Quantum Walks with Non-Abelian Anyons

Lauri Lehman,[1,2] Vaclav Zatloukal,[2] Gavin K. Brennen,[1,2] Jiannis K. Pachos,[2] and Zhenghan Wang[3]

[1]*Centre for Engineered Quantum Systems, Macquarie University, Sydney, NSW 2109, Australia*
[2]*School of Physics and Astronomy, University of Leeds, Leeds LS2 9JT, United Kingdom*
[3]*Microsoft Research, Station Q, University of California, Santa Barbara, California 93106, USA*

(Received 29 October 2010; revised manuscript received 21 April 2011; published 10 June 2011)

We study the single particle dynamics of a mobile non-Abelian anyon hopping around many pinned anyons on a surface, by modeling it with a discrete time quantum walk. During the evolution, the spatial degree of freedom of the mobile anyon becomes entangled with the fusion degrees of freedom of the collective system. Each quantum trajectory makes a closed braid on the world lines of the particles establishing a direct connection between statistical dynamics and quantum link invariants. We find that asymptotically a mobile Ising model anyon becomes so entangled with its environment that its statistical dynamics reduces to a classical random walk with linear dispersion in contrast to particles with Abelian statistics which have quadratic dispersion.

DOI: 10.1103/PhysRevLett.106.230404 PACS numbers: 05.30.Pr, 03.65.Vf, 05.40.Fb

Anyons are pointlike particles with more general statistics than bosons or fermions. They were shown to exist in systems where the physics is constrained to two dimensions [1]. Beyond mere possible existence, they were found to be a good description for low-lying quasiparticle excitations of fractional quantum Hall systems [2,3], and they exactly describe excitations in various strongly correlated two-dimensional spin lattice models [4,5]. Recently, there has been tremendous experimental progress in the preparation and control of systems capable of exhibiting topological order [6–8] with the goal to observe anyonic statistics. This is further motivated by the discovery that braiding some types of non-Abelian anyons can be used for naturally fault-tolerant quantum computing [9]. The quantum physics of anyonic systems is very rich but is only beginning to be explored in its own right. For example, there have been investigations of the equilibrium properties of dynamically interacting non-Abelian anyons in chains [10] and two-dimensional lattices [11,12]. These studies are restricted to static anyons. The behavior of dynamically propagating anyons influenced by their nontrivial statistics remains in general an open problem.

Here we describe a discrete time quantum walk, which captures some of the nonequilibrium physics of moving non-Abelian anyons interacting purely due to particle statistics. This coarse graining in time and space retains the main properties of anyons since there are no dynamic or geometric phases involved. Quantum walks are inspired by the Feynman path integral approach to coherently explore all possible paths in a quantum mechanical evolution when computing observables. In Ref. [13], the authors introduced the general formalism for quantum walks with anyons, and it was shown that, while for Abelian anyons the dispersion is quadratic as in the usual quantum walk, the non-Abelian walk appears to have richer behavior induced by decoherence. Quantum walks differ from classical walks because in the latter the value of a random variable determines the direction of motion of the walk, while in the former the evolution is fully coherent with a spin-$1/2$ degree of freedom (DOF), dubbed a coin, providing a controlled hoping of the walker either left or right allowing coherences to build up between spatial amplitudes. Generically, a transition from coherent quantum to classical random behavior with linear dispersion occurs when a quantum walk strongly decoheres due to interaction with an environment [14,15]. We show that, even without dynamical interactions, statistical entanglements between non-Abelian anyons are sufficient to induce such a transition. Specifically, we solve for the asymptotic distribution of the Ising model non-Abelian anyons. The latter appear as quasiparticle excitations in the Pfaffian wave function description of the $\nu = 5/2$ filled fractional quantum Hall state which is believed to be the most likely physical system to first observe non-Abelian statistics [3,16]. It is the purpose of this work to determine the behavior of such a system by analytical methods, thus opening the way for modeling complex systems that are of interest to statistical physics [17].

The setup [see Fig. 1(a)] is a chain of n Ising model anyons canonically ordered on the surface with $n-1$ pinned anyons and one mobile *walker* anyon. The walker is a spinor particle, and it hops between neighboring sites with spatial index $s = 1, \dots n-1$: the components with $|0\rangle$ or $|1\rangle$ coin state crossing over or under the pinned anyon in between. The distance between sites is set to unity, though for purely topologically interactions the distance scale is irrelevant. The total Hilbert space decomposes as $\mathcal{H} = \mathcal{H}_{\text{space}} \otimes \mathcal{H}_{\text{coin}} \otimes \mathcal{H}_{\text{fusion}} \simeq \mathbb{C}^{n-1} \otimes \mathbb{C}^2 \otimes \mathbb{C}^D$, which becomes infinite dimensional in the asymptotic limit. Fusion DOFs denumerate the number of distinct measurement outcomes of topological charge when pairs of anyons are fused together [9]. Its size grows like

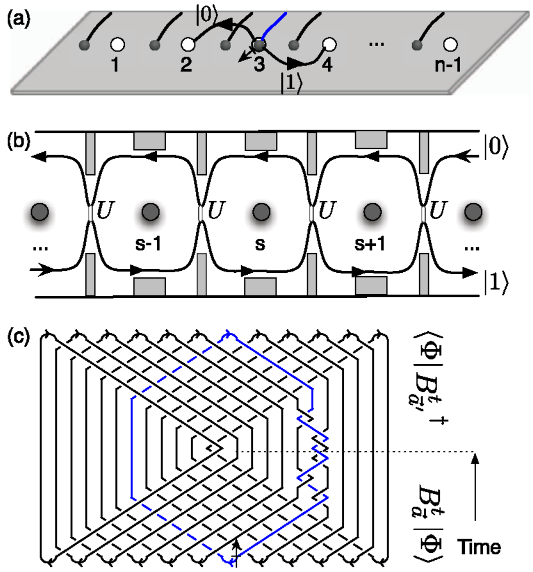

FIG. 1 (color online). Anyonic quantum walk. (a) Setup schematic: n vacuum pairs of anyons (gray dots) are prepared in state $|\Phi\rangle$. Half of each pair is aligned on a chain, and one walker anyon with a spin is free to braid around the others fixed in place. (b) Potential physical realization using a chained version of the two point contact interferometer in fractional quantum Hall systems [18]. The tunneling matrix U and dynamical and Aharanov-Bohm phases can be tuned by adjusting the gate voltages indicated by the thin and wide rectangles. (c) Link representation of the world lines for a Markov closed quantum trajectory [$\vec{a} = (1, 0, 0, 1, 1)^T$, $\vec{a}' = (0, 1, 1, 0, 1)^T$] that contributes to the spatial distribution $p(3, 5)$. The link shown is proper and has one Borromean ring; i.e., if any of three linked components were cut, the others would become disentangled. Here $arf(L) = 1$.

$D \sim d^n$, where d is the quantum dimension of the anyons. For n Ising anyons with total trivial charge, we have $d = \sqrt{2}$ and $D = 2^{\lceil n/2 \rceil - 1}$.

The dynamics is modeled by a composition of two discrete unitary steps $W = TU$, where U acts on the coin and T is a conditional braiding operator. It moves the walker to the right or left depending on the coin state:

$$T = \sum_{s=1}^{n-2} |s-1\rangle_{\mathrm{space}}\langle s| \otimes |0\rangle_{\mathrm{coin}}\langle 0| \otimes b_{s-1} + |s+1\rangle_{\mathrm{space}} \times \langle s| \otimes |1\rangle_{\mathrm{coin}}\langle 1| \otimes b_s,$$

where $\{b_s\}_{s=1}^{n-1}$ is a set of unitary generators of the Ising braid group. Notice that the chirality of the mobile anyonic charge current is fixed counterclockwise by this walk. To make T unitary, we assume periodic boundary conditions ($|0\rangle_{\mathrm{space}} \equiv |n-1\rangle_{\mathrm{space}}$) but will be concerned with walks satisfying $\lfloor n/2 \rfloor < t$ so that winding around the surface is not an issue. We note that in fact the physics studied here could be realized in a continuous time evolution by using the setup shown in Fig. 1(b) depicting a chained version of the double point contact interferometer circuit proposed in Ref. [18], and recently tested experimentally [7], for measuring anyonic statistics in $\nu = 5/2$ fractional quantum Hall samples. In such a realization, the pinned anyons would be quasiholes confined at antidots arranged on a chain in the sample, and chiral edge currents run above and below the chain. The walker anyon (analogous to the probe quasihole of the interferometer) would have its coin DOF carried by its propagation mode: the top (bottom) edge current designating mode $|0\rangle$ ($|1\rangle$). At the regions where the edge currents pinch together, there is a tunneling described by the matrix $U = \begin{pmatrix} u^* & v \\ -v^* & u \end{pmatrix}$. This realizes the coin step where u and v are the tunable reflection and transmission tunneling matrix elements ($|v|$ should be small enough to ensure that tunneling is due to quasiholes only and not composite particles) [18]. The system could be initialized by dragging a probe anyon from the bulk to the $|0\rangle$ edge of the initial site and applying a bias voltage along the edges to generate the continuous time evolution.

The system's initial state is $|\Psi(0)\rangle = |s_0 = \lceil \frac{n}{2} \rceil\rangle_{\mathrm{space}}|0\rangle_{\mathrm{coin}}|\Phi\rangle_{\mathrm{fusion}}$, where $|\Phi\rangle$ is the vacuum configuration of the n pairs of anyons with half the members braided to the right. After t iterations, the state is $|\Psi(t)\rangle = W^t|\Psi(0)\rangle$, and the reduced state of the spatial DOF of the walker is

$$\rho_{\mathrm{space}}(t) = \mathrm{tr}_{\mathrm{coin}}\mathrm{tr}_{\mathrm{fusion}}|\Psi(t)\rangle\langle\Psi(t)| = \sum_{\vec{a},\vec{a}'} \mathrm{tr}\,\mathcal{U}^t_{\vec{a}\vec{a}'}\mathrm{tr}\,\mathcal{Y}^t_{\vec{a}\vec{a}'}|2|\vec{a}| - t + s_0\rangle\langle 2|\vec{a}'| - t + s_0|, \quad (1)$$

where $\mathcal{U}^t_{\vec{a}\vec{a}'} = (\prod_{r=1}^t P_{a_r}U)|0\rangle_{\mathrm{coin}}\langle 0|(\prod_{r=1}^t P_{a'_r}U^\dagger)$ and $\mathcal{Y}^t_{\vec{a}\vec{a}'} = B^t_{\vec{a}}|\Phi\rangle\langle\Phi|B^{t\,\dagger}_{\vec{a}'}$. Here $|\Phi\rangle$ corresponds to the Markov trace state [see Fig. 1(c)]. The coin histories are given by the vectors $\vec{a}, \vec{a}' \in \{0,1\}^{\otimes t}$, and the projectors for each outcome are $P_{a_j} = |a_j\rangle_{\mathrm{coin}}\langle a_j|$. The braid word for a given coin history is

$$B^t_{\vec{a}} = \prod_{r=0}^{t-1} b_{s_0 + a_{t-r} + 2(\sum_{j=1}^{t-r-1} a_j) - (t-r)}. \quad (2)$$

The spatial distribution of the walker anyon depends on the trace over the coin and fusion DOFs. To evaluate the former, for simplicity we chose the Hadamard coin flip operation $U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ so that $\mathrm{tr}\,\mathcal{U}^t_{\vec{a}\vec{a}'} = \frac{1}{2^t}(-1)^{z(\vec{a},\vec{a}')}$ and $z(\vec{a},\vec{a}') = \sum_{j=1}^{t-1} a'_j a'_{j+1} + a_j a_{j+1}$ is the sum of pairs of consecutive right moves (or 1 outcomes of the coin). Note that our results are essentially the same for any nondiagonal choice of unitary U. The trace over the fusion DOF can be related to the Kauffman bracket of a link L, denoted $\langle L\rangle$ and defined below. The link is the Markov trace over the braid words for the forward and backward time evolution histories as dictated by Eq. (1):

$$L = (B^{t\,\dagger}_{\vec{a}'} B^t_{\vec{a}})^{\mathrm{Markov}}. \quad (3)$$

Moreover, $\mathrm{tr}\,\mathcal{Y}^t_{\vec{a}\vec{a}'} = \langle (B^{t\,\dagger}_{\vec{a}'} B^t_{\vec{a}})^{\mathrm{Markov}}\rangle / d^{n-1}$, where d is the quantum dimension of the anyons [13].

Henceforth, we focus on diagonal elements of the spatial probability distribution $p(s,t) \equiv \langle 2s - t + s_0|\rho_{\mathrm{space}}(t)|2s - t + s_0\rangle$, where $s = 1, \ldots, t$ is used to denote spatial location, the signed distance from the origin being $2s - t$.

We have the constraint that $|\vec{a}| = |\vec{a}'|$ (i.e., the final position of the walker for the braids $B_{\vec{a}}$, $B_{\vec{a}'}$ is the same), and the trace over the coin DOF is nonzero only if $a_t = a_t'$ (i.e., the final step is in the same direction for both paths). The result is

$$p(s,t) = \frac{1}{d^{(n-1)}2^t} \sum_{\{\vec{a},\vec{a}';|\vec{a}|=|\vec{a}'|=s,a_t=a_t'\}} (-1)^{z(\vec{a},\vec{a}')}\langle L\rangle, \quad (4)$$

which for $d = 1$ and $\langle L\rangle = 1$ reduces to the usual quantum walk distribution. The probably distribution is a sum over all Feynman paths from the origin to a final spatial point where the first factor in the sum carries the dynamical phase information and the second factor the topological information.

The problem of computing the distribution thus reduces to computing statistics of a quantum link invariant. The Kauffman bracket of a link $\langle L\rangle(A)$ is a Laurent polynomial in the argument A that is an invariant for framed, unoriented links and is framing dependent. It can be related to the Jones polynomial V_L, which is an invariant for framed, oriented links but which is framing independent:

$$\langle L\rangle(A)|_{A\to q^{-1/4}} = (-q^{3/4})^{w(L)}V_L(q),$$

where $w(L)$ is the writhe of L, which is zero for links considered here [19]. The Jones polynomial of a link L with variable $q = e^{i2\pi/(k+2)}$ was shown by Witten [20] to be equal to the expectation value of the product of the path-ordered Wilson loops along the components of the links of L in the $SU(2)_k$ Chern-Simons theory calculated by using the Chern-Simons partition function. In our model the links are the closed world lines traced by anyons, and when the anyons correspond to spin-1/2 irreps of the quantum group $SU(2)_k$, then $p(s,t)$ is a sum over weighted Jones polynomials. At the special value $q = i$, i.e., $k = 2$ where the spin-1/2 irrep anyons are described by the Ising model, the Jones polynomial of a link can be related to a simpler knot invariant known as the *arf* invariant [21]. Specifically,

$$V_L(i) = \begin{cases} \sqrt{2}^{[\#(L)-1]}(-1)^{arf(L)} & \text{if } L \text{ proper,} \\ 0 & \text{if } L \text{ not proper,} \end{cases}$$

where the number of components $\#(L) = n$ here. An oriented link is proper if each component L_k evenly links the union of other components, i.e., $\sum_{j\neq k} lk(L_j, L_k) = 0 \bmod 2\ \forall\ j$ [$lk(L_j, L_k)$ is the linking number [19], which can be computed in polynomial time]. The advantage of this expression is that $arf(L) \in \{0,1\}$ of a link can be computed in polynomial time in the crossing number of a braid presentation. Hence, unlike the generic case, the Jones polynomials at value $q = i$ can be evaluated in polynomial time [22]. However, the number of links contributing to the weight $p(s,t)$ is $\binom{t-1}{s}^2 + \binom{t-1}{s-1}^2$, which is exponential in t so an efficient computation is not *a priori* available.

There is structure to the links in the anyonic quantum walk trajectories that simplifies the computation significantly. The only term that contributes to the *arf* is the triple component invariant c_3, which counts the number of Borromean links [19]. This allows us to express the probability distribution in terms of simple properties of the anyonic walk:

$$p(s,t) = \sum_{\substack{\{\vec{a},\vec{a}'\in\{0,1\}^{\otimes t};\\ |\vec{a}|=|\vec{a}'|=s,a_t=a_t'\}}} \begin{cases} \frac{(-1)^{z(\vec{a},\vec{a}')+\tau(\vec{a},\vec{a}')}}{2^t} & L \text{ proper,} \\ 0 & L \text{ not proper,} \end{cases} \quad (5)$$

where the link L is defined in Eq. (3) and $\tau(\vec{a},\vec{a}')$ is a sum over link invariants [19]. When all links are proper and $\tau(\vec{a},\vec{a}')$ is even, then $p(s,t)$ becomes equal to $p_{\mathrm{QW}}(s,t)$, the quantum walk distribution. When all nonmirror paths (i.e., $\vec{a} \neq \vec{a}'$) are nonproper, then $p(s,t)$ becomes equal to $p_{\mathrm{RW}}(s,t)$, the classical random walk distribution.

To probe the behavior of $p(s,t)$, we initially perform exact numerical simulations. Details of this calculation are given in Ref. [19]. We have calculated the distribution for walks up to $t = 25$. The variance $\sigma^2(t) = \langle s^2\rangle - \langle s\rangle^2$, where the expectation value is $\langle O(s)\rangle \equiv \sum_s O(s)p(s,t)$, is plotted in Fig. 2, and it quickly approaches the linear random walk variance. Using the total variation distance between two distributions $p(s,t)$ and $f(s,t)$ defined as $\Delta(p,f) \equiv \frac{1}{2}\sum_s|p(s,t) - f(s,t)|$, at $t = 25$ we find $\Delta(p, p_{\mathrm{QW}}) = 0.34$ while $\Delta(p, p_{\mathrm{RW}}) = 0.04$.

We now demonstrate explicitly that asymptotically the Ising anyonic walk behaves classically. The essential reason is the rapidly decreasing density of proper links in the regime where the quantum walk distribution has dominant support. In order to upper bound the variance, we can assume that $\tau(\vec{a},\vec{a}')$ of all the proper links are even and that there is no correlation between being proper and $z(\vec{a},\vec{a}')$ for nonmirror paths. Calling the resulting distribution $\tilde{p}(s,t)$, we have

$$\tilde{p}(s,t) = p(s,t)_{\mathrm{RW}} + p_{\mathrm{prop}}(s,t)[p(s,t)_{\mathrm{QW}} - p(s,t)_{\mathrm{RW}}], \quad (6)$$

where $p_{\mathrm{prop}}(s,t)$ is the density of proper links for nonmirror paths ($\vec{a} \neq \vec{a}'$). Since the walker's speed is constant, the maximum possible variance is quadratic, achieved up to a constant less than 1 by the quantum walk [23]. So this choice of distribution can only make the estimate of the variance of the anyonic walker larger, i.e., $\tilde{\sigma}^2(t) \geq \sigma^2(t)$.

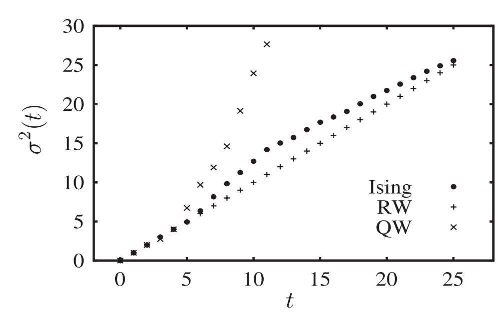

FIG. 2. Numerical results for the variance of the spatial distribution $p(s,t)$ [Eq. (5)] for the Ising anyonic walk and the corresponding classical and quantum walk evolutions with the same initial state.

In Ref. [23] it was shown that the distribution of the quantum walk with the same Hadamard coin flip and initial state as occurs here can be very well approximated asymptotically by the function $p'_{\mathrm{QW}}(\alpha, t)d\alpha = (1-\alpha)d\alpha/\pi(1-\alpha^2)\sqrt{1-2\alpha^2}$, where $\alpha = (s_0 - s)/t$ is restricted to the interval $[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}]$. Outside this interval the distribution falls off exponentially with t as does $p_{\mathrm{QW}}(s, t)$. By restricting to this interval and using the fact that a proper link must have all linking numbers even, it can be shown [19] that the density of nonmirror proper links is $p_{\mathrm{prop}}(s, t) < C/t^2$ for some constant C independent of s and t. The position moments with respect to $\tilde{p}(s, t)$ are

$$
\begin{aligned}
\langle (s - s_0) \rangle &\approx \int_{1/\sqrt{2}}^{-1/\sqrt{2}} p_{\mathrm{prop}}(s, t) t \alpha p'_{\mathrm{QW}}(\alpha, t) d\alpha \\
&< C(1 - 1/\sqrt{2})/t, \\
\langle (s - s_0)^2 \rangle &\approx t(1 - C/t^2) \\
&\quad + \int_{1/\sqrt{2}}^{-1/\sqrt{2}} p_{\mathrm{prop}}(s, t) t^2 \alpha^2 p'_{\mathrm{QW}}(\alpha, t) d\alpha \\
&< t(1 - C/t^2) + (1 - 1/\sqrt{2})C.
\end{aligned}
$$

Thus $\tilde{\sigma}^2(t) < t + O(1)$.

Similarly, to obtain a lower bound for the dispersion, assume a distribution of the form of Eq. (6) but choose a probability distribution $f(s, t)$ with minimum variance to replace $p_{\mathrm{QW}}(s, t)$ to account for destructive interference from correlations between $z(\vec{a}, \vec{a}')$ and $\langle L \rangle$. Picking $f(s, t) = \delta_{s, s_0}$ (zero variance) and calling the resulting distribution $\tilde{\tilde{p}}(s, t)$, then $\tilde{\tilde{\sigma}}(t)^2 > t(1 - C/t^2)$. By the inequalities, $\tilde{\tilde{\sigma}}^2(t) \leq \sigma^2(t) \leq \tilde{\sigma}^2(t)$, $\lim_{t\to\infty} \sigma^2(t)/t = 1$. Hence, asymptotically the Ising anyonic walk has linear dispersion with coefficient 1 like the classical random walk.

In conclusion, we have studied the dynamical behavior of a mobile non-Abelian anyon which becomes entangled with its environment purely by statistical interactions. We find that for the case of Ising anyons the decoherence in the spatial DOF is strong enough to completely wash out the quantum mechanical interferences and reduce the dynamics to a classical stochastic process. This is in sharp contrast to coherent quantum walk dynamics with Abelian anyons, since there the fusion DOF is one-dimensional. It would be of interest to extend this analysis to other non-Abelian anyons such as spin-$1/2$ irreps for other $SU(2)_k$ models. It is known that for $k > 2$ and $k \neq 4$, the braiding evolutions densely span the fusion space, while for $k = 2$ (Ising anyons) and $k = 4$ they do not [24]. It has been shown that for a t step quantum walk subject to decoherence in its position at a rate $p_{\mathrm{meas}} > C'/t$ for some constant C', the evolution approaches classical behavior [25]. We conjecture that since the braiding generators that entangle new fusion degrees of freedom mimic measurement with probability $p_{\mathrm{meas}} \sim 1/k^2$ (for $k \gg 1$), then for $t > k^2$ the walk would behave classically. In the limit $k \to \infty$ the particles are fermions, and we recover coherent quantum walk behavior.

L. L. and V. Z. thank the EU grant EMALI for support during their visit at Leeds University. J. K. P. thanks the Royal Society. G. K. B. received support from the Australian Research Council Discovery Projects funding scheme DP1094758 and from the European Community's Seventh Framework Program (FP7/2007-2013) under Grant Agreement No. 247687 (Integrating Project AQUTE).

[1] J. M. Leinaas and J. Myeheim, Nuovo Cimento Soc. Ital. Fis. B **37**, 1 (1977).
[2] D. Arovas, J. R. Schrieffer, and F. Wilczek, Phys. Rev. Lett. **53**, 722 (1984); B. I. Halperin, Phys. Rev. Lett. **52**, 1583 (1984).
[3] G. Moore and N. Read, Nucl. Phys. **B360**, 362 (1991).
[4] X.-G. Wen, *Quantum Field Theory of Many-Body Systems* (Oxford University, New York, 2004).
[5] A. Yu. Kitaev, Ann. Phys. (N.Y.) **303**, 2 (2003); **321**, 2 (2006).
[6] M. Dolev *et al.*, Nature (London) **452**, 829 (2008);
[7] R. L. Willett, L. N. Pfeiffer, and K. W. West, Proc. Natl. Acad. Sci. U.S.A. **106**, 8853 (2009).
[8] I. Bloch, Nature (London) **453**, 1016 (2008).
[9] M. H. Freedman, A. Kitaev, M. J. Larsen, and Z. Wang, Bull. Am. Math. Soc. **40**, 31 (2002).
[10] A. Feiguin *et al.*, Phys. Rev. Lett. **98**, 160409 (2007); S. Trebst *et al.*, Phys. Rev. Lett. **101**, 050401 (2008).
[11] A. W. W. Ludwig, D. Poilblanc, S. Trebst, and M. Troyer, New J. Phys. **13**, 045014 (2011).
[12] V. Lahtinen and J. K. Pachos, Phys. Rev. B **81**, 245132 (2010).
[13] G. K. Brennen, D. Ellinas, V. Kendon, J. K. Pachos, I. Tsohantjis, and Z. Wang, Ann. Phys. (N.Y.) **325**, 664 (2010).
[14] V. Kendon, Math. Struct. Comput. Sci. **17**, 1169 (2007).
[15] T. A. Brun, H. A. Carteret, and A. Ambainis, Phys. Rev. Lett. **91**, 130602 (2003).
[16] C. Nayak and F. Wilczek, Nucl. Phys. **B479**, 529 (1996).
[17] S. Nechaev, in *Topological Aspects of Low Dimensional Systems*, Proceedings of the Les Houches Summer School, Session LXIX (Springer-Verlag, Berlin, 1999), p. 643.
[18] P. Bonderson, K. Shtengel, and J. Slingerland, Ann. Phys. (N.Y.) **323**, 2709 (2008).
[19] See supplemental material at http://link.aps.org/supplemental/10.1103/PhysRevLett.106.230404 for details of the computation of the link invariants and density of proper links.
[20] E. Witten, Commun. Math. Phys. **121**, 351 (1989).
[21] H. Murakami, Math. Sem. Notes **11**, 335 (1983).
[22] F. Jaeger, D. L. Vertigan, and D. J. A. Welsh, Math. Proc. Cambridge Philos. Soc. **108**, 35 (1990).
[23] A. Nayak and A. Vishwanath, arXiv:quant-ph/0010117.
[24] M. Freedman, M. J. Larsen, and Z. Wang, Commun. Math. Phys. **228**, 177 (2002).
[25] V. Kendon and B. Tregenna, Phys. Rev. A **67**, 042315 (2003); P. C. Richter, Phys. Rev. A **76**, 042306 (2007).