simple root system form a single Weyl orbit and generate the root lattice, so $\alpha(t) = 1$ for all roots. This implies that t lies in the center of G and therefore that $\rho(t)$ is scalar, contrary to hypothesis. ∎

One can also formulate the N-eigenvalue property for complex Lie groups.

Definition 2.5. Let $G_{\mathbb{C}}$ be a reductive complex Lie group and (ρ, V) a faithful irreducible complex representation of $G_{\mathbb{C}}$. Then $(G_{\mathbb{C}}, V)$ satisfies the N-*eigenvalue property* if there exists a semisimple *generating element* $g_{\mathbb{C}} \in G_{\mathbb{C}}$ whose conjugacy class generates a Zariski-dense subgroup of $G_{\mathbb{C}}$, and such that the spectrum of $\rho(g_{\mathbb{C}})$ consists of N-eigenvalues satisfying the no-cycle condition.

Lemma 2.6. Let $G_{\mathbb{C}}$ be a reductive complex Lie group and (ρ, V) a faithful irreducible complex representation of $G_{\mathbb{C}}$. Let G be a maximal compact subgroup of $G_{\mathbb{C}}$. Then (G, V) satisfies the N-eigenvalue property. □

Proof. Let $T_{\mathbb{C}}$ denote the Zariski-closure of the cyclic group $\langle g_{\mathbb{C}} \rangle$ and $T \subset T_{\mathbb{C}}$ the (unique) maximal compact subgroup. As T can be regarded as the set of (real) points of a real algebraic group whose complex points give $T_{\mathbb{C}}$, T is Zariski-dense in $T_{\mathbb{C}}$. We can decompose the restriction of V to $T_{\mathbb{C}}$ as a direct sum of eigenspaces V_χ associated to characters χ of $T_{\mathbb{C}}$. There must be exactly N such eigenspaces, since any coincidence among $\chi_1(g_{\mathbb{C}}), \ldots, \chi_{N+1}(g_{\mathbb{C}})$ gives the same coincidence for the characters on all of $T_{\mathbb{C}}$. The condition that $\chi_i(t) \neq \chi_j(t)$ is open and nonempty in $T_{\mathbb{C}}$ as is the condition that $\{\chi_1(t), \ldots, \chi_N(t)\}$ satisfy the no-cycle condition. It follows that T contains an element g which satisfies both conditions.

As all maximal compact subgroups of $G_{\mathbb{C}}$ are conjugate, without loss of generality we may assume $T \subset G$. We can regard G as the group of real points of a real linear algebraic group whose complex points give $G_{\mathbb{C}}$ and $T \subset G$ as a Zariski-closed subgroup. Let $H \subset G$ denote the smallest normal Zariski-closed subgroup of G containing g, or equivalently, T. Thus H can be regarded as the group of real points of an algebraic group which is a normal subgroup of the algebraic group with real locus G. Let $H_{\mathbb{C}}$ denote the group of $\mathbb{C}$-points of this subgroup. If $H \neq G$, then $H_{\mathbb{C}} \neq G_{\mathbb{C}}$, so $g_{\mathbb{C}} \in T_{\mathbb{C}} \subset H_{\mathbb{C}}$ is contained in a proper normal subgroup of $g_{\mathbb{C}}$, contrary to hypothesis. It follows that g is a generating element for (G, V). ■

3 The 3-eigenvalue problem

In this section, we give an explicit solution of the $\leq$ 3-eigenvalue problem, assuming throughout that G is a compact Lie group which is infinite modulo center.

Proposition 3.1. If (G, V) is a pair satisfying the 2-eigenvalue property, Φ denotes the root system of G, and ϖ the highest weight of V in the notation of [5, 6], then (Φ, ϖ) is one of the following:

(1) (A_r, ϖ_i), $1 \leq i \leq r$,
(2) (B_r, ϖ_r),
(3) (C_r, ϖ_1),
(4) (D_r, ϖ_i), $i = 1, r-1, r$. □

Proof. This is the statement of [13, Theorem 1.1]. ■

Before treating the general 3-eigenvalue problem, we make a detailed study of the A_r case.

Lemma 3.2. Let (ρ, V) be an irreducible representation of $SU(n)$ with highest weight ϖ, and t a noncentral element of $SU(n)$. Suppose there are at most three eigenvalues of $\rho(t)$ and they satisfy the no-cycle property. Then one of the following is true.

(1) For $1 \le i \le n-1$, $\varpi = \varpi_i$, and t has characteristic polynomial $(x-\lambda)^{n-1}(x-\lambda^{1-n})$; the eigenvalues of $\rho(t)$ are λ^i, λ^{i-n}.

(2) For $1 \le i \le n-1$, $\varpi = \varpi_i$, and t has characteristic polynomial $(x-\lambda_1)^{n-2}(x-\lambda_2)^2$; the eigenvalues of $\rho(t)$ are λ_1^i, $\lambda_1^{i-1}\lambda_2$, and $\lambda_1^{i-2}\lambda_2^2 = \lambda_1^{i-n}$.

(3) For $i \in \{1, 2, n-2, n-1\}$, $\varpi = \varpi_i$, and t has eigenvalues λ_1 and λ_2; the spectrum of $\rho(t)$ is $\{\lambda_1, \lambda_2\}$, $\{\lambda_1^2, \lambda_1\lambda_2, \lambda_2^2\}$, $\{\lambda_1^{-2}, \lambda_1^{-1}\lambda_2^{-1}, \lambda_2^{-2}\}$, or $\{\lambda_1^{-1}, \lambda_2^{-1}\}$, if i is 1, 2, $n-2$, or $n-1$, respectively.

(4) For $1 \le i \le n-1$, $\varpi = \varpi_i$, and t has characteristic polynomial $(x-\lambda^{n-2})(x-\lambda\mu)(x-\lambda\mu^{-1})$; the eigenvalues of $\rho(t)$ are λ_1^i, $\lambda_1^i\mu$, and $\lambda_1^i\mu^{-1}$.

(5) For $i = 1$ or $i = n-1$, $\varpi = \varpi_i$, and t has eigenvalues $\lambda_1, \lambda_2, \lambda_3$; the eigenvalues of $\rho(t)$ are the λ_j or the λ_j^{-1} if $i = 1$ or $i = n-1$, respectively.

(6) For $i = 1$ or $i = n-1$, $\varpi = 2\varpi_i$, and t has eigenvalues λ_1 and λ_2, each of multiplicity at least 2; the eigenvalues of $\rho(t)$ are $\{\lambda_1^2, \lambda_1\lambda_2, \lambda_2^2\}$ or $\{\lambda_1^{-2}, \lambda_1^{-1}\lambda_2^{-1}, \lambda_2^{-2}\}$ if i is 1 or $n-1$, respectively.

(7) The highest weight ϖ is $\varpi_1 + \varpi_{n-1}$, and t has eigenvalues λ_1 and λ_2, each of multiplicity at least 2; the eigenvalues of $\rho(t)$ are λ_1/λ_2, 1, and λ_2/λ_1.

(8) For $1 \le i \le j \le n-1$, $\varpi = \varpi_i + \varpi_j$, and t has characteristic polynomial $(x-\lambda)^{n-1}(x-\lambda^{1-n})$; the eigenvalues of $\rho(t)$ are λ^{i+j}, λ^{i+j-n}, and λ^{i+j-2n}.

In particular, only case (5) can give three eigenvalues not in geometric progression. □

Proof. By **Proposition 2.4**, if $\rho(t)$ has $N \le 3$ eigenvalues, ϖ is a sum of at most $N-1$ fundamental weights. If $\varpi = \varpi_i$ and t has eigenvalues $\lambda_1, \dots, \lambda_n$, the eigenvalues of $\rho(t)$ are

$$\left\{ \prod_{s\in S} \lambda_s \mid S \subset \{1, \dots, n\},\ |S| = i \right\}. \tag{3.1}$$

Duality exchanges ϖ_i and ϖ_{n-i}, so without loss of generality we may assume $i \le n/2$. If $\lambda_1, \dots, \lambda_4$ are all distinct, and $n \ge i+3$ (in particular, this holds if $n \ge 5$), then

$$\{\lambda_j\lambda_5\lambda_6\cdots\lambda_{3+i} \mid 1 \le j \le 4\} \tag{3.2}$$

already contains four distinct elements. If $n = 4$ and $i = 2$, two products $\lambda_i\lambda_j$ and $\lambda_k\lambda_l$ are distinct unless $\{i, j\}$ and $\{k, l\}$ are complementary sets, in which case the equality implies $\lambda_i\lambda_j = \pm 1$. At least one of $\lambda_1\lambda_j$, $2 \leq j \leq 4$ is neither 1 nor -1, so there must be at least four elements in the set $\{\lambda_1\lambda_2, \ldots, \lambda_3\lambda_4\}$. If

$$\lambda_1 = \lambda_2 = \lambda_3 \neq \lambda_4 = \lambda_5 = \lambda_6, \tag{3.3}$$

and $i \geq 3$, then

$$\{\lambda_1^j \lambda_4^{3-j} \lambda_7 \cdots \lambda_{3+i} \mid 0 \leq j \leq 3\} \tag{3.4}$$

contains a nonconstant 4-term geometric progression in the spectrum of $\rho(t)$, contrary to hypothesis. If

$$\lambda_1 = \lambda_2 \neq \lambda_3 = \lambda_4 \neq \lambda_5 \neq \lambda_1, \tag{3.5}$$

then

$$\begin{aligned}\{\lambda_1^2\lambda_2\lambda_6 \cdots \lambda_{2+i}, \lambda_1\lambda_2^2\lambda_6 \cdots \lambda_{2+i}, \lambda_1\lambda_2\lambda_3\lambda_6 \\ \cdots \lambda_{2+i}, \lambda_1^2\lambda_3\lambda_6 \cdots \lambda_{2+i}, \lambda_2^2\lambda_3\lambda_6 \cdots \lambda_{2+i}\}\end{aligned} \tag{3.6}$$

contains at least four distinct elements unless $\lambda_1\lambda_3 = \lambda_2^2$ and $\lambda_2\lambda_3 = \lambda_1^2$, in which case it does not satisfy (1.1). The remaining possibilities are that t has two distinct eigenvalues, one of multiplicity 1; two distinct eigenvalues, one of multiplicity 2; two distinct eigenvalues of arbitrary multiplicity, and i (or $n - i$) is ≤ 2; three distinct eigenvalues, two of them of multiplicity 1; or three distinct eigenvalues of arbitrary multiplicity, and i (or $n - i$) is 1.

These give rise to cases (1), (2), (3), (4), and (5), respectively. If $\lambda = \varpi_i + \varpi_j$, $i \leq j$, is among the weights appearing in V_ϖ, then $\varpi_{i-1} + \varpi_{j+1}$ also appears, where we define $\varpi_0 = \varpi_n = 0$. Thus if $\varpi = \varpi_i + \varpi_j$, $i \leq j$, then either ϖ_{i+j}, ϖ_{2n-i-j} or $\varpi_1 + \varpi_{n-1}$ is among the weights of V_ϖ, as $i + j$ is less than, greater than, or equal to n.

First we consider the case $i + j = n$. If t has three distinct eigenvalues $\lambda_1, \lambda_2, \lambda_3$, then

$$\left|\left\{\frac{\lambda_1}{\lambda_2}, \frac{\lambda_2}{\lambda_1}, \frac{\lambda_1}{\lambda_3}, \frac{\lambda_3}{\lambda_1}, \frac{\lambda_2}{\lambda_3}, \frac{\lambda_3}{\lambda_2}\right\}\right| \leq 3 \tag{3.7}$$

implies that the set violates (1.1) with $n = 3$. Thus, t has exactly two eigenvalues λ_1 and λ_2. If $i \geq 2$ and λ_1 and λ_2 each occurs with multiplicity ≥ 2, then

$$\left\{\frac{\lambda_1^2}{\lambda_2^2}, \frac{\lambda_1}{\lambda_2}, 1, \frac{\lambda_2}{\lambda_1}, \frac{\lambda_2^2}{\lambda_1^2}\right\} \tag{3.8}$$

is contained in the spectrum of $\rho(t)$ since the Weyl orbits of $\varpi_1 + \varpi_{n-1}$ and $\varpi_2 + \varpi_{n-2}$ are subsets of the weights of V_ϖ. As $\lambda_1 \neq \lambda_2$, either this set contains 5 distinct elements or it violates (1.1). The remaining cases are (7) and the $i + j = n$ case of (8).

If $i + j \neq n$, replacing V_ϖ by its dual if necessary, we can assume that $i + j < n$. If $3 \leq i + j \leq n - 3$, then ϖ_{i+j} is a weight of V_ϖ, so by the analysis above, t has two eigenvalues, one with multiplicity one, and we are in case (8). If $i + j = 2$, we see that $2\varpi_1$ and ϖ_2 are both weights of V_ϖ, so if $\lambda_1, \lambda_2, \lambda_3$ are eigenvalues of t,

$$\{\lambda_1^2, \lambda_2^2, \lambda_3^2, \lambda_2\lambda_3, \lambda_3\lambda_1, \lambda_1\lambda_2\} \tag{3.9}$$

is contained in the spectrum of $\rho(t)$, contrary to assumption. If there are exactly two eigenvalues, we get (6) and the $i = j = 1$ case of (8).

If $i + j = n - 2$, then V_ϖ contains all the weights of $V_{\varpi_{n-2}}$, so t may have only two eigenvalues, λ_1 and λ_2, by the analysis of the case that ϖ is a fundamental weight, above. If each occurs with multiplicity ≥ 2 and (without loss of generality) λ_1 occurs with multiplicity ≥ 3, then

$$\left\{\frac{\lambda_2}{\lambda_1^3}, \frac{1}{\lambda_1^2}, \frac{1}{\lambda_1\lambda_2}, \frac{1}{\lambda_2^2}\right\} \tag{3.10}$$

is a 4-term geometric progression contained in the spectrum of $\rho(t)$ contrary to hypothesis. If $i + j = n - 1$, then V_ϖ contains all the weights of $V_{\varpi_{n-1}}$. If λ_1 and λ_2 are eigenvalues of t of multiplicity ≥ 2, then the spectrum of $\rho(t)$ contains the 4-term geometric progression

$$\left\{\frac{\lambda_2}{\lambda_1^2}, \frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{\lambda_1}{\lambda_2^2}\right\}. \tag{3.11}$$

If t has three distinct eigenvalues $\lambda_1, \lambda_2, \lambda_3$, then the spectrum of $\rho(t)$ contains

$$\left\{\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}, \frac{\lambda_1}{\lambda_2\lambda_3}, \frac{\lambda_2}{\lambda_1\lambda_3}, \frac{\lambda_3}{\lambda_1\lambda_2}\right\} \tag{3.12}$$

which either violates the no-cycle condition or contains more than 3 elements. It follows that t has exactly two eigenvalues, one of multiplicity $n-1$. So all of these possibilities are subsumed in case (8). ∎

Theorem 3.3. If (G, V) is an indecomposable pair satisfying the 3-eigenvalue property, Φ denotes the root system of the derived group D of G°, and ϖ the highest weight of V, then (Φ, ϖ) is either one of the pairs enumerated in Proposition 3.1 or one of the following.

(1) $(A_r, \varpi_i + \varpi_j)$, $1 \le i \le j \le r$,
(2) (B_r, ϖ_i), $1 \le i \le r-1$,
(3) $(B_r, 2\varpi_r)$,
(4) (C_r, ϖ_i), $2 \le i \le r$,
(5) $(C_r, 2\varpi_1)$,
(6) (D_r, ϖ_i), $2 \le i \le r-2$,
(7) (D_r, ϖ), $\varpi \in \{2\varpi_{r-1}, \varpi_{r-1} + \varpi_r, 2\varpi_r\}$,
(8) (E_6, ϖ_i), $i = 1, 3, 6$,
(9) (E_7, ϖ_i), $i = 1, 7$,
(10) (F_4, ϖ_4),
(11) (G_2, ϖ_2),

If there exists a generating element with three eigenvalues which do not form a geometric progression, then (Φ, ϖ) is (A_r, ϖ_1) or (A_r, ϖ_r). □

Proof. By Proposition 2.4, the root system is simple and if $\varpi = \sum_i a_i \varpi_i$ and the highest root is $\sum_i b_i \alpha_i$, then $\sum a_i b_i \le 2$. By [5, Planches], this reduces the possibilities to those listed, together with

(12) (D_r, ϖ), $\varpi \in \{2\varpi_1, \varpi_1 + \varpi_{r-1}, \varpi_1 + \varpi_r\}$,
(13) (E_6, ϖ), $\varpi \in \{2\varpi_1, \varpi_2, \varpi_5, 2\varpi_6, \varpi_1 + \varpi_6\}$,
(14) (E_7, ϖ), $\varpi \in \{\varpi_2, \varpi_6, 2\varpi_7\}$,
(15) (E_8, ϖ), $\varpi \in \{\varpi_1, \varpi_8\}$,
(16) (F_4, ϖ_1).

To see that the classical cases (1)–(7) above are achieved, we let $G = D$ and V the indicated representation, and we choose the generating element as follows. For A_r, we let g be the image of the diagonal element $\mathrm{diag}(\lambda^{-r}, \lambda, \dots, \lambda) \in SU(r+1)$ in G. For B_r, we let g denote the image of an element in $\mathrm{Spin}(2r+1)$ whose image in $SO(2r+1)$ is $\mathrm{diag}(\lambda, \lambda^{-1}, 1, \dots, 1)$. For C_r, we let g denote the image of the element $(\lambda, \lambda^{-1}, 1, \dots, 1)$ in $Sp(2r)$. For D_r, we let g denote the image of an element in $\mathrm{Spin}(2r)$ whose image in $SO(2r)$ is $\mathrm{diag}(\lambda, \lambda^{-1}, 1, \dots, 1)$.

Next we show that the excluded cases (12)–(16) above do not occur. For D_r, we consider an element g whose image in $SO(2r)$ has eigenvalues $\lambda_1^{\pm 1}, \dots, \lambda_r^{\pm 1}$. In $V_{2\varpi}$, the eigenvalues of g are $\lambda_i^{\pm 2}, \lambda_i^{\pm 1}\lambda_j^{\pm 1}$, and 1. It is easy to see these represent at least 5 distinct values. A similar analysis rules out the remaining cases in (12).

For E_6 and F_4 we use the existence of equal rank semisimple subgroups of the form A_2^k. As these subgroups share a maximal torus with their ambient groups, every generating element g can be conjugated into the subgroup. We use the branching rules tabulated in [22] to compute the restrictions of G-representations via $SU(3)^k \to G$; since the center of $SU(3)^k$ has exponent 3, and since we know that there are no 2-eigenvalue solutions for F_4 and E_6, there can be no $\leq$ 3-eigenvalue solutions coming from central elements of $SU(3)^k$ and satisfying (1.1). If $M(\lambda) \in SU(3)$ has eigenvalues $\lambda, \lambda, \lambda^{-2}$, then $M(\lambda) \times M(\lambda^{-1})$ maps to an element of F_4 which has eigenvalues $\lambda^{-3}, 1, \lambda^3$ for V_{ϖ_4}. The restriction of F_4 to $SU(3)^2$ is

$$V_{2\mu_2} \boxtimes V_{\mu_1} \oplus V_{2\mu_1} \boxtimes V_{\mu_2} \oplus V_{\mu_1+\mu_2} \boxtimes V_0 \oplus V_0 \boxtimes V_{\mu_1+\mu_2}; \tag{3.13}$$

the image of any element noncentral in both factors has at least four eigenvalues from the first summand; the image of any element central in the second factor but not the first has at least four eigenvalues from the first two summands; the image of any element central in the first factor but not in the second has at least four eigenvalues or the eigenvalues $\{1, e^{\pm 2\pi i/3}\}$ from the first two summands. For (E_6, ϖ_1), the image of $M(\lambda) \times M(\lambda) \times 1$ has eigenvalues $\{\lambda^{-2}, \lambda, \lambda^4\}$, and it is not difficult to see that this is essentially the only way to get three eigenvalues. For (E_6, ϖ_2), the image of $M(\lambda) \times M(\lambda) \times 1$ has eigenvalues $\{\lambda^{-3}, 1, \lambda^3\}$. To see that the excluded cases (13) do not give solutions to the 3-eigenvalue problem, we note that

$$\begin{aligned} V_{\varpi_2}|_{A_2^3} &= V_{\mu_1+\mu_2} \boxtimes V_{\mu_2} \boxtimes V_{\mu_2} \oplus V_{\mu_2} \boxtimes V_{\mu_1+\mu_2} \boxtimes V_{\mu_1} \oplus \cdots; \\ V_{2\varpi_1}|_{A_2^3} &= V_{\mu_1+\mu_2} \boxtimes V_{\mu_2} \boxtimes V_{\mu_2} \oplus V_{\mu_2} \boxtimes V_{\mu_1+\mu_2} \boxtimes V_{\mu_1} \oplus \cdots; \\ V_{\varpi_1+\varpi_6}|_{A_2^3} &= V_{2\mu_1} \boxtimes V_{\mu_1} \boxtimes V_{\mu_2} \oplus V_{\mu_1} \boxtimes V_{2\mu_1} \boxtimes V_{\mu_1} \oplus \cdots. \end{aligned} \tag{3.14}$$

These summands are already enough to guarantee that if (E_6, ϖ_2), $(E_6, 2\varpi_1)$, or $(E_6, \varpi_1 + \varpi_6)$ satisfies the 3-eigenvalue condition, any generating element in A_2^3 must be central in two of the three factors and have eigenvalues $\lambda, \lambda, \lambda^{-2}, \lambda^3 \neq 1$, in the third. However, if $\omega^3 = 1$, neither

$$\{\lambda^{-3}, 1, \lambda^3, \omega\lambda, \omega\lambda^{-2}\} \tag{3.15}$$

Table 3.1

Φ	ϖ	$\{\mu_i\}$	ζI e-values	$M(\lambda)$ e-values
E_7	ϖ_1	$\mu_1+\mu_7, \mu_4$	± 1	$\lambda^{-8}, \lambda^{-4}, 1,$ λ^4, λ^8
E_7	ϖ_2	$\mu_1+\mu_5, \mu_3+\mu_7,$ $2\mu_1, 2\mu_7$	$\pm i$	$\lambda^{-14}, \lambda^{-10}, \lambda^{-6},$ $\lambda^{-2}, \lambda^2, \lambda^6, \lambda^{10}, \lambda^{14}$
E_7	ϖ_6	$\mu_1+\mu_3, \mu_5+\mu_7,$ $\mu_1+\mu_7, \mu_2+\mu_6$	± 1	$\lambda^{-12}, \lambda^{-8}, \lambda^{-4}, 1, \lambda^4,$ λ^8, λ^{12}
E_7	ϖ_7	μ_2, μ_6	$\pm i$	$\lambda^{-6}, \lambda^{-2}, \lambda^2, \lambda^6$
E_7	$2\varpi_7$	$0, \mu_4, \mu_2+\mu_6,$ $2\mu_2, 2\mu_6$	± 1	$\lambda^{-12}, \lambda^{-8}, \lambda^{-4}, 1, \lambda^4,$ λ^8, λ^{12}
E_8	ϖ_1	$\mu_3, \mu_6, \mu_1+\mu_8$	$1, e^{\pm 2\pi i/3}$	$\lambda^{-9}, \lambda^{-6}, \lambda^{-3}, 1, \lambda^3,$ λ^6, λ^9
E_8	ϖ_8	$\mu_1+\mu_2, \mu_1+\mu_5,$ $\mu_1+\mu_8, \mu_2+\mu_7,$ $\mu_4+\mu_8, \mu_7+\mu_8$	$1, e^{\pm 2\pi i/3}$	$\lambda^{-15}, \lambda^{-12}, \lambda^{-9}, \lambda^{-6},$ $\lambda^{-3}, 1,$ $\lambda^3, \lambda^6, \lambda^9, \lambda^{12}, \lambda^{15}$

nor

$$\{\lambda^2, \lambda^{-1}, \lambda^{-4}, \omega\lambda, \omega\lambda^{-2}\} \tag{3.16}$$

can have order ≤ 3 and satisfy the no-cycle property.

For E_n, $n \geq 7$, we use the equal rank subgroups A_n. Again, [22] gives the restriction of V_ϖ to $SU(n+1)$. Table 3.1 lists all irreducible components of these restrictions for all possible ϖ. It also specifies the eigenvalues in V_ϖ for the image of the scalar matrix ζI and the matrix $M(\lambda)$.

It follows that neither scalar matrices nor matrices of the form $M(\lambda)$ give rise to 3-eigenvalue solutions. By Lemma 3.2, the only possible solutions to the 3-eigenvalue problem for E_7 and E_8 are the pairs (E_7, ϖ_1), (E_7, ϖ_7), and (E_8, ϖ_1). For the first, an element of $SU(8)$ with eigenvalues λ, λ, λ, λ, λ, λ, λ^{-3}, λ^{-3} maps to an element of E_7 with eigenvalues $\lambda^{-4}, 1, \lambda^4$. For the second, an element of $SU(8)$ with eigenvalues λ, λ, λ, λ, λ^{-1}, λ^{-1}, λ^{-1}, λ^{-1} maps to an element of E_7 with eigenvalues $\lambda^{-2}, 1, \lambda^2$. For (E_8, ϖ_1), the only possibility is an element of $SU(9)$ with λ_1 of multiplicity 7 and λ_2 of multiplicity 2. This maps to an element of E_8 with two three-term geometric progressions of eigenvalues: λ_1^3, $\lambda_1^2\lambda_2$, $\lambda_1\lambda_2^2$; and $\lambda_1\lambda_2^{-1}$, 1, $\lambda_1^{-1}\lambda_2$. To have three eigenvalues in all, we must have $\lambda_1^3 = \lambda_1\lambda_2^{-1}$, which together with $\lambda_1^7\lambda_2^2 = 1$ implies that the eigenvalues are all equal, which we have already seen is not a possibility.

The case of G_2 is trivial.

When there are three eigenvalues not in geometric progression, the representations cannot be self-dual, and if $\phi = A_r$, then $\varpi \in \{\varpi_1, \varpi_r\}$ by **Lemma 3.2**. The only remaining cases for which V_ϖ is not self-dual are $(D_r, V_{2\varpi_{r-1}})$ and its dual (when r is odd) and (E_6, ϖ_1) and its dual. In the first case, as r is odd, the Weyl orbit of ϖ_1 lies in the set of weights of both $V_{2\varpi_{r-1}}$ and $V_{2\varpi_r}$. The eigenvalues contributed by these weights come in mutually inverse pairs; if there are ≤ 3 but not three in geometric progression, then there must be two: λ and λ^{-1}, which are distinct from one another. Then the Weyl orbit of ϖ_3 also lies in the set of weights of V_ϖ, so $\lambda^3, \lambda, \lambda^{-1}, \lambda^{-3}$ are all eigenvalues of $\rho(t)$, which is absurd. In the second case, restricting from E_6 to $SU(6) \times SU(2)$, we get

$$V_{\varpi_1} \boxtimes V_{\varpi_1} \oplus V_{\varpi_4} \boxtimes V_0. \tag{3.17}$$

The second summand contributes ≤ 2 eigenvalues or 3 eigenvalues not in geometric progression, so an inverse image (g_1, g_2) of the generating element must be a scalar ζ in $SU(6)$ (and therefore $\zeta^6 = 1$). The eigenvalues of g in the first summand are $\{\zeta\lambda, \zeta^4, \zeta\lambda^{-1}\}$ which are in geometric progression, contrary to assumption. ■

4 The asymptotic N-eigenvalue condition

In this section we consider what can be said when the eigenvalues of a generating element are sufficiently general. One hypothesis which is strong enough for our purposes is that the eigenvalues are distinct rth roots of unity where r is a sufficiently large prime. We consider a somewhat more general condition.

Proposition 4.1. Let $T \cong U(1)^d$ be a torus and U an open neighborhood of the identity in T. There exists a finite set S of characters $\chi : T \to U(1)$ and an integer m such that if n is a positive integer and $t \in T$ an n-torsion point, at least one of the following must be true.

(1) There exists $\chi \in S$ such that $\chi(t) \neq 1$ has order $\leq m$.

(2) There exists an integer k relatively prime to n such that $t^k \in U$. □

Proof. We use induction on dimension, the proposition being trivial in dimension 0.

By Urysohn's lemma, there exists a continuous function $f : T \to [0, 1]$ such that $f(x) = 0$ for $x \notin U$ and $f(x) = 1$ in some neighborhood of the identity. It is well known (see, e.g., [28, Chapter VII, Theorem 1.7]) that finite linear combinations of characters are dense in the L^∞ norm on the set of continuous functions on T. It follows that there exists a real-valued finite sum $f(x) := \sum_{\chi\in S} a_\chi \chi(x)$ such that $f(x) < 0$ for all $x \in T \setminus U$ and $a_0 = \int f(x)dx > 0$. Enlarging S if necessary, we may assume without loss of generality that if $n\chi \in S$ for some positive integer n, then $\chi \in S$.

Suppose $\chi(t) = 1$ for some nontrivial character $\chi \in S$. Let $\lambda \in S$ denote a primitive character in S and k a positive integer such that $\chi = k\lambda$. If m is taken greater than the value of k associated with any character in S, either (1) is satisfied or $\lambda(t) = 1$. As λ is primitive, $\ker\lambda$ is a subtorus of T. As there are only finitely many subtori arising in this way, the proposition follows by induction.

We may therefore assume that the order of $\chi(t)$ is greater than m for each $\chi \in S$. We have

$$\sum_{\{k\in[0,n]\cap\mathbb{Z}|(k,n)=1\}} f(t^k) = a_0\phi(n) + \sum_{\chi\in S\setminus\{0\}} a_\chi \sum_{\{k\in[0,n]\cap\mathbb{Z}|(k,n)=1\}} \chi(t^k). \tag{4.1}$$

If n_χ is the order of $\chi(t)$, then

$$\sum_{\{k\in[0,n]\cap\mathbb{Z}|(k,n)=1\}} \chi(t^k) = \frac{\phi(n)}{\phi(n_\chi)} \sum_{\{k\in[0,n_\chi]\cap\mathbb{Z}|(k,n_\chi)=1\}} \chi(t^k) = \frac{\mu(n)\phi(n)}{\phi(n_\chi)}. \tag{4.2}$$

Choosing m large enough that for all $n_\chi > m$,

$$\sum_{\chi\in S\setminus\{0\}} |a_\chi| \le \phi(n_\chi)a_0, \tag{4.3}$$

we conclude that

$$\sum_{\{k\in[0,n]\cap\mathbb{Z}|(k,n)=1\}} f(t^k) \ge 0 \tag{4.4}$$

and therefore that $t^k \in U$ for some k prime to n. ■

Theorem 4.2. For every integer $N \ge 2$ there exists an integer m such that if (G, V) satisfies the N-eigenvalue property with a generator g with eigenvalues $\lambda_1, \ldots, \lambda_N$, and G is finite modulo its center, then the group $\langle\lambda_i\lambda_j^{-1}\rangle$ generated by ratios of eigenvalues of $\rho(g)$ contains a nontrivial root of unity of order less than m. □

Proof. If G is finite modulo its center and acts irreducibly on V, then either G° is trivial or it consists of all scalars of absolute value 1. In the latter case, we can replace g by $\det(g)^{1/\dim(V)}g$ for any choice of root, and the resulting conjugacy class still satisfies the N-eigenvalue property, generates a subgroup of $G \cap SL(V)$ (which is finite), and determines the same group of eigenvalue ratios $\langle \lambda_i \lambda_j^{-1} \rangle$. Without loss of generality, therefore, we may assume G is finite.

Any automorphism of $\mathbb{C}$ determines an automorphism of the abstract group $GL_n(\mathbb{C})$ for each n. Consider the quotient $T = U(1)^n/U(1)$ of the diagonal unitary matrices by the unitary scalar matrices. Let $U \subset T$ denote the image of A^n in T, where A is the arc from $-\pi/6$ to $\pi/6$, and let n be the order of the group generated by the eigenvalues of g. We apply Proposition 4.1 to obtain m large enough that our hypotheses imply the existence of a field automorphism σ of $\mathbb{C}$ such that all the eigenvalues of $\sigma(\rho(g))$ lie in an arc of length $\leq \pi/3$ on the unit circle. By [3, Theorem 8], this implies that the representation $\sigma\circ\rho$ is imprimitive. As the conjugacy class of g generates G, the element g itself must satisfy the hypothesis of Lemma 2.1, and therefore the spectrum of $\sigma(\rho(g))$ does not satisfy (1.1). As this property is stable under Galois action, the spectrum of $\rho(g)$ fails to satisfy (1.1), contrary to hypothesis. ■

Corollary 4.3. For every integer $N \geq 2$ there exists an integer m such that if (G, V) satisfies the N-eigenvalue property with a generator g of prime order r, then $r < m$ or G is infinite modulo its center. □

We remark that it is probably possible to prove a stronger version of this corollary, in which a good bound is given for m, using [42] as a starting point.

5 Application to Hodge-Tate theory

Let $\bar{\mathbb{Q}}_p$ be an algebraic closure of $\mathbb{Q}_p$, and $\mathbb{C}_p$ denote the completion of $\bar{\mathbb{Q}}_p$. Let K and L be subfields of $\bar{\mathbb{Q}}_p$ finite over $\mathbb{Q}_p$, and let $\Gamma_K := \mathrm{Gal}(\bar{\mathbb{Q}}_p/K)$. Let $V_L \cong L^d$ be a finite-dimensional L-vector space and $\rho_L : \Gamma_K \to GL(V_L)$ a continuous representation. Then Γ_K acts on both factors of $V_{\mathbb{C}_p} := V_L \otimes_L \mathbb{C}_p$. The representation is said to be *Hodge-Tate* if $V_{\mathbb{C}_p}$ decomposes as a direct sum of factors $V_{i\mathbb{C}_p}$ such that Γ_K acts on V_i through the ith tensor power of the cyclotomic character. If X is a complete nonsingular variety over K and $\bar{X}$ is obtained from X by extending scalars to $\bar{\mathbb{Q}}_p$, then $V_L := H^k(\bar{X}, L)$ is Hodge-Tate for all nonnegative integers k, and the factors $V_{i\mathbb{C}_p}$ are nonzero only if $0 \leq i \leq k$ [8].

Let G_L denote the Zariski-closure of the image of $\rho_L(\Gamma_K)$ in GL_d. By the axiom of choice, any two uncountable algebraically closed fields of characteristic zero whose cardinalities are the same are isomorphic. Therefore, $\mathbb{C} \cong \mathbb{C}_p$, and extending scalars, we can view $G_{\mathbb{C}}$ as a complex algebraic group. Let G denote a maximal compact subgroup of $G_{\mathbb{C}}$. The inclusion $G_{\mathbb{C}} \subset GL(V_{\mathbb{C}})$ gives G a complex representation which we denote (ρ, V). If ρ_L is absolutely irreducible, then $V_{\mathbb{C}}$ is an irreducible representation of $G_{\mathbb{C}}$ and therefore of G.

Although G_L need not be connected, by passing to a finite extension K' of K (i.e., replacing Γ_K by a normal open subgroup) we can replace G_L by its identity component. Therefore, in trying to understand what Lie algebras and Lie algebra representations can arise from Hodge-Tate structures with specified weights, without loss of generality we may assume that G_L is connected.

Definition 5.1. Let $G_{\mathbb{C}}$ be a connected reductive algebraic group over $\mathbb{C}$, and V a faithful complex representation of $G_{\mathbb{C}}$. Then, $(G_{\mathbb{C}}, V)$ is of N-*eigenvalue type* if for every almost simple normal subgroup $H_{\mathbb{C}}$ of $G_{\mathbb{C}}$ and every irreducible factor W of $V|_{H_{\mathbb{C}}}$, the image of $H_{\mathbb{C}}$ in $GL(W)$ satisfies the N_W-eigenvalue property for some $N_W \leq N$.

Lemma 5.2. Let $G_{\mathbb{C}}$ be a connected reductive complex Lie group and (ρ, V) a faithful representation. Let $g_{i\mathbb{C}} \in G_{\mathbb{C}}$ be semisimple elements generating a Zariski-dense subgroup of $G_{\mathbb{C}}$, such that the spectrum of $\rho(g_{i\mathbb{C}})$ has N-eigenvalues satisfying the no-cycle condition. Then $(G_{\mathbb{C}}, V)$ is of N-eigenvalue type. □

Proof. Let $D_{\mathbb{C}}$ denote the derived group of $G_{\mathbb{C}}$. The universal cover $\widetilde{D}_{\mathbb{C}}$ factors into simply connected, almost simple complex groups $G_{j\mathbb{C}}$. Every irreducible factor W of V restricts to an irreducible representation of $\widetilde{D}_{\mathbb{C}}$ which decomposes as $W_1 \otimes \cdots \otimes W_k$, where W_j is an irreducible representation of $G_{j\mathbb{C}}$.

Each $g_{i\mathbb{C}}$ in our generating set factors as $d_{i\mathbb{C}} z_{i\mathbb{C}}$, where $z_{i\mathbb{C}}$ lies in the center of $G_{\mathbb{C}}$. We choose $\widetilde{d}_{i\mathbb{C}} \in \widetilde{D}_{\mathbb{C}}$ lying over $d_{i\mathbb{C}}$, and let $g_{ij\mathbb{C}}$ denote the $G_{j\mathbb{C}}$ coordinate of $\widetilde{d}_{i\mathbb{C}}$. For each j, there exists W such that W_j is nontrivial and i such that $g_{ij\mathbb{C}}$ does not lie in the center of $G_{j\mathbb{C}}$. As $g_{i\mathbb{C}}$ is semisimple, the same is true of $d_{i\mathbb{C}}$ and therefore $\widetilde{d}_{i\mathbb{C}}$ and therefore $g_{ij\mathbb{C}}$. Moreover, it has at most N-eigenvalues on W_j and they satisfy (1.1), since if S and T are sets of complex numbers and the product set satisfies (1.1), then $|S|, |T| \leq |ST|$, and $|S|$ and $|T|$ satisfy (1.1). As $g_{ij\mathbb{C}}$ is not in the center of $G_{j\mathbb{C}}$, the conjugacy class of $\rho_j(g_{ij\mathbb{C}})$ generates a noncentral normal subgroup of the almost simple group $\rho_j(G_{j\mathbb{C}})$ and therefore generates the whole group. ■

Theorem 5.3. If V_L is an absolutely irreducible Hodge-Tate representation of G_K with N distinct weights, then $(G^\circ_{\mathbb{C}}, V)$ is of N-eigenvalue type. □

Proof. The grading of $V_{\mathbb{C}}$ which assigns $V_{i\mathbb{C}}$ degree i uniquely determines a cocharacter $h : \mathbb{G}_m \to G_{\mathbb{C}}$ such that $\rho \circ h$ acts isotypically on $V_{i\mathbb{C}}$ by the ith power character.

By [26], G_L° is the smallest L-algebraic subgroup of GL_d which contains $h(\mathbb{G}_m)$. Thus $\{h^\sigma(\mathbb{G}_m) \mid \sigma \in \mathrm{Aut}_L(\mathbb{C})\}$ generates $G_{\mathbb{C}}^\circ$. If $u \in \mathbb{C}^\times$ is of infinite order, then any element $g_{j\mathbb{C}} \in h^{\sigma_j}(u)$ (Zariski-topologically) generates $h^{\sigma_j}(\mathbb{G}_m)$. Together, the $g_{j\mathbb{C}}$ generate $G_{\mathbb{C}}^\circ$. There are exactly N distinct eigenvalues of $\rho(g_{j\mathbb{C}})$ and they satisfy the no-cycle condition. The theorem now follows from Lemma 5.2. ■

Theorem 5.4. Assume that the Fontaine-Mazur conjecture [10, Conjecture 5a] holds. If X is a complete nonsingular variety over a number field K, k is a nonnegative integer, $G_{\mathbb{C}}$ is the complexification of the Zariski-closure of $\mathrm{Gal}(\bar{K}/K)$ in $\mathrm{Aut}(H^k(\bar{X}, \mathbb{Q}_p))$, and $V = \mathrm{Aut}(H^k(\bar{X}, \mathbb{Q}_p)) \otimes_{\mathbb{Q}_p} \mathbb{C}$, then $(G_{\mathbb{C}}^\circ, V)$ is of k-eigenvalue type. □

Proof. As X has good reduction over K, there exists a rational integer M such that X is the generic fiber of a smooth proper scheme $\mathcal{X}$ over $\mathcal{O}_K[1/M]$, where $\mathcal{O}_K$ is the ring of integers of K. Thus, the homomorphism $\mathrm{Gal}(\bar{K}/K) \to \mathrm{Aut}(H^k(\bar{X}, \mathbb{Q}_p))$ factors through $\rho : \Gamma_{K,Mp} \to GL_n(\mathbb{Q}_p)$, the Galois group over K of the maximal subfield of $\bar{K}$ unramified over any prime of $\mathcal{O}_K$ not dividing Mp.

For each prime ν of $\mathcal{O}_K$ dividing Mp, we fix an embedding $\bar{K} \hookrightarrow \bar{K}_\nu$ and therefore an embedding $\Gamma_{G_\nu} \hookrightarrow \Gamma_{K,Mp}$. Let G, regarded as an algebraic group over $\mathbb{Q}_p$, be the Zariski-closure of $\rho(\Gamma_{K,Mp})$ in GL_n, G_ν the Zariski-closure of $\rho(\Gamma_{G_\nu})$, and G_p the normal subgroup of G generated by G_ν° for all ν lying over p. Replacing K by a finite extension, we may assume that G_ν is connected for all such ν, so G_p is generated by conjugates of the G_ν. By Theorem 5.3, the complexification $G_{p\mathbb{C}}$, together with its natural n-dimensional representation, is of k-eigenvalue type. If G_p is of finite index in G, the theorem follows. Otherwise, there exists a homomorphism $\Gamma_{K,Mp} \to G(\mathbb{Q}_p)/G_p(\mathbb{Q}_p)$ with Zariski-dense, and therefore infinite p-adic analytic image. By construction, this homomorphism is unramified at all primes over ν. Such a homomorphism cannot exist according to the Fontaine-Mazur conjecture. ■

Corollary 5.5. If the Fontaine-Mazur conjecture is true, then for every complex nonsingular variety X over a number field K, the Zariski-closure of the image of $\mathrm{Gal}(\bar{K}/K)$ in $\mathrm{Aut}(H^2(\bar{X}, \mathbb{Q}_p))$ has no factor of type E_8. □

6 Application to braid group representations

Artin's braid group $\mathcal{B}_m$ is generated by $\sigma_1, \ldots, \sigma_{m-1}$ subject to relations

$$\sigma_i\sigma_j = \sigma_j\sigma_i \quad \text{if } |i-j| \geq 2, \qquad \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \quad \text{for } 1 \leq i \leq m-1. \tag{6.1}$$

In [13], the closed images of the unitary $q = e^{2\pi i/\ell}$ Hecke algebra representations of the braid groups are completely analyzed (completing a program initiated by Jones) for $\ell \geq 5$ and $\ell \neq 6$. In this section, we will carry out a similar analysis. We also discuss the situations in which the braid group representations arising from quantum groups at roots of unity satisfy the 3-eigenvalue condition.

6.1 Set-up

Given an irreducible unitary representation (ρ, V) of $\mathcal{B}_m$, there are three distinct possibilities for $G = \overline{\rho(\mathcal{B}_m)}$:

(1) $G/Z(G)$ is finite,
(2) $SU(V) \subset G$,
(3) $G/Z(G)$ is infinite, but $SU(V) \not\subset G$.

While the first (finite group) and third (nondense) possibilities are interesting, we will focus on the second. There are a number of reasons for doing this. Firstly, we will see that $SU(V) \subset G$ is the generic situation, while the other (nondense) cases require a case-by-case analysis that we will carry out in a separate work. Also, density is crucial for applications to quantum computing—our original motivation. Lastly, the application of **Theorem 3.3** leads most directly to the conclusion $SU(V) \subset G$, that is, by showing that (G, V) is an indecomposable pair satisfying the 3-eigenvalue property for which the three eigenvalues do not form a geometric progression. Nearly all of the finite group/nondense examples come from pairs having eigenvalues in geometric progression which will be considered in a forthcoming paper by the first two authors. We proceed with the following program.

(1) Determine which representations have exactly three eigenvalues.
(2) Determine conditions for the representations from (1) to be unitary.
(3) Determine when the three eigenvalues from (1) and (2) satisfy the no-cycle condition. This will give us all pairs (G, V).
(4) Determine when the three eigenvalues from (1) and (2) are not in geometric progression. Although this does not ensure density, it does guarantee the pair (G, V) is indecomposable, as three eigenvalues coming from a decomposable pair must be in geometric progression by **Lemma 2.2**.
(5) Determine when G is infinite modulo the center for the cases not excluded by (1)–(4).

6.2 BMW-algebra representations of the braid groups

We apply the strategy outlined above to BMW-algebras, first recalling what is well known and then proceeding to the subsequent steps.

6.2.1 Definitions and combinatorial results. Most of the material here can be found in [38], and we summarize the details germane to the problem, carrying out steps (1) and (2) in the above program.

The Birman-Wenzl-Murakami (BMW) algebras are a sequence of finite-dimensional algebras equipped with Markov traces. They can be described as quotients of the group algebra $\mathbb{C}(r,q)\mathcal{B}_m$ of Artin's braid group, where r and q are complex parameters. The precise definition of the BMW-algebra $\mathcal{C}_m(r,q)$ is as follows.

Definition 6.1. Let $g_1, g_2, \ldots, g_{m-1}$ be invertible generators satisfying the braid relations (B1) and (B2) above and

(R1) $(g_i - r^{-1})(g_i - q)(g_i + q^{-1}) = 0$,

(R2) $e_i g_{i-1}^{\pm 1} e_i = r^{\pm 1} e_i$, where

(E) $(q - q^{-1})(1 - e_i) = g_i - g_i^{-1}$ defines e_i.

The relations (R2) can be best understood by pictures where g_i is the braid generator σ_i and e_i is the ith generator of the Temperley-Lieb algebra. Relation (R1) shows that the image of g_i in any representation of $\mathcal{C}_m(r,q)$ has 3 eigenvalues: r^{-1}, q, and $-q^{-1}$. When $r \neq \pm q^n$ and q is not a root of unity, each BMW-algebra $\mathcal{C}_m(r,q)$ is finite-dimensional and semisimple with simple components labelled by Young diagrams with $m - 2j \geq 0$ boxes for $j \in \mathbb{N}$. In other words, the BMW-algebra is a direct sum of full matrix algebras. For each simple component $\mathcal{C}_{m,\lambda}$, let $V_{m,\lambda}$ be the unique nontrivial simple $\mathcal{C}_{m,\lambda}$-module. Then the branching rule for restricting $V_{m,\lambda}$ to $\mathcal{C}_{m-1}(r,q)$ is

$$V_{m,\lambda} \cong \bigoplus_{\mu \leftrightarrow \lambda} V_{m-1,\mu}, \tag{6.2}$$

where $V_{m-1,\mu}$ is a simple $\mathcal{C}_{m-1}(r,q)$-module and μ is a Young diagram with $m-1-2j \geq 0$ boxes obtained from λ by adding/removing a box to/from λ. This description of BMW-algebra branching rules can be neatly encoded in a graph called the *Bratteli diagram*. The graph consists of vertices labelled by (m,λ) with $|\lambda| = m - 2k$ arranged in rows (labelled by integers m). Vertices in adjacent rows are connected if the their labels differ by 1 in the first entry and by one box in the second. The dimension of $V_{m,\lambda}$ can thus be computed by adding up the dimensions of the $V_{m-1,\mu}$ whose labels are connected to (m,λ) by an edge. We obtain representations of $\mathcal{B}_m$ on $\bigoplus_\lambda V_{m,\lambda}$ via the map $\sigma_i \to g_i \in \mathcal{C}_m(r,q)$.

We are interested in obtaining unitary representations of $\mathcal{B}_m$ from BMW-algebras, so we must consider semisimple quotients with r and q specialized at roots of unity. Specifically, we let $r = q^n$ for $-1 \neq n \in \mathbb{Z}$ and $q = e^{\pi i/\ell}$ ($\ell \neq 1$), that is, a primitive 2ℓth root of unity. If a given irreducible representation is unitary for $q = e^{\pi i/\ell}$, it will remain so for $q = e^{-\pi i/\ell}$. For other choices of primitive roots of unity, we cannot expect to have unitarity. The quotient of each specialized BMW-algebra by the annihilator of the trace $A_m := \{a \in \mathcal{C}_m(r, q) : \mathrm{tr}(ab) = 0 \text{ for all } b\}$ is semisimple and we denote it $\mathcal{C}_m(q^n, q)$ (where q is understood to be $e^{\pi i/\ell}$). The branching rules and simple decomposition described above for the generic case still essentially apply to $\mathcal{C}_m(q^n, q)$, except that some components no longer appear, and fewer Young diagrams are needed to describe the persisting components (for all m). Precisely which components survive depends on the values ℓ and n, and the derivation can be found in [38], the results of which we will describe below. For now, it is enough to note that each simple component (sector) that does survive the quotient gives us an irreducible representation of $\mathcal{B}_m$. Let $\rho^{(n,\ell)}_{(m,\lambda)}$ acting on $V^{(n,\ell)}_{(m,\lambda)}$ be the representation of $\mathcal{B}_m$ corresponding to the simple component of $\mathcal{C}_m(q^n, q)$ labelled by λ. Since the conjugacy class of $\rho^{(n,\ell)}_{(m,\lambda)}(\sigma_1)$ generates the closed image of $\mathcal{B}_m$ topologically, there is a chance that the pair

$$\left(\overline{\rho^{(n,\ell)}_{(m,\lambda)}(\mathcal{B}_m)}, V^{(n,\ell)}_{(m,\lambda)}\right) \tag{6.3}$$

satisfies the 3-eigenvalue property.

As a first step we need to know the conditions under which the image of $\sigma_1 \in \mathcal{B}_m$ under $\rho^{(n,\ell)}_{(m,\lambda)}$ has 3 distinct eigenvalues. The answer is well known to experts (see [38]): *for* $m \geq 3$, *the image of* σ_1 *under the irreducible representation* $\rho^{(n,\ell)}_{(m,\lambda)}$ *has 3 distinct eigenvalues precisely when* $|\lambda| < m$ *and* $\mathcal{C}_{3,\square}$ *is three dimensional.* This is equivalent to the requirement that the corresponding simple component $\mathcal{C}_{m,\lambda}$ contains the simple component $\mathcal{C}_{3,\square}$. This is most easily seen by considering the Bratteli diagram as described above. It is shown in [38] that $\mathcal{C}_m(q^n, q)/\mathcal{A}_m \cong \mathcal{I}_m \oplus \overline{\mathcal{H}}_m(q^2)$ where $\overline{\mathcal{H}}_m(q^2)$ is a quotient of the Iwahori-Hecke algebra of type A_{m-1}, and $\mathcal{I}_m$ is the ideal generated by e_{m-1} (see [38]). The Young diagrams labelling simple components of $\overline{\mathcal{H}}_m(q^2)$ have m boxes, whereas those of $\mathcal{I}_m$ have $m - 2j$ boxes for some $j \geq 1$. The representations of $\mathcal{B}_m$ corresponding to the Iwahori-Hecke algebra part of $\mathcal{C}_m(q^n, q)$ have been studied in [13] where they are analyzed using the solution to the 2-eigenvalue problem. Thus the image of σ_1 on the irreducible representation $V_{m,\lambda}$ ($m \geq 3$) has (exactly) 3 distinct eigenvalues precisely when $|\lambda| < m$ and $\mathcal{C}_{3,\square}$ is three dimensional in which case the eigenvalues are $\{q^{-n}, q, -q^{-1}\}$. We can eliminate many redundant cases using isomorphisms (see [31]):

$$\mathcal{C}_m(q^n, q) \cong \mathcal{C}_m(-q^{-n}, q) \cong \mathcal{C}_m(-q^n, -q) \cong \mathcal{C}_m(q^{-n}, q^{-1}). \tag{6.4}$$

We describe the restrictions more precisely in the following, which is a reformulation of several results in [25, 38]. Denote by λ_i (resp., λ'_i) the number of boxes in the ith row (resp., column) of the Young diagram λ.

Proposition 6.2. Let $q = e^{\pi i/\ell}$ and $m \geq 3$.

(1) The matrix algebra $\mathcal{C}_{3,\square}$ is a simple three dimensional subalgebra of $\mathcal{C}_m(q^n, q)$, provided one of the following conditions holds:
  - (a) $n = 1$ and $\ell \geq 3$,
  - (b) $n = 2$ and $\ell \geq 4$,
  - (c) $3 \leq n \leq \ell - 3$ (so $\ell \geq 6$),
  - (d) $4 - \ell \leq n \leq -4$, n is even, and ℓ is odd (so $\ell \geq 9$),
  - (e) $5 - \ell \leq n \leq -5$, n is odd, and ℓ is even (so $\ell \geq 10$).

  Moreover, this list is exhaustive up to the isomorphisms (6.4).

(2) The λ for which $\mathcal{C}_{m,\lambda}$ may appear as a simple component in some $\mathcal{C}_m(q^n, q)$ are in the following sets of (n, ℓ)-admissible Young diagrams corresponding to each of the 5 cases above:
  - (a) $\{[1^2]\} \cup \{[k] : k \in \mathbb{N}\}$,
  - (b) $\{[1^3]\} \cup \{[k], [k, 1] : 1 \leq k \leq \ell - 1\}$,
  - (c) $\{\lambda : \lambda_1 + \lambda_2 \leq \ell - n + 1 \text{ and } \lambda'_1 + \lambda'_2 \leq n + 1\} \cup \{[\ell - n + 1, 1^{n-1}]\}$,
  - (d) $\{\lambda : \lambda_1 + \lambda_2 \leq 1 - n \text{ and } \lambda'_1 \leq (\ell + n - 1)/2\}$,
  - (e) $\{\lambda : \lambda_1 \leq (-1 - n)/2 \text{ and } \lambda'_1 \leq (\ell + n - 1)/2\}$.

(3) Thus the image of σ_1 under the irreducible representation $\rho^{(n,\ell)}_{(m,\lambda)}$ with $|\lambda| < m$ has 3 distinct eigenvalues provided n and ℓ satisfy one of the conditions of (1) and λ is in the corresponding set of admissible Young diagrams in (2). These representations are unitary except possibly in case (d). □

Remark 6.3. Observe that the set in (2)(a) is infinite and independent of ℓ. The other four labelling sets are finite, and it is easy to see that the corresponding Bratteli diagrams are periodic. In the case $n = 2$, there is a slight exception to the rule for constructing the Bratteli diagram: the diagrams labelled by $[\ell - 1, 1]$ and $[\ell - 1]$ are *not* connected by an edge (see [38, Proposition 6.1]). The fact that the representations in (a),(b),(c), and (e) are unitary was proved in [38]. The full (reducible) representations of $\mathcal{B}_m$ factoring over $\mathcal{C}_m(q^n, q)$ corresponding to case (d) were shown in [25] to be nonunitarizable *for any* q when $\ell > 2(-n + 1)$. This leaves only finitely many possible ℓ for each fixed n, and even in these cases one can use the techniques of [25] to show that for $q = e^{\pi i/\ell}$, one does not get unitarity except in degenerate cases. Restricting to the irreducible sectors, one may get unitarizable representations, but not uniformly, so that for $m \gg 0$, no irreducible sector is unitary.

6.2.2 Cycles and geometric progressions. The eigenvalues of any of the irreducible representations satisfying the conditions of Proposition 6.2 are $\{q, -q^{-1}, q^{-n}\}$, with $q = e^{\pi i/\ell}$. Steps (3) and (4) of the program can be accomplished with simple computations. We have the following lemma.

Lemma 6.4. Let n, ℓ and λ be as in Proposition 6.2. Then the eigenvalues of $\rho_{(m,\lambda)}(\sigma_1)$

(1) fail the no-cycle property if and only if $n = 1$ or $(n, \ell) = (3, 6)$, and

(2) are in geometric progression if and only if $n \in \{3, \ell - 3, \pm\ell/2\}$. □

Proof. The only way $\{q, -q^{-1}, q^{-n}\}$ can fail the no-cycle condition is if it contains a coset of $\{\pm 1\}$ or $\{1, e^{2\pi i/3}, e^{4\pi i/3}\}$. With the restrictions in Proposition 6.2 that $\ell \geq n+2$ for $n > 0$ and $\ell \geq 4 - n$ for $n < 0$ as well as $\ell \geq 3$, one checks that only $n = 1$ and $(n, \ell) = (3, 6)$ fail no-cycle. For the eigenvalues to be in geometric progression (still satisfying the conditions of Proposition 6.2), we check the solutions of $\lambda_1\lambda_2 - (\lambda_3)^2 = 0$ for the three possible assignments of λ_3. These yield the three solutions for n above. ■

Remark 6.5. All of the exceptional cases $n \in \{1, 3, \ell - 3, \pm\ell/2\}$ will be considered in a future work. As we remarked above, the case $n = 1$ is unique in that the labelling set of irreducible sectors is infinite. In fact, it is not hard to see, using the classification of m-dimensional irreducible representations of $\mathcal{B}_m$ found in [11], that one obtains some finite group images for every m when $n = 1$. By the isomorphisms of BMW-algebras corresponding to $r \leftrightarrow -r^{-1}$, we see that the two cases $n = 3$ and $n = \ell - 3$ are actually the same. Moreover, it can be shown that the (specialized quotient) BMW-algebras $\mathcal{C}_m(q^3, q)$ can be embedded (diagonally) in quotients of the tensor squares of Iwahori-Hecke algebras $\mathcal{H}_m(q^2)$. This indicates that the corresponding pairs may be tensor decomposable. In the subcase $(n, \ell) = (3, 6)$, the work of Jones in [15] shows that the images are all finite groups (essentially $PSL(2m, 3)$). The case $n = -\ell/2$ sometimes also has finite group images, for example, when $(n, \ell) = (-5, 10)$, see [16].

6.2.3 Infinite images and density. Finally, we need to determine, for representations not excluded by the steps (1)–(4) above, the values of m, ℓ, n, and λ for which the image of $\mathcal{B}_m$ under the unitary irreducible representation $\rho^{(n,\ell)}_{(m,\lambda)}$ in $\mathcal{C}_m(q^n, q)$ with $q = e^{\pi i/\ell}$ is infinite modulo the center. Proposition 6.2 implies that a sufficient condition for $\rho^{(n,\ell)}_{(m,\lambda)}$ to have infinite image is that the three-dimensional representation $\rho^{(n,\ell)}_{(3,\square)}$ has an infinite image. So as a first step, we study this condition. For convenience of notation we denote this representation simply by ρ despite its dependence on the parameters. A nonunitary realization of ρ is given by

$$\sigma_1 \longrightarrow A := \begin{pmatrix} \frac{1}{q^n} & \frac{q^2-1}{q} & 0 \\ 0 & \frac{q^2-1}{q} & i \\ 0 & -i & 0 \end{pmatrix}, \qquad \sigma_2 \longrightarrow B := \begin{pmatrix} 0 & 0 & -i \\ 0 & \frac{1}{q^n} & \frac{-i(q^2-1)}{q^{n+1}} \\ i & 0 & \frac{q^2-1}{q} \end{pmatrix} \tag{6.5}$$

found in [2].

Blichfeldt [3] has determined the irreducible finite subgroups of $PSL(3,\mathbb{C})$. Six are primitive groups of orders 36, 60, 72, 168, 216, and 360, and the imprimitive subgroups come in two infinite families isomorphic to extensions of S_3 and $\mathbb{Z}_3$ by abelian groups.

Definition 6.6. A group Γ is *primitive* if Γ has a faithful irreducible representation which cannot be expressed as a direct sum of subspaces which Γ permutes nontrivially.

By **Lemma 2.1**, a sufficient condition for $G = \overline{\rho(\mathcal{B}_3)}$ to be primitive is that the spectrum of $\rho(\sigma_1)$ satisfies the no-cycle property. So by **Lemma 6.4**, the image of ρ is only imprimitive in the excluded cases $n = 1$ and $(n, \ell) = (3, 6)$. So we may assume that the G is primitive. We wish to determine when G is infinite modulo the center. By rescaling the images of the generators σ_i by the cube root of the determinant of $\rho(\sigma_i)$, we may assume that $G \subset SL(3,\mathbb{C})$, and to determine the image modulo the center it suffices to consider the projective image. Thus $G/Z(G) \subset PSL(3,\mathbb{C})$, and we may apply Blichfeldt's classification. We state his result and include some useful information about orders of elements in the following proposition.

Proposition 6.7. The primitive subgroups of $PSL(3,\mathbb{C})$ are as follows.

(1) The *Hessian* group H of order 216 or a normal subgroup of H of order 36 or 72. The Hessian group is the subgroup of A_9 generated by $(124)(568)(397)$ and $(456)(798)$, and has elements of order $\{1, 2, 3, 4, 6\}$.

(2) The simple group $PSL(2, 7) \subset A_7$ of order 168. The orders of elements are $\{1, 2, 3, 4, 7\}$.

(3) The simple group A_5 having elements of orders $\{1, 2, 3, 5\}$.

(4) The simple group A_6 having elements of orders $\{1, 2, 3, 4, 5\}$. □

Using this result we have the following theorem.

Theorem 6.8. Let n and ℓ be chosen so that $\rho^{(n,\ell)}_{(3,\square)}$ is a three-dimensional unitary irreducible representation of $\mathcal{B}_3$ with eigenvalues not in geometric progression and satisfying the no-cycle condition. That is, n and ℓ satisfy the hypotheses of **Proposition 6.2**(1)(b), (c), or (e) in addition to $n \notin \{3, \ell-3, \pm\ell/2\}$. Let $m \geq 3$ and $|\lambda| < m$ with λ (n,ℓ)-admissible. The closure of the group $\rho^{(n,\ell)}_{(m,\lambda)}(\mathcal{B}_m)$ is infinite modulo the center with two exceptions: if $(n,\ell) \in \{(-5,14),(-9,14)\}$ with $(m,\lambda) \in \{(3,\square),(4,[0])\}$, then the projective images are isomorphic to $PSL(2,7)$. Excluding these cases, if the dimension of the representation $\rho^{(n,\ell)}_{(m,\lambda)}$ is k, then the closure of the image of $\mathcal{B}_m$ contains $SU(k)$. $\square$

Proof. Knowing the specific eigenvalues of $\rho(\sigma_1)$, we compute its projective order $t(n,\ell)$ as a function of ℓ and n to be

$$t(n,\ell) = \begin{cases} \ell/2 & \text{if } \ell \equiv 2 (\mathrm{mod}\, 4),\ n \equiv 3 (\mathrm{mod}\, 4), \\ \ell & \text{if } \ell \equiv 0 (\mathrm{mod}\, 4),\ n \text{ even or} \\ & \ell \equiv 2 (\mathrm{mod}\, 4),\ n \equiv 1 (\mathrm{mod}\, 4), \\ 2\ell & \text{otherwise.} \end{cases} \tag{6.6}$$

Under the stated hypotheses on n and ℓ we consider cases, comparing with the list of possible orders of elements in Blichfeldt's classification.

(1) If ℓ is odd, then $\ell \geq 5$ in which case $t(n,\ell) \geq 10$ which is too large.

(2) If $\ell \equiv 0 (\mathrm{mod}\, 4)$, then $\ell = 8$ is the smallest value not yet excluded which gives $t(n,\ell) \geq 8$ which is again too large.

(3) If $\ell \equiv 2 (\mathrm{mod}\, 4)$, then $\ell \geq 6$ and $t(n,\ell) \geq 12$ unless n is odd. If $n \equiv 1 (\mathrm{mod}\, 4)$, then $\ell \geq 10$ which gives us $t(n,\ell) = \ell \geq 10$ which does not appear on the list. When $\ell \equiv 2 (\mathrm{mod}\, 4)$ and $n \equiv 3 (\mathrm{mod}\, 4)$ with $n > 0$, we must have $n \geq 7$ which forces $\ell \geq 18$ since $n \neq \ell/2$. For $\ell \equiv 2 (\mathrm{mod}\, 4)$ and $n \equiv 3 (\mathrm{mod}\, 4)$ with $n < 0$ we must have $\ell \geq 14$ (since $\ell = 10$ leads to $n = -5 = -\ell/2$), which has the two possible values $n = -5$ or $n = -9$ which we claim gives rise to finite images. Observe that $t(-5,14) = t(-9,14) = 7$.

To show that the projective images for $(-5,14)$ and $(-9,14)$ are both $PSL(2,7)$, we first observe that by the isomorphism of (6.4) with $r \leftrightarrow -r^{-1}$ and $q^{-5} \leftrightarrow q^{-14+5} = q^{-9}$ these two cases have the same images. Then we use the explicit matrices A and B above to define $S = B^{-1}$ and $T = BAB$ which then (projectively) satisfy the relations $S^7 = (S^4T)^4 = (ST)^3 = T^2 = I_{3\times 3}$ defining $PSL(2,7)$. It is immediate from the Bratteli diagram that the representation of $\mathcal{B}_4$ corresponding to $(4,[0])$ is irreducible and isomorphic to that of $(3,\square)$ when restricted to $\mathcal{B}_3$. Moreover, the representations of $\mathcal{B}_4$ corresponding to diagrams $[1^2]$ and $[2]$ each contain the representation of $\mathcal{B}_3$ corresponding to the Young diagram $[1^2,1]$ which was shown in [13] to have infinite image (modulo the center). For all of the infinite image cases, the hypotheses of Theorem 3.3 are satisfied and the eigenvalues are not in geometric progression, so density follows. $\blacksquare$

6.3 Quantum groups

In this subsection we consider braid group actions on centralizer algebras of representations of quantum groups at roots of unity. We find and analyze examples in which we may apply Theorem 3.3. We follow the general strategy in Section 6.1, but we note that as the representation spaces available to us are not necessarily simple subquotients of braid group algebras (unlike BMW-algebras), there is a subtlety regarding irreducibility.

6.3.1 Braid group action on centralizer algebras. The Drinfeld-Jimbo quantum group $U := U_q\mathfrak{g}$ associated to a simple Lie algebra $\mathfrak{g}$ is a ribbon Hopf-algebra. The so-called *universal* R*-matrix* that intertwines the coproduct with the opposite coproduct on U can be used to construct representations of the braid group $\mathcal{B}_n$ on the morphism space $\mathrm{End}_U(V^{\otimes n})$ for any finite-dimensional highest weight U-module V as follows. Fix such a U-module V and define $\check{R} = P_V \circ R|_{V\otimes V} \in \mathrm{End}_U(V^{\otimes 2})$ to be the U-isomorphism afforded us by composing the image of the universal R-matrix acting on $V \otimes V$ with the "flip" operator $P_V : v_1 \otimes v_2 \to v_2 \otimes v_1$. Then define isomorphisms for each $1 \leq i \leq n-1$:

$$\check{R}_i := \mathbf{1}^{\otimes(i-1)} \otimes \check{R} \otimes \mathbf{1}^{\otimes(n-i-1)} \in \mathrm{End}_U\left(V^{\otimes n}\right) \tag{6.7}$$

so that the $\check{R}_i$ satisfy the braid group relations. Then define a representation of $\mathcal{B}_n$ on $\mathrm{End}_U(V^{\otimes n})$ by $\sigma_i \cdot f = \check{R}_i \circ f$.

Lusztig has defined a modified form of the quantum group U so that one may specialize the quantum parameter q to $e^{\pm\pi i/\ell}$. In fact, one may choose any q so that q^2 is a primitive ℓth root of unity, but we will restrict our attention to $q = e^{\pi i/\ell}$ since these values (sometimes) yield unitary representations (see [39]), which remain unitary for $\overline{q}$. The full representation category of U at roots of unity is not semisimple, but has a semisimple subquotient category. This process is essentially due to Andersen and his coauthors (see [1] and references therein). This yields a semisimple ribbon category $\mathcal{F}$ (see [32] for the definitions) with finitely many simple objects labelled by highest weights in a truncation of the dominant Weyl chamber, called the *Weyl alcove*. The braid group still acts on $\mathrm{End}_U(V^{\otimes n})$ for any object V as above, and for each quantum group we look for simple objects V_λ so that the images of the braid generators on the irreducible subrepresentations of $\mathrm{End}_U(V_\lambda^{\otimes n})$ have 3 eigenvalues. Because the tensor product rules for objects labelled by weights near the upper wall of the Weyl alcove depends on ℓ, we do not explicitly determine all V_λ giving rise to pairs with the 3-eigenvalue property, and restrict our attention to weights near 0. As in the BMW-algebra setting, we will always have an irreducible three-dimensional representation of $\mathcal{B}_3$ to which we may reduce most questions.

We sketch the idea (see, e.g., [30, Section 3]): if V is a simple object in (a finite semisimple ribbon category) $\mathcal{F}$ such that $V \otimes V \cong V_0 \oplus V_1 \oplus V_2$ is the decomposition into 3 inequivalent simple objects, then $\mathrm{End}_U(V^{\otimes 3})$ has a three dimensional irreducible subrepresentation isomorphic to $\mathrm{Hom}_U(V^{\otimes 3}, W)$ for a simple object W appearing in $V^{\otimes 3}$ with multiplicity three. Provided the (categorical) q-dimension of each of W, V, and V_i are nonzero, then this representation is irreducible and the image of σ_1 will have three distinct eigenvalues. As in the BMW-algebra situation, we can construct a Bratteli diagram encoding the containments of the semisimple finite-dimensional algebras:

$$\mathrm{End}_U(V) \subset \mathrm{End}_U(V \otimes V) \subset \cdots \subset \mathrm{End}_U\left(V^{\otimes n}\right) \ldots . \tag{6.8}$$

The simple components of $\mathrm{End}_U(V^{\otimes n})$ will be isomorphic to $\mathrm{Hom}_U(V^{\otimes n}, V_\mu)$ where V_μ is a simple object appearing in the decomposition of $V^{\otimes n}$. The edges of the Bratteli diagram are determined by decomposing $V_\gamma \otimes V$ where V_γ is a simple subobject of $V^{\otimes(n-1)}$. There are techniques known for obtaining these decompositions, for example, Littelman's path basis technique [21], or crystal bases. However, when we consider the action of the braid group $\mathcal{B}_n$ on the spaces $\mathrm{Hom}_U(V^{\otimes n}, V_\mu)$, we have no guarantee that the action is irreducible. This is because $\mathrm{End}_U(V^{\otimes n})$ might not be generated by the image of $\mathcal{B}_n$.

6.3.2 Density results. We proceed to find pairs (X_r, λ) so that the ribbon category corresponding to the quantum group $U_q\mathfrak{g}(X_r)$ of Lie type X_r has simple object V_λ with $V_\lambda^{\otimes 3} \cong V_0 \oplus V_1 \oplus V_2$ as above. We find that (A_r, ϖ_2), $(A_r, 2\varpi_1)$, (B_r, ϖ_1), (C_r, ϖ_1), (D_r, ϖ_1), and (E_6, ϖ_1) do satisfy these conditions (where the weights ϖ_i are labelled as in [5, 6]). With these in hand, we compute the eigenvalues of the images of σ_i in the corresponding representations. We use the following result found in [20, Corollary 2.22(3)], originally due to Reshetikhin. The form $\langle \cdot, \cdot \rangle$ is the symmetric inner product on the root lattice normalized so that the square lengths of *short* roots is 2, and the weight ρ is the half sum of the positive roots.

Proposition 6.9. Suppose that $V = V_\varpi$ is an irreducible representation of the quantum group $U_q\mathfrak{g}$ and that $V \otimes V_\lambda$ is multiplicity free for all V_λ appearing in some $V^{\otimes n}$. Then for any V_ν appearing in $V^{\otimes 2}$, it holds that

$$\check{R}_i|_{V_\nu} = \pm q^{(1/2)\langle \nu, \nu+2\rho\rangle - \langle \varpi, \varpi+2\rho\rangle} \mathbf{1}_{V_\nu}, \tag{6.9}$$

where the sign is $+1$ if V_ν appears in the symmetrization of $V^{\otimes 2}$ and -1 if V_ν appears in the antisymmetrization of $V^{\otimes 2}$. $\square$

Table 6.1 Eigenvalues of $\check{R}_i$.

(X_r, λ)	$S^2(V_\lambda)$	$\bigwedge^2(V_\lambda)$	Eigenvalues
(A_r, ϖ_2)	$V_{2\varpi_2}$	$V_{\varpi_1+\varpi_3} \oplus V_{\varpi_4}$	$q^{4/(r+1)+1}\{q, -q^{-1}, -q^{-5}\}$
$(A_r, 2\varpi_1)$	$V_{2\varpi_2} \oplus V_{4\varpi_1}$	$V_{2\varpi_1+\varpi_2}$	$-q^{4/(r+1)-1}\{-q^{-1}, -q^5, q\}$
(B_r, ϖ_1)	$V_{2\varpi_1} \oplus \mathbb{1}$	V_{ϖ_2}	$\{q^2, q^{-4r}, -q^{-2}\}$
(C_r, ϖ_1)	$V_{2\varpi_1}$	$V_{\varpi_2} \oplus \mathbb{1}$	$\{q, -q^{-1}, -q^{-2r-1}\}$
(D_r, ϖ_1)	$V_{2\varpi_1} \oplus \mathbb{1}$	V_{ϖ_2}	$\{q, q^{2r-1}, -q^{-1}\}$
(E_6, ϖ_1)	$V_{2\varpi_1} \oplus V_{\varpi_6}$	V_{ϖ_3}	$q^{1/3}\{q, q^{-9}, -q^{-1}\}$

We record the results in Table 6.1, where the notation follows [5, 6]. The symbol $\mathbb{1}$ denotes the unit object in the category. The necessary computations are standard and can be done by hand, for example, using the technique of [21]. The braid group representations corresponding to Lie types B, C, and D are the same as those factoring over specializations of BMW-algebras, due to q-Schur-Weyl-Brauer duality, see [38]. For this reason we ignore these cases in the following weaker version of Theorem 6.8.

Theorem 6.10. Let (X_r, λ) be as in Table 6.1 with $X = A_r$ or E_6. Then the following hold.

(1) For (A_r, ϖ_2): provided $r \geq 3$ and $\ell \geq \max(r+3, 7)$, $\mathrm{Hom}_U((V_\lambda)^{\otimes 3}, V_{\varpi_2+\varpi_4})$ is unitary, irreducible, and three dimensional and the image of σ_1 has 3 distinct eigenvalues. If V_μ appears in $V_{\varpi_2+\varpi_4} \otimes V_\lambda^{\otimes n-3}$, then $\mathrm{Hom}_U(V^{\otimes n}, V_\mu)$ *contains* an irreducible unitary representation of $\mathcal{B}_n$ with the 3-eigenvalue property. When $\ell \notin \{10, 14\}$, the eigenvalues of the image of σ_1 are not in geometric progression and the images of $\mathcal{B}_n$ are infinite modulo the center and so are dense in these cases.

(2) For $(A_r, 2\varpi_1)$: $\mathrm{Hom}_U((V_\lambda)^{\otimes 3}, V_{2\varpi_1+2\varpi_2})$ is unitary, irreducible, and three dimensional provided $r \geq 1$ and $\ell \geq r+5$. If V_μ appears in $V_{2\varpi_1+2\varpi_2} \otimes V_\lambda^{\otimes n-3}$, then $\mathrm{Hom}_U(V^{\otimes n}, V_\mu)$ *contains* an irreducible unitary representation of $\mathcal{B}_n$ with the 3-eigenvalue property. When $\ell \notin \{6, 10\}$, the eigenvalues of the image of σ_1 are not in geometric progression and the images of $\mathcal{B}_n$ are infinite modulo the center and so are dense in these cases.

(3) For (E_6, ϖ_1): $\mathrm{Hom}_U((V_\lambda)^{\otimes 3}, V_{\varpi_1+\varpi_6})$ is unitary, irreducible, and three dimensional provided $\ell \geq 14$. If V_μ appears in $V_{\varpi_1+\varpi_6} \otimes V_\lambda^{\otimes n-3}$, then $\mathrm{Hom}_U(V^{\otimes n}, V_\mu)$ *contains* an irreducible unitary representation of $\mathcal{B}_n$ with the 3-eigenvalue property. Provided $\ell \neq 18$, the eigenvalues of the image of σ_1 are not in geometric progression and the images of $\mathcal{B}_n$ are infinite modulo the center and so are dense in these cases. $\square$

Proof. For the object labelled by V_ν to be in the fundamental alcove, we must have $\langle \nu + \rho, \theta \rangle < \ell$ where θ is the highest root. This condition together with the requirement that the eigenvalues be distinct yield the first restrictions in each case. The unitarity of the representations is shown in [39]. In each case the representation spaces $\mathrm{Hom}_U(V^{\otimes n}, V_\mu)$ described in the theorem contain the three-dimensional representation spaces, so by restriction to $\mathcal{B}_3$ we see that the $\mathcal{B}_n$ representations must contain an irreducible unitary subrepresentation with the 3-eigenvalue property. Geometric progressions appear in each of the three cases if and only if $\ell = 10$ in the first case, $\ell = 6$ or 10 in the second case, and $\ell = 18$ in the last case. Computing projective orders of the images of σ_1 and comparing as in the proof of Theorem 6.8, we find that the only finite group image that arises is in the first case with $\ell = 14$. With these exceptions, the hypotheses of Theorem 3.3 are satisfied and we may conclude that the images are dense. ■

Remark 6.11. To get sharper results we would need to describe the decompositions of the $\mathcal{B}_n$ representations $\mathrm{Hom}_U(V^{\otimes n}, V_\mu)$ that appear in the above theorem. This is in general quite complicated. In fact, the type E_6 case appears in an exceptional series discussed in [40] (and extended slightly in [24]). These give new semisimple finite-dimensional quotients of the braid group algebras analogous to BMW-algebras about which little is known.

6.4 Concluding remarks

In comparing this work to the 2-eigenvalue paper, it may be noted that we do not provide applications of our results to the distribution of values of the Kauffman polynomial in analogy with those given for the Jones polynomial in [13, Section 5]. That is, we do not consider the set of values $F_L(a, z)$ for fixed a and z and varying L. These values can be described as linear combinations of traces of any braid with closure L in the different irreducible factors of a BMW-algebra, just as in [13]. The difficulty is that our information on the closures of braid groups in BMW-algebras is less detailed than the corresponding information for Hecke algebras. In particular, we have not completely determined these closures for the irreducible factors of the BMW-representations which are excluded in the statement of Theorem 6.8. Neither have we determined the equivalences and dualities existing between different irreducible factors in a fixed BMW-algebra. We certainly expect the limiting distributions to be Gaussian as for the Jones polynomial, but we do not yet have enough information to ensure that this is so.

In 1990s, Vertigan (see [36, Theorems 6.3.5 and 6.3.6]) analyzed the classical computational complexity of exactly evaluating various knot polynomials at fixed complex values. With a few exceptions, all evaluations are #P-hard. At these few exceptional values, the link invariants have classical topological interpretations and can be computed in polynomial time. These results fit very well with the analysis of closed images of the braid group representations. In the case of unitary Jones representations of the braid groups at q, the closed image is dense in the corresponding special unitary groups exactly when computing the link invariants is #P-hard at q, while the finite image cases correspond to polynomial time computations. Part of the appeal of working out the exceptions to Theorem 6.8 is the hope of relating these cases to interesting special values of the Kauffman polynomial.

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