Classification of symmetric polynomials of infinite variables: Construction of Abelian and non-Abelian quantum Hall states

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The classification of complex wave functions of infinite variables is an important problem since it is related to the classification of possible quantum states of matter. In this paper, we propose a way to classify symmetric polynomials of infinite variables using the pattern of zeros of the polynomials. Such a classification leads to a construction of a class of simple non-Abelian quantum Hall states which are closely related to parafermion conformal field theories.

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I. INTRODUCTION

A. Functions of infinite variables

One of the most important problems in condensed matter physics is to understand how particles are organized in the ground state. Almost all the low energy properties of a system are determined by such an organization. Mathematically, the ground state of \(N\) particles is described by a wave function—a complex function of \(N\) variables \(\Phi(r_1, \ldots, r_N)\), where \(r_i\) is the coordinate of the \(i\)th particle. Thus, the problem of understanding the patterns of the many-particle organizations (or in physical terms, of understanding the quantum phases of many-particle systems) is to classify the complex wave functions \(\Phi(r_1, \ldots, r_N)\) in the \(N \to \infty\) limit.

Such a classification problem is one of the most fundamental problems in physics since it determines the possible quantum phases of many-particle systems. Due to the success of the Landau symmetry breaking theory in describing phases and phase transitions,\(^1\) for a long time physicists believe that the phases of matter are classified by their symmetry properties. Mathematically, this is equivalent to believing that the wave functions are classified by their symmetry properties, such as, for example, whether the wave function is invariant under translation \(\Phi(r) \to \Phi(r+a)\) or not. Under such a belief, the wave functions with the same symmetries are grouped into one class and such a class represents a single phase of matter. This is why group theory becomes an important mathematical foundation in physics.

However, after the discovery of fractional quantum Hall (FQH) states,\(^2,3\) it was realized that symmetry is not enough to classify all the possible organizations encoded in the wave functions \(\Phi(r_1, \ldots, r_N)\). This is because the wave functions that describe different FQH states have exactly the same symmetry. Thus, the wave functions of FQH states contain new kinds of organizations of particles that has nothing to do with symmetry.\(^4,5\) The new organizations of the particles are called topological orders.

Intuitively, what is new in the FQH wave functions is that the wave functions contain a long-range quantum entanglement.\(^6,7\) This is why the FQH wave functions describe new states of matter that cannot be described by symmetries. The wave functions with long-range entanglements and the corresponding topological orders not only appear in FQH systems, but they also appear in various quantum spin systems. Understanding this new class of wave functions and the resulting new states of matter is currently a very active research direction in condensed matter physics.\(^8\)–\(^28\)

To gain a deeper and more precise understanding of topological orders and the associated long-range entanglements, we need to solve the related mathematical problem of classifying \(\Phi(r_1, \ldots, r_N)\) in the \(N \to \infty\) limit. This is a difficult problem which is not well studied in mathematics. The problem is not even well defined. However, this does not mean that the problem is not important. It is common not to have a well defined problem when we wander into an unknown territory. The first task of knowing the unknown is usually to come up with a proper definition of the problem.

In this paper, we will not attempt to classify generic complex wave functions \(\Phi(r_1, \ldots, r_N)\). We limit ourselves to a simpler problem of trying to classify FQH states and their topological orders. (For a review on topological order in FQH states, see Refs. 29 and 30.) The corresponding mathematical problem is to classify symmetric and antisymmetric polynomials of \(N\) variables \(\Phi(z_1, \ldots, z_N)\) in the \(N \to \infty\) limit. We will first try to come up with a physically meaningful and mathematically rigorous definition of the problem. Then, we will solve the problem in some simple cases. This leads to a class of “simple” (anti)symmetric polynomials which corresponds to a class of simple FQH states. The constructed FQH states include both Abelian and non-Abelian FQH states.\(^14,15\)

II. FRACTIONAL QUANTUM HALL STATES AND POLYNOMIALS

A. Fractional quantum Hall wave functions

First, we would like to give a brief review on FQH theory. A FQH state is a quantum ground state of two-dimensional electrons in a magnetic field. Such a quantum state is described by a complex wave function,

\[
\Psi(x_1, y_1, x_2, y_2, \ldots, x_N, y_N),
\]

where \((x_i, y_i)\) are the coordinates of the \(i\)th electron and \(N\) is the total number of electrons. In the strong magnetic field
limit, if the filling fraction \( \nu \) of a FQH state is less than 1, then all the electrons are in the lowest Landau level. In this case, the ground state wave function has the following form:

\[
\Psi = \Phi(z_1, \ldots, z_N)\exp\left(-\frac{1}{4} \sum_{i=1}^{N} |z_i|^2\right),
\]

where \( z_i = x_i + iy_i \) and \( \Phi(z_1, \ldots, z_N) \) is a holomorphic function of \( z_i \) [i.e., \( \Phi(z_1, \ldots, z_N) \) does not depend on \( z_i^* \)]. Since \( \Phi(z_1, \ldots, z_N) \) has no poles, \( \Phi(z_1, \ldots, z_N) \) is a polynomial of \( z_i^* \).

Due to the Fermi statistics of the electrons, \( \Phi(z_1, \ldots, z_N) \) must be an antisymmetric polynomial. If we assume the electrons to have Bose statistics, then \( \Phi(z_1, \ldots, z_N) \) must be a symmetric polynomial. Thus, to understand the phases of FQH systems is to classify antisymmetric or symmetric polynomials. Therefore, in this paper, we will assume electrons to have Bose statistics and concentrate on classifying symmetric polynomials.

For a system of \( N \) bosonic electrons, which symmetric polynomial will represent the ground state of the system? It will depend on the interaction between the electrons. If the interaction potential between two electrons has a \( \delta \)-function form

\[
V_\delta(z_1, z_2) = \delta(z_1 - z_2),
\]

then the ground state is described by the symmetric polynomial

\[
\Phi_{1/2} = \prod_{i<j} (z_i - z_j)^2.
\]

Such a state has a vanishing total potential energy \( V_{\text{tot}} = 0 \), where

\[
V_{\text{tot}} = \int \prod i d^2z_i \Psi^* \{ (z_i) \} \sum_{i<j} V_i(z_i, z_j) \Psi^* (z_i).
\]

The vanishing of the total potential energy \( V_{\text{tot}} \) requires that the wave function \( \Psi^* \{ z_i \} \) to be zero as \( z_i \rightarrow z_j \). Since the average energy \( V_{\text{tot}} \approx 0 \) for any wave functions, the vanishing \( V_{\text{tot}} \) for \( \Phi_{1/2} \) indicates that \( \Phi_{1/2} \) is the ground state.

If the interaction potential between two electrons is given by

\[
V_2(z_1, z_2) = v_0 \delta(z_1 - z_2) + v_2 \frac{\partial^2}{\partial z_1^2} \delta(z_1 - z_2) \frac{\partial^2}{\partial z_1^2},
\]

with \( v_0 > 0 \) and \( v_2 > 0 \), then the ground state will be

\[
\Phi_{1/4} = \prod_{i<j} (z_i - z_j)^4.
\]

For interaction (2), the vanishing of the total potential energy \( V_{\text{tot}} \) not only requires that the wave function \( \Psi^* \{ z_i \} \) to be zero as \( z_i \rightarrow z_j \), but it also requires \( \Psi^* \{ z_i \} \) to vanish faster than \( (z_i - z_j)^2 \) as \( z_i \rightarrow z_j \). This means that the symmetric polynomial must have a fourth order zero as \( z_i \rightarrow z_j \). One such polynomial is given by \( \Phi_{1/4} = \prod_{i<j} (z_i - z_j)^4 \), which has the lowest total power of \( z_i^* \).

More complicated ground states can be obtained through more complicated interactions. For example, consider the following three-body interaction between electrons:

\[
V_{\text{PF}}(z_1, z_2, z_3) = S(v_0 \delta(z_1 - z_2) \delta(z_2 - z_3)) - v_1 \delta(z_1 - z_2) \delta(z_3 - z_2) \delta(z_2 - z_3),
\]

where \( S \) is the total symmetrization operator between \( z_1, z_2, \) and \( z_3 \). Such an interaction selects the symmetric polynomial

\[
\Phi_{\text{PF}} = A \left( \frac{1}{z_1 - z_2} \frac{1}{z_2 - z_3} \frac{1}{z_3 - z_4} \prod_{i<j} (z_i - z_j) \right)
\]
to describe the ground state (which has a vanishing total potential energy \( V_{\text{tot}} \)). Here, \( A \) is the total antisymmetrization operator between \( z_1, \ldots, z_N \).

Three symmetric polynomials \( \Phi_{1/2}, \Phi_{1/4}, \) and \( \Phi_{\text{PF}} \) contain different topological orders and correspond to three different phases of an \( N \)-electron system in the \( N \rightarrow \infty \) limit. They are the filling fraction \( \nu = 1/2 \) Laughlin state,\(^{3} \) the filling fraction \( \nu = 1/4 \) Laughlin state, and the filling fraction \( \nu = 1/4 \) Pfaffian state.\(^{14} \) We would like to find a classification of symmetric polynomials such that the above three symmetric polynomials belong to three different classes.

### B. Ideal Hamiltonian and zero-energy state

The above three examples share some common properties. The Hamiltonians described by the interaction potentials \( V_1, V_2, \) and \( V_{\text{PF}} \) are all positive definite and contain zero-energy eigenstates. The zero-energy eigenstates of \( V_1, V_2, \) and \( V_{\text{PF}} \) are exact ground states of the corresponding Hamiltonians. Since the interaction potentials are constructed from \( \delta \) functions and their derivatives, the exact ground states (the zero-energy states) for such type of potentials are characterized by the pattern of zeros, i.e., the orders of zeros of the ground state wave function, as we bring two or more electrons together.

In this paper, we will concentrate on such ideal Hamiltonians and their exact zero-energy ground states. From this point of view, classifying FQH states corresponds to classifying patterns of zeros in symmetric polynomials. In other words, for each pattern of zeros, we can define an ideal Hamiltonian such that the symmetric polynomials with the given pattern of zeros will be the zero-energy ground states of the Hamiltonian. Such symmetric polynomials will describe a phase of a FQH system provided that the Hamiltonian has a finite energy gap.

Clearly, for the ideal Hamiltonians, apart from the zero-energy ground states, other eigenstates of the Hamiltonian all have nonzero and positive energies. However, this does not imply the Hamiltonian to have a finite energy gap. Only when the minimal energy of the excitations has a finite non-zero limit as electron number approaches to infinity, does the Hamiltonian have a finite energy gap. Thus, to classify the
FQH states, we not only need to classify the patterns of zeros and the associated ideal Hamiltonians, we also need to judge if the constructed ideal Hamiltonian has a finite energy gap or not. At the moment, we do not have a good way to make such a judgment. Therefore, here, we will concentrate on classifying the patterns of zeros and the associated zero-energy states.

III. PATTERN OF ZEROS

A. Derived polynomials and their \( D_{ab} \) characterization

In order to classify translation invariant symmetric polynomials of \( N \) variables \( \Phi(z_1, \ldots, z_N) \) in the \( N \to \infty \) limit, we need to define the polynomials for any \( N \). The key in our definition is to introduce “local conditions.” These local conditions apply to polynomials of any numbers of variables.

From the discussion in Sec. II A, we see that one way to implement the local condition is to let one variable approach a value of \( \bar{z} \). Note that \( \Phi \) is a type of zeros. So, if the constructed ideal Hamiltonian has a finite energy gap, we may look at the lowest total power of \( \bar{z} \) and each variable \( z_i \).

To get a feeling what a consistent set of \( \{D_{ab}\} \) may look like, let us consider the following symmetric polynomials (the Laughlin state):

\[
\Phi_{1/q}(\{z_i\}) = \prod_{i<j} (z_i - z_j)^q,
\]

where \( q \) is an even integer. Such a symmetric polynomial leads to the following derived polynomial:

\[
P_{1/q}(\{z_i^{(a)}\}) = \left\{ \prod_{a \neq b} \prod_{i \neq j} (z_i^{(a)} - z_j^{(b)})^{q_{ab}} \right\} \left( \prod_a \prod_{i<j} (z_i^{(a)} - z_j^{(a)})^{q_{aa}} \right).
\]

B. \( S_a \) characterization of polynomials

There is another way to implement local conditions on a translation invariant symmetric polynomial \( \Phi(z_i) \). We introduce a sequence of integers \( S_a \), where \( a = 0, 1, 2, \ldots \), and require that the minimal total powers of \( z_1, \ldots, z_a \) in \( \Phi(z_i) \) is given by \( S_a \). Thus, in addition to \( \{D_{ab}\} \), we can also use \( \{S_a\} \) to characterize a symmetric polynomial. For a translation invariant symmetric polynomial, \( \Phi(0, z_2, \ldots, z_N) \neq 0 \). Thus, \( S_1 = 0 \).

The two characterizations, \( \{D_{ab}\} \) and \( \{S_a\} \), are closely related. One way to see the relation is to put the symmetric polynomial \( \Phi(z_1, \ldots, z_N) \) on a sphere as discussed in Appendix A. Let \( N_\phi \) be the maximum power of \( z_i \) in \( \Phi(z_1, \ldots, z_N) \). Then, \( \Phi(z_1, \ldots, z_N) \) can be put on a sphere with \( N_\phi \) flux quanta and each variable \( z_i \) carries an angular momentum \( J = N_\phi/2 \).

From the discussion near the end of Appendix A, we find that each type-\( a \) particle described by \( z_i^{(a)} \) in \( P(\{z_i^{(a)}\}) \) carries a definite angular momentum, which is denoted as \( I_a \). Since the lowest total power of \( z_1, \ldots, z_a \) is \( S_a \), the minimal total \( L \)
quantum number for those variables is \(-aJ+S_a\). Therefore, the angular momentum of the \(\tilde{z}_l^{(a)}\) variable is

\[ J_a = aJ - S_a. \quad (8) \]

Since \(\tilde{z}_l^{(1)} = z_l\), we find that \(J_1 = 1\).

Again, according to the discussion near the end of Appendix A, if we fuse two variables \(\tilde{z}_l^{(a)}\) and \(\tilde{z}_l^{(a+b)}\), the type \((a+b)\) particle described by \(\tilde{z}_l^{(a+b)}\) will carry an angular momentum

\[ J_{a+b} = J_a + J_b - D_{ab}. \quad (9) \]

We see that \(D_{ab}\) can be expressed in terms of \(S_a\) as follows:

\[ D_{ab} = S_{a+b} - S_a - S_b. \quad (10) \]

The conditions on \(D_{ab}\) [Eq. (5)] can be translated into the conditions on \(S_a\):

\[ S_{2a} = \text{even}, \quad S_{a+b} = S_a + S_b. \quad (11) \]

From the recursive relation \(J_{a+1} = J_a + J_1 - D_{a+1}\), we find \(S_{a+1} = S_a + D_{a+1}\). Using \(S_1 = 0\), we see that \(S_a\) can also be calculated from \(D_{ab}\):

\[ S_a = \sum_{b=1}^{a-1} D_{b,1}. \quad (12) \]

Due to the one-to-one correspondence between \(\{D_{ab}\}\) and \(\{S_a\}\), we will also call the sequence \(\{S_a\}\) a pattern of zeros.

C. Boson occupation characterization

The symmetric polynomial \(\Phi(z_1, \ldots, z_N)\) can be written as a sum of polynomials described by boson occupations

\[ \Phi((z)_l) = \sum_{\{\tilde{n}\}} C_{\{\tilde{n}\}} \Phi_{\{\tilde{n}\}}((z)_l), \]

where \(\Phi_{\{\tilde{n}\}}\) is a boson occupation state with \(\tilde{n}\) bosons occupying the \(\tilde{z}_l^l\) orbital. Mathematically, \(\Phi_{\{\tilde{n}\}}((z)_l)\) is given by

\[ \Phi_{\{\tilde{n}\}}(z_1, \ldots, z_N) = \sum_{P} \prod_{i=1}^{N} \tilde{z}_P^{l_i}, \quad (13) \]

where \(P\) is a one-to-one mapping from \(\{1, \ldots, N\} \rightarrow \{1, \ldots, N\}\); \(S_P\) is the sum over all those one-to-one mapping; and \(l_i\), where \(i = 1, 2, \ldots\), is a sequence of ordered integers such that the number of \(l\) valued \(l_i\) 's is \(n_i\).

What kinds of boson occupations \(\{\tilde{n}\}\) appear in the above sum? Let us set \(z_l = 0\) in \(\Phi((z)_l)\). Since \(\Phi(0, z_2, \ldots, z_N) \neq 0\) due to the translation invariance, there must be a boson occupation \(\{\tilde{n}\}\) in the above sum that contains one boson occupying the \(z_0^{l_0}\) orbital. Now let us assume that a boson occupies \(z_0^{l_0}\) and bring the second particle \(z_2\) to 0; the minimal power of \(z_2\) in \(\Phi(0, z_2, \ldots, z_N)\) is \(D_{11}\):

\[ \Phi(0, z_2, \ldots, z_N) \sim z_2^{D_{11}} P_2(z_2, z_4, \ldots) + o(z_2^{D_{11}}). \]

Thus, among those \(\{\tilde{n}\}\) which have one boson occupying the \(z_0^{l_0}\) orbital, there must be \(\{\tilde{n}\}\) that contains a second boson occupying the \(z_2^2\) orbital where \(l_2 = D_{11} = S_2 - S_1\). Next, let us assume that two bosons occupy the \(z_0^{l_0}\) and \(z_2^2\) orbitals and we bring the third particle \(z_3\) to 0; the minimal power of \(z_3\) is \(D_{21}\):

\[ P_2(z_2, z_4, \ldots) \sim z_3^{D_{21}} P_3(z_4, z_5, \ldots) + o(z_3^{D_{21}}). \]

Thus, among those \(\{\tilde{n}\}\) which have two bosons occupying the \(z_0^{l_0}\) and \(z_2^2\) orbitals, there must be an \(\{\tilde{n}\}\) that contains a third boson occupying the \(z_3^3\) orbital where \(l_3 = D_{21} = S_3 - S_2\). This way we can show that there must contain \(\{\tilde{n}\}\) such that the \(a\)th boson occupies the orbital \(z_2^{la}\) with \(l_a = S_a - S_{a-1}\). Here, \(a = 1, 2, \ldots\), and \(l_1 = 0\). Let \(n_l\) be the numbers of \(l_a = S_a - S_{a-1}\) that satisfy \(l_a = l\). We see that the boson occupation state \(\Phi_{\{\tilde{n}\}}((z)_l)\) happens to be the state with \(\tilde{a}\) boson occupying the orbital \(z_l^l\). This allows us to show that \(\Phi(z_1, \ldots, z_N)\) has the form

\[ \Phi((z)_l) = \Phi_{\{n_l\}}((z)_l) + \sum_{\{n_l\}} C_{\{n_l\}} \Phi_{\{n_l\}}((z)_l), \quad (14) \]

or in other words,

\[ \langle \Phi_{\{n_l\}} | \Phi \rangle = 0. \quad (15) \]

The two sequences, \(\{S_a\}\) and \(\{n_l\}\), have a one-to-one correspondence. We will call \(\{n_l\}\) the boson occupation description of the pattern of zeros \(\{S_a\}\).

The boson occupation distributions \(\{\tilde{n}\}\) that appear in the sum in Eq. (14) satisfy certain conditions. First, the boson occupation \(\{\tilde{n}\}\) can be described by a pattern of zeros \(\{S_a\}\). Then, the conditions on \(\{\tilde{n}\}\) can be stated as \(S_a \geq S_a\). Thus, the minimal total power of \(z_1, \ldots, z_a\) in \(\Phi_{\{\tilde{n}\}}((z)_l)\) is \(S_a\), which is equal or bigger than \(S_a\).

Haldane* conjectured that \(\tilde{n}\)'s in expression (14) can be obtained from \(n_l\) by one or many squeezing operations. A squeezing operation is a two-particle operation that moves one particle from the orbital \(z_l^l\) to the orbital \(z_l^l\) and the other from \(z_2^2\) to \(z_2^2\), where \(l_1 < l_1' \leq l_2 < l_2'\), and \(l_1 + l_2 = l_1' + l_2'\). We can show that if \(\tilde{n}\) is obtained from \(n_l\) by squeezing operations, then the minimal total power of \(z_1, \ldots, z_a\) in \(\Phi_{\{\tilde{n}\}}((z)_l)\) is equal or bigger than \(S_a\). This is consistent with the above discussion.

Let \(P_{aJ_a}\) be a projection operator acting on the state \(\Phi\) on a sphere. \(P_{aJ_a}\) projects into the subspace where \(a\) particles in \(\Phi\) have a total angular momentum equal to \(J_a\) or less. We see that for a symmetric polynomial \(\Phi(z_1, \ldots, z_N)\) described by a pattern of zero \(S_a\), it satisfies

\[ P_{N, J_a} \cdots P_{3, J_3} P_{2, J_2} \Phi(z_1, \ldots, z_N) = \Phi(z_1, \ldots, z_N), \]

where \(J_a = aJ - S_a\). This allows us to obtain

\[ P_{N, J_a} \cdots P_{3, J_3} P_{2, J_2} \Phi_{\{n_l\}}(z_1, \ldots, z_N) \neq 0, \quad (16) \]

where \(n_l\) is the boson occupation description of \(S_a\).

IV. CONSISTENT CONDITIONS ON THE PATTERN OF ZEROS

For a translation invariant symmetric polynomial \(\Phi((z)_l)\), the corresponding pattern of zeros \(\{D_{ab}\}\) and \(\{S_a\}\) satisfies
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The complex function $D_{ab}$ gives us a complex function $f(z_i^{(a)})$ in the loop. The crosses mark the zeros of $f(z_i^{(a)})$ not at $z_i^{(b)}$ and $z_i^{(c)}$.

some special properties. Here, we would like to find those properties as much as possible. Those properties will be called consistent conditions on the pattern of zeros. If we find all the consistent conditions, a set of integers $(D_{ab})$ or $\{S_a\}$ that satisfies those consistent conditions will correspond

A. Concave condition

If we fix all variables $z_i^{(a)}$ except $z_i^{(a)}$, then the derived polynomial $P_a(z_i^{(a)})$ gives us a complex function $f(z_i^{(a)})$. The complex function $f(z_i^{(a)})$ has isolated zeros at $z_i^{(b)}$’s and possibly also at some other points.

Let us move $z_i^{(a)}$ around two points $z_i^{(b)}$ and $z_i^{(c)}$. The phase of the complex function $f(z_i^{(a)})$ will change by $2\pi W_{a,b,c}$, where $W_{a,b,c}$ is an integer (see Fig. 1). Since $f(z_i^{(a)})$ has an order $D_{ab}$ zero at $z_i^{(a)}$ and an order $D_{ac}$ zero at $z_i^{(a)}$, the integer $W_{a,b,c}$ satisfies

$$W_{a,b,c} \geq D_{ab} + D_{ac}$$

because $f(z_i^{(a)})$ has no poles. Now, let $z_i^{(a)} \to z_i^{(a)}$ to fuse into $z_i^{(b+c)}$. In this limit, $W_{a,b,c}$ becomes the order of zeros between $z_i^{(a)}$ and $z_i^{(b+c)}$: $W_{a,b,c} = D_{a,b,c}$. Thus, we obtain the following conditions on $D_{ab}$:

$$D_{a,b,c} \geq D_{ab} + D_{ac}.$$  \hspace{1cm} (17)

Concave condition (17) is equivalent to a condition on $S_a$:

$$S_{a,b,c} + S_a + S_b + S_c \geq S_{a,b} + S_{b,c} + S_{a,c}.$$  \hspace{1cm} (18)

We note that the Laughlin state $\Phi_{1/n}(z_i)$ has $\prod_{i<j}(z_i - z_j)^l$ saturates the above conditions: $D_{a,b,c} = D_{ab} + D_{ac}$ or $S_{a,b,c} + S_a + S_b + S_c \geq S_{a,b} + S_{b,c} + S_{a,c}$. (18)

B. Symmetry condition

If we fix all variables $z_i^{(a)}$ except $z_i^{(a)}$, $z_i^{(b)}$, and $z_i^{(c)}$, then the derived polynomial $P_a(z_i^{(a)})$ gives us a complex function $f(z_i^{(a)})$. Let us assume $z_i^{(a)}$, $z_i^{(c)}$, and $z_i^{(b)}$ are close to each other and far away from all other variables. $f(z_i^{(a)})$ has a $D_{ab}$th order zero as $z_i^{(a)} \to z_i^{(b)}$ and a $D_{ab}$th order zero as $z_i^{(a)} \to z_i^{(c)}$. Thus, $D_{a,b,c} = D_{ab} + D_{ac}$ is the number of zeros of $f(z_i^{(a)})$ in the same neighborhood that are not at $z_i^{(b)}$ and $z_i^{(c)}$. (see Fig. 2).

Here, we would like to assume that $\Phi(z_i)$ satisfies the unique-fusion condition. In this case, the type-$a$ variables in the derived polynomials have “no shapes” and can be treated as points. Since other variables are far away, the zeros of $f(z_i^{(a)}), z_i^{(b)}$, and $z_i^{(c)}$ must satisfy certain symmetry conditions (see Fig. 2).

We see that when $a=b=c$, $f(z_i^{(a)}), z_i^{(b)}$, and $z_i^{(c)}$ form an equilateral triangle. Thus, the zeros of $f(z_i^{(a)}), z_i^{(b)}$, and $z_i^{(c)}$ (when viewed as a function of $z_i^{(a)}$) that are marked by the crosses must appear in pairs. We find that $D_{a,a,a} + D_{a} - D_{a}$ must be even, or equivalently

$$S_{a,a} - S_{a} = \text{even}. \hspace{1cm} (19)$$

C. $n$-cluster condition

The structure of symmetric polynomials with infinite variables is very complicated and hard to manage. Here, we would like to introduce an $n$-cluster condition that makes a polynomial with infinite variables behave more like a polynomial with a finite number of variables. A symmetric polynomial satisfies the $n$-cluster condition if after we fuse the variables of $\Phi(z_i)$ into $n$-variable clusters, the derived polynomial

$$P(z_i^{(a)}, \ldots, z_N^{(a)}) \sim \prod_{i<j}(z_i^{(a)} - z_j^{(a)})^l$$  \hspace{1cm} (20)

has a simple Laughlin form where $l$ is a positive integer.

To see the structure of cluster form more clearly, let us assume that the polynomial $\Phi(z_1, \ldots, z_N)$ describes a FQH state with filling fraction $\nu$. This means that as a homogenous polynomial, the total order of the $z_i$, $S_{N}$, in $\Phi(z_1, \ldots, z_N)$ satisfies

$$S_{N} = 1/2\nu N^2 + O(N).$$

This motivates us to write $\Phi(z_1, \ldots, z_N)$ as

$$\Phi(z_1) = G(z_1)\Phi_{\nu}(z_1).$$

FIG. 2. Pattern of zeros of $f(z_i^{(a)})$ (when viewed as a function of $z_i^{(a)}$): (a) $D_{a,b,c}=D_{ab}-D_{ac}=1$, (b) $D_{a,b,c}=D_{ab}-D_{ac}=2$, and (c) $a=b=c$ and $D_{a,a,a}=D_{ac}$. The dashed lines form two equilateral triangles. The zeros that are not located at any variable are marked by crosses.
\[ \Phi_{\nu}(z_j) = \prod_{i < j} (z_j - z_i)^{\nu - 1}. \]  

(21)

Here, \( G(z_j) \) satisfies
\[ G(\lambda z_1, \ldots, \lambda z_k) = \lambda^{\nu+N} G(z_1, \ldots, z_k), \quad s_N = O(N) \ldots. \]

Note that \( G(z_j) \) is, in general, not a single-valued function since \( \Phi_{\nu}(z_j) \) is, in general, not single valued. However, the product of \( G(z_j) \) and \( \Phi_{\nu}(z_j) \) is a single-valued symmetric polynomial.

We can fuse the variables in \( G(z_j) \) to obtain a derived function \( G(z_i^{(a)}) \) [just as how we obtain the derived polynomial \( P(z_i^{(a)}) \) from the original symmetric polynomial \( \Phi(z_j) \)]. Similarly, we can also fuse the variables in \( \Phi_{\nu}(z_j) \) to obtain a derived function \( \Phi_{\nu}(z_i^{(a)}) \) as follows:
\[ \Phi_{\nu}(z_i^{(a)}) = \prod_{i,j,a < b} (z_i^{(a)} - z_j^{(b)})^{\nu + \nu_i}. \]

(22)

Thus, the derived polynomial \( P(z_i^{(a)}) \) can be expressed as
\[ P(z_i^{(a)}) = G(z_i^{(a)}) \Phi_{\nu}(z_i^{(a)}). \]  

(23)

Equation (23) can be viewed as a definition of \( G(z_i^{(a)}) \).

Assuming \( \Phi(z_j) \) has an \( n \)-cluster form, then if we fuse the variables in \( G(z_j) \) into \( n \)-variable clusters, the derived function
\[ G(z_i^{(a)}) = 1. \]

Here, we will require \( G(z_i^{(a)}) \) to satisfy more strict conditions,
\[ G(z_i^{(a)}) = G(z_i^{(a\%n)}), \]
\[ G(...)z_i^{(a)}(...) = G(...). \]

(24)

The second condition states that \( G(...)z_i^{(a)}(...) \) does not depend on \( z_i^{(a)} \) if \( a \% n = 0 \). If \( G \) satisfies Eq. (24), we will say the corresponding symmetric polynomial \( \Phi(z_j) = G(z_j)\Phi(z_j) \) to have an \( n \)-cluster form.

For a symmetric polynomial \( \Phi(z_j) \) of an \( n \)-cluster form, its pattern of zeros \( D_{ab} \) can be written as
\[ D_{ab} = \nu - 1 ab + d_{ab}, \]  

(25)

where \( d_{ab} \) satisfy
\[ d_{ab} = d_{ba}, \]
\[ d_{ab} = 0 \quad \text{if} \quad b \% n = 0, \]
\[ d_{ab} = d_{ab}. \]  

(26)

The pattern of zeros \( D_{ab} \) that satisfies the above conditions is said to have an \( n \)-cluster form. Note that \( \nu - 1 ab \) in Eq. (25) describe the pattern of zeros in the derived function \( \Phi_{\nu}(z_i^{(a)}) \) [see Eq. (22)] and \( d_{ab} \) describes the pattern of zeros in the derived function \( G(z_i^{(a)}) \).

Setting \( (a, b) = (n, n) \) and \( (a, b) = (1, n) \) in Eq. (25), we find that
\[ \nu - 1 n^2 = \text{even}, \quad \nu - 1 n = \text{integer}. \]

or
\[ \nu - 1 = \frac{m}{n}, \quad mn = \text{even}. \]  

(27)

We also find that
\[ D_{a,b+n} = D_{a,b} + am. \]  

(28)

Let
\[ s_a = S_a - \frac{1}{2} a(a - 1). \]  

(29)

We find that [see Eq. (10)]
\[ d_{ab} = s_{a+b} - s_a - s_b. \]  

(30)

Cluster conditions (26) become
\[ s_{a+n} - s_a = s_n = 0, \]
\[ s_{a+n} - s_a = s_{b+n} - s_b. \]  

(31)

Since \( S_1 = s_1 = 0 \) [see Eq. (8)], we find that \( s_{a+1} = s_a \) and \( s_{a+n} - s_a = s_{a+n} - s_a, \) Thus,
\[ s_{a+k+n} = k s_n + s_a, \quad \text{where} \quad a = 1, 2, \ldots, \infty. \]  

(32)

This allows us to obtain \( s_a \) for any \( a > 0 \) from \( s_1, s_2, \ldots, s_n. \) Similarly, all the \( S_a's \) can be determined from \( S_1, S_2, \ldots, S_n: \)
\[ S_{a+k+n} = s_{a+k+n} + \frac{m}{2n} (a + k n)(a + k n - 1) \]
\[ = k s_n + s_a + \frac{m}{2n} (a + k n)(a + k n - 1) \]
\[ = S_a - \frac{m}{2n} a(a - 1) + k \left[ S_n - \frac{m}{2n} (n - 1) \right] \]
\[ + \frac{m}{2n} (a + k n)(a + k n - 1) \]
\[ = S_a + k S_n + \frac{k(k - 1) n m}{2} + k m a. \]  

(33)

The above result is actually valid for any positive integer \( a. \)

It is convenient to introduce
\[ h_i^a = s_a - \frac{a}{n} s_n = S_a - \frac{a S_n}{n} - \frac{am}{2} a^2 \frac{m}{2n}. \]  

(34)

From Eq. (32), we can show that \( h_i^a \) is periodic:
\[ h_i^{a+k} = h_i^a. \]

Since \( s_1 = 0, \) we see that \( h_i^a = -a s_n/n \) and \( s_a = h_i^{a} - a h_i^{a}. \) From Eq. (29), we see that \( S_a \) can be calculated from \( h_i^{a}: \)
\[ S_a = h_i^{a} - a h_i^{a} + \frac{a(a - 1) m}{2n}. \]

(35)

Equations (34) and (35) imply that the two sequences of numbers, \( \{S_n\} \) and \( \{h_i^a\}, \) have a one-to-one correspondence.
and can faithfully represent each other. In this paper, we will use both sequences to characterize the symmetric polynomials. The \( h_n \) characterization turns out to have a close relation to the conformal field theory (CFT) description of the FQH states (see Appendix B)\(^{14,35-37}\).

If \( S_n \) has \( n \)-cluster form (33), then the corresponding boson occupation numbers \( n_i \) have some nice properties. From Eq. (33), we see that \( S_{a+n} - S_{a-1+n} = S_a - S_{a-1} + m \). Thus, \( l_i = l_{i-1} + 1 \). This means that the boson occupation in the orbitals \( z_i \) has a periodic structure: every time we skip \( n \) bosons, skip \( m \) orbitals. Or, in other words, if we know the occupation distribution of the first \( n \) bosons, the occupation distribution of second \( n \) bosons can be obtained from that of first \( n \) bosons by shifting the orbital index \( l \) by \( m \). Thus, the occupation numbers \( n_i \) satisfy \( n_i = n_{i+m} \) [see Eq. (52)]. Also, each \( m \) orbital contain \( n \) bosons. Due to the one-to-one correspondence between \( S_n \) and \( n \), we can also use \( n_0, \ldots, n_{m-1} \) to describe the pattern of zeros in \( \Phi(z_i) \).

**D. Translation invariance**

To study the translation invariance of the symmetric polynomial \( \Phi(z_1, \ldots, z_N) \), let us put the polynomial on a sphere (see Appendix A) and study its rotation invariance. In fact, in this paper, when we mention translation invariance, we actually mean rotation invariance on a sphere.

Let \( N_\phi \) be the number of flux quanta going through the sphere. Then, each variable \( z_i \) in \( \Phi(z_1, \ldots, z_N) \) carries an angular momentum \( J = N_\phi / 2 \). What is the total angular momentum of \( \Phi(z_1, \ldots, z_N) \)? In general, \( \Phi(z_1, \ldots, z_N) \) does not carry a definite angular momentum. Therefore, here we will calculate the maximum angular momentum of \( \Phi(z_1, \ldots, z_N) \) from the pattern of zeros \( D_{ab} \).

The maximum angular momentum is nothing but the angular moment of \( z^{(N)} \)—the particle obtained by fusing all the \( N \) electrons together. The angular moment of \( z^{(N)} \) is given by [see Eqs. (8) and (12)]

\[
J_N = J_{tot} = NJ - \sum_{a=1}^{N-1} D_{a,1} = NJ - S_N.
\]  

(36)

If \( J_N \) is zero for a symmetric polynomial \( \Phi(z_i) \), then \( \Phi(z_i) \) is invariant under the \( O(3) \) rotation of the sphere. In other words, \( \Phi(z_i) \) is translation invariant.

However, for an arbitrary choice of \( N \) and \( J \), \( J_N \) is not zero in general. \( J_N \) can be zero only for certain combinations of \( (N,J) \). For the filling fraction \( \nu = 1/q \) Laughlin state, \( J_N = NJ - qN - 1 \). We find \( J_N \) is zero if

\[
2J = N_\phi = qN - 1.
\]  

(37)

This is the relation between the number of magnetic flux quanta, \( N_\phi \), and the number of electrons, \( N \), of the \( 1/q \) Laughlin state if the Laughlin state is to fill the sphere completely (which gives rise to a rotation invariant state).

Assume that the symmetric polynomial \( \Phi(z_1, \ldots, z_N) \) has an \( n \)-cluster form described by the data \((m; S_{a+1}, S_{a+2}, \ldots)\). If we put the polynomial on a sphere, the maximum total angular momentum of \( \Phi(z_1, \ldots, z_N) \) is given by Eq. (36). If \( N = nN_c \), we find from Eq. (33) that

\[
S_{nN_c} = S_{n(N_c - 1)m} = N_c S_n + \frac{mn N_c (N_c - 1)}{2},
\]  

\[
J_{tot} = JN N_c - N_c S_n - \frac{mn N_c (N_c - 1)}{2}.
\]  

(38)

When \( N = nN_c \), \( \Phi(z_1, \ldots, z_N) \) can give rise to the Laughlin wave function (20) after fusing \( z_i \)'s into \( N_c z^{(N_c)} \) [see Eq. (20)]. Since it is always possible to fill the sphere with the Laughlin state, this implies that there exists an integer \( 2J \) to make \( J_{tot} = 0 \). Such an integer is given by

\[
2J = N_\phi = \frac{2S_{nN_c}}{nN_c} = \frac{2S_a + m(N_c - 1)}{n}.
\]  

(39)

This requires that

\[
2S_a = n \bmod n.
\]  

(40)

To summarize, the \( n \)-cluster condition requires that if \( N \bmod n = 0 \) and \((N_\phi, N)\) satisfies Eq. (39), then the symmetric polynomial \( \Phi(z_1, \ldots, z_N) \) must represent a rotation invariant state on sphere. The existence of such rotation invariant state requires \( S_a \) to satisfy Eq. (40).

**V. CONSTRUCTION OF IDEAL HAMILTONIANS**

We have seen that the pattern of zeros in an electron wave function \( \Phi(z_i) \) can be described by a set of integers \( S_2, S_3, \ldots \). In this section, we are going to construct an ideal Hamiltonian on sphere to realize such a kind of electron wave function as a ground state of the Hamiltonian.

On a sphere, the set of integers \( S_a \) also has a very physical meaning. For an electron system on a sphere with \( N_\phi \) flux quanta, each electron carries an orbital angular momentum \( J = N_\phi / 2 \) if the electrons are in the first Landau level.\(^{31}\) For a cluster of \( a \) electrons, the maximum allowed angular momentum is \( aJ \). However, for the wave function \( \Phi(z_i) \) described by \( S_a \), the maximum allowed angular momentum is just \( J_a = aJ - S_a \). The pattern of zeros forbs the appearance of angular momenta \( aJ - S_a + 1, aJ - S_a + 2, \ldots, aJ \) for any \( a \)-electron clusters in \( \Phi(z_i) \).

Such a condition can be easily enforced by a Hamiltonian. Let \( P^{(a)}_{S_a} \) be a projection operator that acts on \( a \)-electron Hilbert space. \( P^{(a)}_{S_a} \) projects onto the subspace of \( a \) electrons with total angular momenta \( aJ - S_a + 1, \ldots, aJ \). Now consider the Hamiltonian\(^{19,38,39}\)

\[
H_{(S_a)} = \sum_a \sum_{a \text{-electron clusters}} P^{(a)}_{S_a},
\]  

(41)

where \( \Sigma_{a \text{-electron clusters}} \) sum over all \( a \)-electron clusters. The wave function \( \Phi(z_i) \) with a pattern of zeros described by \( S_a \), if it exists, will be the zero-energy ground state of the above Hamiltonian.

We note that the Hamiltonian \( H_{(S_a)} \) is well defined for any choice of \( S_a \). However, for a generic choice of \( S_a \), the zero-energy ground state of \( H_{(S_a)} \) may not be the one with a pattern of zeros described by \( S_a \). This is because when we say that the wave function \( \Phi(z_i) \) has a pattern of zeros described by \( S_a \), we mean two things:
(a) The angular momenta $aJ - S_a + 1, \ldots, aJ$ do not appear for any $a$-electron clusters in $\Phi((z_i))$.

(b) The angular momenta $aJ - S_a$ do not appear for $a$-electron clusters in $\Phi((z_i))$.

The zero-energy ground state of $H_{(S_a)}$ satisfies condition (a). However, sometimes, we may find that condition (a) also implies that $aJ - S_a$ does not appear for $a$-electron clusters in $\Phi((z_i))$ for certain values of $a$. This means that the zero-energy ground state of $H_{(S_a)}$ is actually described by a pattern of zeros $\tilde{S}_a$ which satisfy $\tilde{S}_a \leq S_a$. However, for a certain special set of $\{S_a\}_a$ that describe the pattern of zeros of an existing symmetric polynomial, we have $\tilde{S}_a \neq S_a$. For those $S_a$, the zero-energy ground state of $H_{(S_a)}$ is described by the pattern of zeros of $\{S_a\}_a$ itself.

We have seen that for a FQH state described by a pattern of zeros $\{S_a\}_a$, a state of $a$-electron clusters has a nonzero projection into the space $\mathcal{H}_{a,S_a}$ where $\mathcal{H}_{a,S_a}$ is a space with a total angular momentum $aJ - S_a$. However, different positions of $S_a$ can lead to different images in the space $\mathcal{H}_{a,S_a}$. Let $\mathcal{H}_a$ be the subspace of $\mathcal{H}_{a,S_a}$ that is spanned by those images. In general, $\mathcal{H}_{a,S_a} \neq \mathcal{H}_a$. So, in general, the zero-energy ground state of the ideal Hamiltonian $H_{(S_a)}$ may not be unique. In an attempt to construct an ideal Hamiltonian for which the FQH state $\Phi$ is the unique ground state, we can add additional projection operators and introduce a new ideal Hamiltonian

$$H_{(S_a)} = \sum_a \sum_{\text{a-electron clusters}} (P_{S_a}^{(a)} + P_{\tilde{S}_a}^{(a)}),$$

where $P_{\tilde{S}_a}^{(a)}$ is a projection operator into the space $\mathcal{H}_{a,S_a}$ and $\mathcal{H}_{a,S_a}$ is a subspace of $\mathcal{H}_{a,S_a}$ formed by vectors that are perpendicular to $\mathcal{H}_a$.

VI. SUMMARY OF GENERAL RESULTS

In Secs. III and V, we have considered a subclass of symmetric polynomials $\Phi((z_i))$ (of infinity variables) that satisfy (a) a unique-fusion condition (see discussion in Sec. III A), (b) an n-cluster condition (see discussion in Sec. IV C), and (c) the translation invariance

$$\Phi((z_i)) = \Phi((z_i - z_j)).$$

The unique-fusion condition requires that when we fuse the variables $z_i$ together to obtain new polynomials, we will always get the same polynomial no matter how we fuse the variables together. The n-cluster condition requires that if we fuse all the variables $z_i$ into clusters of $n$ variables each, the resulting polynomial of the clusters has the Jastrow form

$$\Pi_i \prod_{j < k} (z_i^{(a)} - z_j^{(a)}).$$

We find that each translation invariant symmetric polynomial $\Phi((z_i))$ of the $n$-cluster form and satisfying the unique-fusion condition is characterized by a set of non-negative integers $(m;S_2,\ldots,S_n)$. However, not all sets of non-negative integers $(m;S_2,\ldots,S_n)$ can be realized by such symmetric polynomials. The $(m;S_2,\ldots,S_n)$ that correspond to existing translation invariant symmetric polynomials (that satisfy the $n$-cluster and the unique-fusion conditions) must satisfy certain conditions.

First, $m$ and $S_a$ must satisfy [see Eqs. (27) and (40)]

$$m > 0, \quad mn = \text{even},$$

$$2S_a = 0 \mod n.$$  

From $m, S_2, \ldots, S_n$ and $S_1 = 0$, we can determine $S_a$ for any $a > 1$ [see Eq. (33)]

$$S_{avz} = S_a + kS_n + \frac{k(k-1)mn}{2} + kma.$$  

Those $S_a$ must satisfy [see Eqs. (11) and (18)]

$$\Delta_2(a,a) = \text{even},$$

$$\Delta_2(a,b) \equiv 0, \quad \Delta_3(a,b,c) \equiv 0,$$

where

$$\Delta_2(a,b) = S_{avb} - S_a - S_b,$$

$$\Delta_3(a,b,c) = S_{avb + c} - S_{avb} - S_{b + c} - S_{avc} + S_a + S_b + S_c.$$  

$S_a$'s also satisfy another condition which is harder to describe. To describe the new condition, we first note that the sequence $\{S_a\}_a$ can be encoded by another sequence of non-negative integers $n_i$, where $i = 0, 1, \ldots$. To obtain $n_i$ from $S_a$, we introduce $l_a^{(i)} = S_a - S_{a-1}$ for $a = 1, 2, \ldots$. Then, $n_i$ is the number of $l_a^{(i)}$'s that satisfy $l_a^{(i)} = i$. The two sequences, $\{S_a\}_a$ and $\{n_i\}_i$, have a one-to-one correspondence and can faithfully represent each other. The number $n_i$ can be regarded as the boson occupation number that was used to characterize FQH states in the thin cylinder limit. $n_i$ is also used to label Jack polynomials that describe FQH states.

Now, let us introduce $2J + 1$ orbitals $|m_i\rangle$, $m = -J, -J + 1, \ldots, J - 1, J$, which form a representation of SU(2) with an angular momentum $J$. (Here $2J$ is an integer.) We can create a many-boson state $|\{n_i\}\rangle$ by putting $n_i$ bosons into the $m = l - J$ orbitals. Then, $S_a$ must be such that [see Eq. (16)]

$$P_{N,l - J - S_a} \cdots P_{3,3l - S_a} P_{2,2l - S_a} |\{n_i\}\rangle \neq 0,$$

where $P_{a,l}$ is a projection operator that projects into the subspace where any $a$ particles have a total angular momentum equal to $J_a$ or less and $N$ is the number of particles in $|\{n_i\}\rangle$.

We will call $(m;S_2,\ldots,S_n)$ an $S$ vector and denote it as

$$S = (m;S_2,\ldots,S_n).$$

We find that translation invariant symmetric polynomials (that satisfy the n-cluster and the unique-fusion conditions) are labeled by the $S$ vectors that satisfy Eqs. (43), (45), and (47).

We would like to stress that Eqs. (43), (45), and (47) are only necessary conditions for $(m;S_2,\ldots,S_n)$ to describe a translation invariant symmetric polynomial of the $n$-cluster form and satisfying the unique-fusion condition. We do not know if those conditions are sufficient or not. Some $S$ vec-
tors that satisfy Eqs. (43), (45), and (47) may not correspond to an existing symmetric polynomial. Also, there may be more than one symmetric polynomial that are described by the same $S$ vectors that satisfy Eqs. (43), (45), and (47). Each such symmetric polynomial corresponds to a FQH state. By solving Eqs. (43), (45), and (47), we can obtain $(m;S_2,\ldots, S_n)$’s that correspond to the Laughlin states, the Pfaffian state,\cite{14} the parafermion states,\cite{19} and many new non-Abelian states.

We also obtained some additional results. Our numerical studies of Eqs. (43) and (45) suggest that all solutions of the equations satisfy

$$h_a^{\text{sc}} = h_{m-a}^{\text{sc}}, \quad (48)$$

where

$$h_a^{\text{sc}} = S_a - \frac{a S_a}{n} + \frac{am}{2} - \frac{a^2 m}{2n}, \quad (49)$$

although we cannot derive Eq. (48) analytically. Such a relation implies that

$$S_{m-a} = S_a + \frac{n-2a}{n} S_n,$$

$h_m^{\text{sc}}$’s also satisfy

$$h_m^{\text{sc}} = h_{m-1}^{\text{sc}}, \quad h_m^{\text{sc}} \geq 0,$$

where $a \mod n = a \mod n$.

From Eq. (49), we find that $(h_1^{\text{sc}}, \ldots, h_n^{\text{sc}})$ and $(S_2, \ldots, S_n)$ have a one-to-one correspondence. They can faithfully represent each other. Due to the one-to-one relation between $S_i$ and $h_i^{\text{sc}}$, we can also use $n$, $m$, and $h_1^{\text{sc}}, \ldots, h_n^{\text{sc}}$ to characterize the pattern of zero in the symmetric polynomial $\Phi(z_i)$. We will package the data in the form

$$h = \left( \frac{m}{n}; h_1^{\text{sc}}, \ldots, h_n^{\text{sc}} \right),$$

and call $h$ an $h$ vector. We see that patterns of zeros in a symmetric polynomial can also be described by the $h$ vectors.

Each symmetric polynomial described by the pattern of zeros $\{S_i\}$ is related to a CFT generated by simple-current operators which have an Abelian fusion rule (see Appendix B). $(h_1^{\text{sc}}, \ldots, h_n^{\text{sc}})$ turn out to be the scaling dimensions of those simple-current operators. Since $\Delta_3(a,b,c)$ only depends on $h_a^{\text{sc}},$

$$\Delta_3(a,b,c) = h_{ab+bc}^{\text{sc}} - h_{ab}^{\text{sc}} - h_{bc}^{\text{sc}} + h_a^{\text{sc}} + h_b^{\text{sc}} + h_c^{\text{sc}},$$

and $\Delta_3(a,b,c) \geq 0$ is a property of the simple-current CFT.

Condition (47) is hard to check. So let us consider Eqs. (43) and (45) only. One class of solutions of Eqs. (43) and (45) is given by

$$h_a^{\text{sc}} = h_{m/2}^{\text{sc}} = \frac{(a\mod n)[n-(a\mod n)]}{n}.$$  

This class of solutions corresponds to the $Z_n$ parafermion CFT which is generated by simple-current operators $\psi_b$ that satisfy an Abelian fusion rule $\psi_a \psi_b = \psi_{a+b}$ and $\psi_b = \psi_{n/2}$.

The scaling dimensions of $\psi_b$ is given by the above $h_a^{\text{sc}}$. The parafermion states introduced in Ref. 19 are related to such a class of solutions.

A more general class of solutions of Eqs. (43) and (45) corresponds to generalized parafermion CFTs. A generalized parafermion CFT is generated simple-current operators that have the following dimensions:

$$h_a^{\text{sc}} = h_a^{\text{sc,PF}} = \frac{(ka\mod n)(n-(ka\mod n))}{n}.$$  

Those solutions represent a new class of non-Abelian FQH states, which will be called generalized parafermion states.

It turns out that all the solutions of Eqs. (43) and (45) are closely related to parafermion CFTs; i.e., a solution $h_a^{\text{sc}}$ satisfies

$$h_a^{\text{sc}} = \sum_i \frac{k_i}{2} h_{a_i}^{\text{PF}} \mod 1, \quad (50)$$

where $k_i$’s are positive or negative integers and $h_{a_i}^{\text{PF}}$’s are the scaling dimensions of the parafermion operators in some parafermion CFTs labeled by $i$. They are given by

$$h_{a_i}^{\text{PF}} = h_{a_i}^{PF,PF},$$

for certain integers $k$ and $n'$, where $n'$ is a factor of $n$.

If $h_a^{\text{sc}} = h_a^{PF,PF}$, then the solution corresponds to an existing symmetric polynomial generated by a (generalized) parafermion CFT. If $h_a^{\text{sc}} = h_a^{PF,PF}$, then the solution corresponds to the square root of a symmetric polynomial generated by a (generalized) parafermion CFT. Therefore, the later solution does not correspond to any existing symmetric polynomials. Numerical experiments suggest that the later case always have $\Delta_3(a,b,c) = \text{odd}$ for some $a$, $b$, and $c$. This motivates us to introduce the new condition

$$\Delta_3(a,b,c) = \text{even} \quad (51)$$

to exclude those illegal cases. Conditions (43), (45), and (51) provide an easy way to obtain $S_a$’s that may correspond to existing symmetric polynomials. The new condition (51) is a generalization of necessary conditions $\Delta_3(a,a,a) = \text{even}$ [see Eq. (19)], (B8), and $S_a \neq 1$.

Our numerical studies suggest that the solutions of Eqs. (43), (45), and (51) give rise to $h_a^{\text{sc}}$ that satisfy

$$h_a^{\text{sc}} = \sum_i k_i h_{a_i}^{\text{PF}} \mod 2.$$  

Those solutions also have the properties that $m=\text{even}$ and $S_a=\text{even}$. We also find that for $S_a$ satisfying Eqs. (43) and (45), the corresponding $n_l$ is a periodic function of $l$ for $l \equiv 0$ with a period $m$.

$$n_l = n_{l+m}.$$  

The $n_l$’s satisfy
Thus, if $S_1$ and $S_2$ are two solutions of Eqs. (43) and (45), then

$$S = k_1S_1 + k_2S_2$$

is also a solution for any non-negative integers $k_1$ and $k_2$. Therefore, we can divide the solutions of Eqs. (43) and (45) into two classes: primitive solutions and nonprimitive solutions. The primitive solutions are those that cannot be written as a sum of two other solutions. All solutions of Eqs. (43) and (45) are linear combinations of primitive solutions with non-negative integral coefficients.

As an application of the product rule, let us consider a symmetric polynomial of $n$-cluster form $\Phi(\{z_i\})$ which is described by $(m;S_2,\ldots,S_n)$. We can construct a new symmetric polynomial of $n$-cluster form from $\Phi(\{z_i\})$,

$$\tilde{\Phi}(\{z_i\}) = \Phi(\{z_i\}) \prod_{i<j} (z_i - z_j)^q,$$

where $q$ is even. The symmetric polynomial $\tilde{\Phi}(\{z_i\})$ is described by

$$(\tilde{m};\tilde{S}_2,\ldots,\tilde{S}_n) = \left( m + nq; S_2 + q, \ldots, S_n + \frac{qn(n-1)}{2} \right).$$

**VII. GENERAL STRUCTURE OF THE SOLUTIONS**

The $S$ vectors that satisfy Eqs. (43) and (45) have some general properties. In this section, we will discuss those properties.

**A. $n$-cluster polynomial as $\kappa n$-cluster polynomial**

Let $P(\{z_i^{(0)}\})$ be a derived symmetric polynomial of $n$-cluster form described by $(m;S_2,\ldots,S_n)$. From Eq. (26), we see that $P(\{z_i^{(a)}\})$ can also be viewed as a symmetric polynomial of $\kappa n$-cluster form where $\kappa$ is a positive integer. When viewed as a $\kappa n$-cluster polynomial, $P(\{z_i^{(a)}\})$ is described by $(\kappa m,S_2,\ldots,S_n)$, where $S_{m+1},\ldots,S_{\kappa n}$ are obtained from $(m;S_2,\ldots,S_n)$ through Eq. (33).

The filling fraction $\nu=1/q$ Laughlin state $\Phi_{1/q} = \prod_{i<j} (z_i - z_j)^q$ has a one-cluster form. Thus, $\Phi_{1/q}$ can also be viewed as an $n$-cluster polynomial for any positive $n$. When viewed as an $n$-cluster polynomial, the $\nu=1/q$ Laughlin state is described by

$$(m;S_2,\ldots,S_n) = \left( nq; q, \ldots, \frac{qn(n-1)}{2} \right).$$

Such a $\nu=1/q$ Laughlin state always appears as a solution of Eqs. (43) and (45) for any $n$.

**B. Products of symmetric polynomials**

Let $P(\{z_i^{(0)}\})$ and $P'(\{z_i^{(0)}\})$ be two derived symmetric polynomials of $n$-cluster form described by $(m;S_2,\ldots,S_n)$ and $(m';S'_2,\ldots,S'_n)$, respectively. Then, their product $P(\{z_i^{(a)}\})P'(\{z_i^{(a)}\})$ is also a symmetric polynomial of $n$-cluster form. $P(\{z_i^{(0)}\})$ is described by

$$(\tilde{m},\tilde{S}_2,\ldots,\tilde{S}_n) = (m + m'; S_2 + S'_2,\ldots,S_n + S'_n).$$

This is because the pattern of zeros of $\tilde{P}(\{z_i^{(a)}\})$ is related to the patterns of zeros of $P(\{z_i^{(0)}\})$ and $P'(\{z_i^{(0)}\})$ through

$$\tilde{D}_{ab} = D_{ab} + D'_{ab}.$$

Also, the relation between $(m;S_2,\ldots,S_n)$ and $D_{ab}$ is linear [see Eqs. (10), (12), and (33)]. Therefore, if two $S$ vectors, $S'$, and $S''$ describe two existing symmetric polynomials, then their sum $S=S'+S''$ also describes an existing symmetric polynomial, whose fillings fractions are reciprocally additive.

Indeed, the solutions of Eqs. (43) and (45) have a structure that is consistent with the above result. We note that Eqs. (43) and (45) are linear in the $S$ vector $S=(m;S_2,\ldots,S_n)$.

**VIII. SOME EXAMPLES**

In this section, we will give some examples of symmetric polynomials described by the $S$ vector $(m;S_2,\ldots,S_n)$ that satisfy Eqs. (43), (45), and (51).

**A. $n=1$ cases**

If $n=1$, the different patterns of zeros are characterized by an even integer $m$. We find $S_a = mA(a-1)/2$ and $D_{ab} = mab$. Each even $m$ corresponds to a $\nu=1/m$ Laughlin state

$$\Phi_{1/m}(\{z_i\}) = \prod_{i<j} (z_i - z_j)^m.$$

We have introduced three equivalent ways to describe a pattern of zeros: the $S$ vector $(m;S_2,\ldots,S_n)$, the $h$ vector $(\frac{m}{n};h_1^2,\ldots,h_n^2)$, and the boson occupation number $n_j = (n_0,\ldots,n_{m-1})$. For the $\nu=1/m$ Laughlin state those data are given by

$$\Phi_{1/m} : \quad S = (m),$$

$$\begin{pmatrix} m \\ \frac{m}{n} \end{pmatrix} = (m;0),$$

$$(n_0,\ldots,n_{m-1}) = (1,0,\ldots,0).$$

**B. $n=2$ cases**

If $n=2$, the different patterns of zeros are characterized by two integers $m,S_2$. The following two sets of $m$, $S_2$ are the primitive solutions of Eqs. (43) and (45):

$$(m;S_2) = (1;0), \quad (m';S'_2) = (4;2).$$
Let us discuss the solution \((m; S_2)=(1; 0)\) in more detail. The corresponding boson occupation numbers are

\[
(n_0, n_1, \ldots) = (2, 2, 2, \ldots),
\]

where there are two bosons occupying each orbital. Let us check condition (47) for the \(J=1/2\) case wherein there are only two orbitals. This leads to a state \([2, 2]\) with four bosons described by the wave function

\[
\Phi_{[2, 2]} = z_1 z_2 + z_1 z_3 + z_1 z_4 + z_2 z_3 + z_2 z_4 + z_3 z_4.
\]

On sphere the above wave function becomes (see Appendix A)

\[
\Phi_{[2, 2]}^{sp} = S[v_1 v_2 u_3 u_4],
\]

where \(S\) is the symmetrization operator. Since \(S_4=2\), we find that \(J_4=4J-S_4=0\) and \(P_{4J}\) is a projection into the subspace with vanishing total angular momentum. A direct calculation reveals that \(P_{4J}\Phi_{[2, 2]}^{sp}=0\). Thus, \((m; S_2)=(1; 0)\) does not satisfy condition (47) and does not correspond to any translation invariant symmetric polynomial.

Now let consider \(m, S_2\) that satisfy a new condition (51) in addition to Eqs. (43) and (45). The following two sets of \(m, S_2\) are the primitive solutions:

\[
\Phi_{2; 2}^{2; 2} : (m; S_2) = (2; 0),
\]

\[
\left(\begin{array}{c}
(2) \\
(2; 0)
\end{array}\right),
\]

\[
(n_0, \ldots, n_{m-1}) = (2, 0, 0),
\]

and

\[
\Phi_{1/2} : (m; S_2) = (4; 2),
\]

\[
\left(\begin{array}{c}
(4) \\
(2; 0, 0)
\end{array}\right),
\]

\[
(n_0, \ldots, n_{m-1}) = (1, 0, 1, 0).
\]

Here, we also listed the corresponding \(h\) vector \(h = (h_1^{sc}, \ldots, h_m^{sc})\) and the boson occupation numbers \((n_0, \ldots, n_{m-1})\).

Let us discuss the solution \((m; S_2)=(2; 0)\) in more details. We find \((S_1, S_2, S_3, S_4) = (0, 0, 2, 4)\) and

\[
\begin{pmatrix}
D_{11} & D_{12} \\
D_{21} & D_{22}
\end{pmatrix} = \begin{pmatrix}
0 & 2 \\
2 & 4
\end{pmatrix},
\]

which means that we will have no zero if we bring two particles together and a second order zero if we bring a third particle to a two-particle cluster. Such a pattern of zeros describes the following translation invariant symmetric polynomial:

\[
\Phi_{2; 2}^{2; 2}([z_j]) = A \left(\frac{1}{z_1 - z_2 z_3 - z_4} \cdots \right) \prod_{i<j} (z_i - z_j),
\]

where \(z\) is the filling fraction \(\nu=1/2\) bosonic Pfaffian state.\(^{14}\) Here, \(A\) is the antisymmetrization operator. The Pfaffian state can be written as a correlation of the following operator in a CFT:

\[
V_{\nu}(z) = \psi(z)e^{i\phi(z)},
\]

where \(\phi(z)\) is the Majorana fermion operator in the Ising CFT (which is also the \(Z_2\) parafermion CFT). The \(h_4^{1/2} = 1/2\) in the \(h\) vector is the scaling dimension of \(\psi\).

The other solution \((m; S_2)=(4; 2)\) gives rise to the following \(D_{ab}\):

\[
\begin{pmatrix}
D_{11} & D_{12} \\
D_{21} & D_{22}
\end{pmatrix} = \begin{pmatrix}
0 & 4 \\
4 & 8
\end{pmatrix}.
\]

It describes the symmetric polynomial

\[
\Phi_{1/2}([z_j]) = \prod_{i<j} (z_i - z_j)^2,
\]

which is the filling fraction \(\nu=1/2\) bosonic Laughlin state. The \(\nu=1/2\) Laughlin state can be written as a correlation of the following operator in the Gaussian model [or \(U(1)\) CFT]:

\[
V_{\nu}(z) = e^{i\mathbb{Z}\phi(z)}.
\]

We note that the \(\nu=1/2\) bosonic Laughlin state is characterized by a pattern of boson occupation numbers \((n_j) = (1, 0, 1, 0, \ldots)\) and that the \(\nu=1\) bosonic Pfaffian state is characterized by \((n_j) = (2, 0, 2, 0, \ldots)\). Those patterns match the boson occupation distributions of the two states in the thin cylinder limit.\(^{40-44}\) This appears to be a general result: the \(n_j\) that characterize a symmetric polynomial correspond to one of the boson occupation distribution of the same state in the thin cylinder limit. Or more precisely: the \(n_j\) that characterize a symmetric polynomial correspond to the boson occupation distribution of the same state in the thin sphere limit.\(^{45}\)

The solution \((m; S_2)=(4; 0)=2 \times (2; 0)\) gives rise to the following \(D_{ab}\):

\[
\begin{pmatrix}
D_{11} & D_{12} \\
D_{21} & D_{22}
\end{pmatrix} = \begin{pmatrix}
0 & 4 \\
4 & 8
\end{pmatrix}.
\]

It describes a symmetric polynomial which is the square of \(\Phi_{2; 2}^{2; 2}\):

\[
\Phi_{2; 2}^{2; 2}([z_j]) = \Phi_{2; 2}^{2; 2}([z_j])
\]

\[
= \left(\frac{1}{z_1 - z_2 z_3 - z_4} \cdots \right) \prod_{i<j} (z_i - z_j)^2.
\]

Let us consider another translation invariant symmetric polynomial

\[
\Phi_{d_{5/2}}([z_j]) = S \left(\frac{1}{(z_1 - z_2)^2} \frac{1}{(z_3 - z_4)^2} \cdots \right) \prod_{i<j} (z_i - z_j)^2,
\]

where \(S\) is the symmetrization operator. We note that \(\Phi_{d_{5/2}}([z_j])\) and \(\Phi_{2; 2}^{2; 2}([z_j])\) have the same pattern of zeros given by Eq. (58). In the following, we would like to show that

\[
\Phi_{d_{5/2}}([z_j]) \propto \Phi_{2; 2}^{2; 2}([z_j]).
\]
We first note that $\Phi_{Z_2 Z_2}(\{z_i\})$ can be written as a correlation of the following operator in a CFT:

$$V_\varepsilon(z) = \lambda_1(z)\lambda_2(z)e^{i\varepsilon(z)},$$

where $\lambda_1(z)$ is the Majorana fermion operator in an Ising CFT and $\lambda_2(z)$ is the Majorana fermion operator in another Ising CFT. Thus, the operator $V_\varepsilon(z) = \lambda_1(z)\lambda_2(z)e^{i\varepsilon(z)}$ is an operator in Ising $\times$ Ising $\times$ U(1) CFT.

The state $\Phi_{d\varepsilon}(\{z_i\})$ can also be written as a correlation of the following operator in a CFT:

$$V_{\partial\bar{\varepsilon}}(z) = \partial_z\bar{\varepsilon}(z)e^{i\partial_z\bar{\varepsilon}(z)},$$

where $\bar{\varepsilon}(z)$ is the field in a second U(1) CFT. Thus, the operator $V_{\partial\bar{\varepsilon}}(z) = \partial_z\bar{\varepsilon}(z)e^{i\partial_z\bar{\varepsilon}(z)}$ is an operator in U(1) $\times$ U(1) CFT.

From the bosonization of the Ising $\times$ Ising CFT, one can show that the Ising $\times$ Ising CFT is equivalent to the U(1) CFT and $\lambda_1\lambda_2$ has the same $\mathcal{N}$-body correlation functions as $\partial_z\bar{\varepsilon}(z)$. Thus, $\Phi_{d\varepsilon}(\{z_i\}) = \Phi_{Z_2 Z_2}(\{z_i\})$. Numerical calculations have suggested that $\Phi_{d\varepsilon}(\{z_i\})$ has gapless excitations and is unstable.

Next, let us consider the following two polynomials:

$$\Phi_{Z_2 Z_2}(\{z_i\}) = \Phi_{Z_2 Z_2}^e(\{z_i\}) = \prod_{i<j}(z_i - z_j)^2,$$

and

$$\Phi_{Z_2 Z_2}(\{z_i\}) = \Phi_{Z_2 Z_2}^o(\{z_i\}) = \prod_{i<j}(z_i - z_j)^2,$$

where $S_q = S$ when $q$ is even and $S_q = A$ when $q$ is odd. The two symmetric polynomials have the same pattern of zeros $D_{AB}$. However, when $q > 2$, the two polynomials are different. Those polynomials provide us examples that there can be more than one polynomial that have the same pattern of zeros.

### C. $n = 3$ cases

If $n = 3$, the different patterns of zeros are characterized by three integers $m, s_2, s_3$. The following two sets of $m, s_2, s_3$ are the primitive solutions of Eqs. (43) and (45):

$$\Phi_{Z_2 Z_2} : (m; s_2, s_3) = (2; 0, 0),$$

$$\left(\frac{m}{n}; h_{1}, \ldots, h_{n} \right) = \left(\frac{2}{3}, \frac{2}{3}, 0\right),$$

$$\left(n_0, \ldots, n_{m-1} \right) = (3, 0),$$

and

$$\Phi_{1/2} : (m; s_2, s_3) = (6; 2, 6),$$

$$\left(\frac{m}{n}; h_{1}, \ldots, h_{n} \right) = \left(\frac{6}{3}; 0, 0, 0\right),$$

$$\left(n_0, \ldots, n_{m-1} \right) = (1, 0, 1, 0, 1, 0).$$

When $n \equiv 1$, we find that the solutions of Eqs. (43) and (45) automatically satisfy Eq. (51).

From the $h$ vector of the solution $(m; s_2, s_3) = (2; 0, 0)$, we find that the corresponding polynomial $\Phi_{Z_2 Z_2}^e$ describes the $Z_3$ Read–Rezayi parafermion state since $h_{1}^e = h_{2}^e = 2/3$ in the $h$ vector match the scaling dimensions of the simple-current operators in the $Z_3$ parafermion CFT. Such a state has a filling fraction $\nu = 3/2$. Here, we have been using $\Phi_{qimZ_n}$ to denote a $Z_n$ parafermion state. We will follow such a convention for the rest of this paper. The second solution $(m; s_2, s_3) = (6; 2, 6)$ describes the $\nu = 1/2$ Laughlin state.

### D. $n = 4$ cases

When $n = 4$, the different patterns of zeros are characterized by four integers $m, s_2, s_3, s_4$. The primitive solutions of Eqs. (43) and (45) are given by the following three sets of $m, s_2, s_3, s_4$:

$$(m; s_2, s_3, s_4) = (1; 0, 0, 0, 0), (2; 0, 1, 2), (8, 2, 6, 12).$$

The solution $S = (2; 0, 1, 2)$ is the same as solution $S = (1; 0, 0, 0)$ for the $n = 2$ case (i.e., the two solutions give rise to the same sequence $(S_{n}^a, a = 1, 2, 3, \ldots)$). Such a solution does not satisfy Eq. (47), as shown in Sec. VIII B.

The solution $S = (1; 0, 0, 0)$ does not satisfy Eq. (47) either. Let us check condition (47) for the $J = 1/2$ case wherein there are only two orbitals. This leads to a state $\{4, 4\}$ with eight bosons. On a sphere, such a state is given by

$$\Phi_{[4, 4]}^p = \mathcal{S}_{[4, 4]} = \mathcal{S}_{[4, 4]} = \mathcal{S}_{[4, 4]}.$$

Since $S_4 = 4$, we find that $J = 8 - S_4 = 0$ and $P_{s, s}$ is a projection into the subspace with vanishing total angular momentum. Explicit calculation shows that the state $\Phi_{[4, 4]}^p$ has a vanishing projection onto the $J_{\text{tot}} = 0$ subspace.

Thus, we consider the solutions of Eqs. (43), (45), and (51) to exclude those invalid cases. The primitive solutions of Eqs. (43), (45), and (51) are:

$$\Phi_{4/2Z_4} : (m; s_2, \ldots, s_4) = (2; 0, 0, 0),$$

$$\left(\frac{m}{n}; h_{1}^e, \ldots, h_{n}^e \right) = \left(\frac{2}{3}, \frac{3}{4}, 0\right),$$

$$\left(n_0, \ldots, n_{m-1} \right) = (4, 0);$$

$$\Phi_{2/2Z_4} : (m; s_2, \ldots, s_4) = (4; 0, 2, 4),$$

$$\left(\frac{m}{n}; h_{1}^e, \ldots, h_{n}^e \right) = \left(\frac{4}{3}, \frac{1}{2}, 0, \frac{1}{2}\right),$$

$$\left(n_0, \ldots, n_{m-1} \right) = (2, 0, 2, 0);$$

$$\Phi_{1/2} : (m; s_2, \ldots, s_4) = (8; 2, 6, 12),$$

$$\left(\frac{m}{n}; h_{1}^e, \ldots, h_{n}^e \right) = \left(\frac{8}{3}; 0, 0, 0\right),$$

$$\left(n_0, \ldots, n_{m-1} \right) = (8, 0, 0, 0).$$
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\[ (\frac{m}{n}; h_1^{\infty}, \ldots, h_n^{\infty}) = \left( \frac{8}{4}; 0,0,0,0 \right), \]

\[ (n_0, \ldots, n_{m-1}) = (1,0,1,0,1,0,1,0). \]

Among those three primitive solutions, only \( \Phi_{4/2;Z_4} \) is new. From the \( h \) vector in Eq. (63), we find that the solution \((m; S_2, S_3, S_4) = (2,0,0,0,0)\) describes the \( Z_4 \) parafermion state \( \Phi_{4/2;Z_4} \) with \( \nu = 2 \).

The solution \((m; S_2, S_3, S_4) = (4; 0,2,4)\) is the same as the \( \nu = 1 \) bosonic Pfaffian state \((m; S_2) = (2; 0)\) discussed before. So the \( h \) vectors of the two solutions, \((\frac{m}{n}; h_1^{\infty}, \ldots, h_n^{\infty}) = (\frac{4}{4}; 2,0,0,0)\) and \((\frac{m}{n}; h_1^{\infty}, h_n^{\infty}) = (\frac{3}{4}; 2,0,0,0)\), characterize the same state. In fact, the repeated \( \left( \frac{4}{4}; 0,0,0 \right) \) pattern in \((h_1^{\infty}, \ldots, h_n^{\infty})\) implies that \( (\frac{m}{n}; h_1^{\infty}, \ldots, h_n^{\infty}) = (\frac{4}{4}; 2,0,0,0) \) can be reduced to \( (\frac{3}{4}; 2,0,0,0) \). Also, the two solutions lead to the same pattern of boson occupation numbers: \((n) = (2,0,2,0,2,0,2,\ldots)\), again indicating that the two solutions describe the same state.

The solution \((m; S_2, S_3, S_4) = (8; 2,6,12)\) describes the \( \nu = 1/2 \) Laughlin state \( \Phi_{1/2} \) characterized by \((\frac{m}{n}; h_1^{\infty}, \ldots, h_n^{\infty}) = (\frac{1}{4}; 0,0,0,0)\). We have seen that \( h_{1/2} = (\frac{3}{4}; 2) \), \( h_{2/4} = (\frac{3}{2}; 0) \), \( h_{3/6} = (\frac{1}{2}; 0,0,0,0) \), and \( h_{4/8} = (\frac{3}{4}; 0,0,0,0) \) all describe the same \( \nu = 1/2 \) Laughlin state \( \Phi_{1/2} \). All the above solutions share the same pattern of boson occupation numbers: \((n) = (1,0,1,0,1,0,1,0,\ldots)\).

The product state \( \Phi_{4/2;Z_4} \Phi_{2/2;Z_2} \) described by

\[ \Phi_{4/2;Z_4} \Phi_{2/2;Z_2} : (m; S_2, \ldots, S_n) = (6; 0,2,4), \]

\[ (\frac{m}{n}; h_1^{\infty}, \ldots, h_n^{\infty}) = \left( \frac{6}{4}; 1, 5, 4, 0 \right), \]

\[ (n_0, \ldots, n_{m-1}) = (2,0,2,0,0,0,0) \]
is a possible stable \( \nu = 2/3 \) FQH state. Note that the above \( h \) vector is the sum of the \( h \) vectors of the \( Z_2 \) and \( Z_4 \) parafermion states.

E. \( n = 5 \) cases

When \( n = 5 \), conditions (43) and (45) have the following three sets of primitive solutions:

\( \Phi_{5/2;Z_5} : (m; S_2, \ldots, S_n) = (2; 0,0,0,0,0), \)

\[ (\frac{m}{n}; h_1^{\infty}, \ldots, h_n^{\infty}) = \left( \frac{2}{5}, \frac{4}{5}; 4, 6, 4, 0 \right), \]

\[ (n_0, \ldots, n_{m-1}) = (5, 0); \]

\( \Phi_{5/8;Z_2^2} : (m; S_2, \ldots, S_n) = (8; 0,2,6,10), \)

\[ (\frac{m}{n}; h_1^{\infty}, \ldots, h_n^{\infty}) = \left( \frac{8}{5}, \frac{6}{5}; 4, 4, 6, 0 \right), \]

\[ (n_0, \ldots, n_{m-1}) = (2,0,1,0,2,0,0,0); \]

\( \Phi_{1/2} : (m; S_2, \ldots, S_n) = (10; 2,6,12,20), \)

\[ (\frac{m}{n}; h_1^{\infty}, \ldots, h_n^{\infty}) = \left( \frac{10}{5}, 0, 0, 0, 0, 0 \right), \]

\[ (n_0, \ldots, n_{m-1}) = (1,0,1,0,1,0,1,0,1). \]

All other solutions are linear combinations of the above three solutions.

\( (\frac{m}{n}; h_1^{\infty}, \ldots, h_n^{\infty}) = (\frac{8}{5}, \frac{6}{5}, \frac{4}{5}, \frac{3}{5}, \frac{2}{5}, \frac{1}{5}, 0) \) describes a \( Z_5 \) parafermion state \( \Phi_{5/2;Z_5} \) studied by Read and Rezayi.\(^{19} \)

\( (\frac{m}{n}; h_1^{\infty}, \ldots, h_n^{\infty}) = (\frac{8}{5}, \frac{6}{5}, \frac{4}{5}, \frac{3}{5}, \frac{2}{5}, \frac{1}{5}, 0) \) describes a new parafermion state \( \Phi_{5/8;Z_2^2} \) with \( \nu = 5/8 \). The third state \( (\frac{m}{n}; h_1^{\infty}, \ldots, h_n^{\infty}) = (\frac{10}{5}, 0, 0, 0, 0, 0, 0) \) describes the \( \nu = 1/2 \) Laughlin state \( \Phi_{1/2} \).

The \( Z_5 \) parafermion state \( \Phi_{5/2;Z_5} \) can be expressed as a correlation of simple-current operators \( \phi_{1} \) in the \( Z_5 \) parafermion CFT. \( \phi_{1} \) has a scaling dimension of \( h_{5}^{\infty} = 4/5 \). The new parafermion state \( \Phi_{5/8;Z_2^2} \) can be expressed as a correlation of simple-current operators \( \phi_{2} \) in the \( Z_5 \) parafermion CFT. \( \phi_{2} \) has a scaling dimension of \( h_{5}^{\infty} = 6/5 \). In general, the simple-current operator \( \phi_{2} \) of a \( Z_n \) parafermion CFT has a scaling dimension

\[ h_{n}^{\infty} = \frac{l(n-1)}{n}. \]

Here, we have been using \( \Phi_{1/im;Z_4^3} \) to denote a generalized \( Z_n \) parafermion state. We will follow such a convention in the rest of this paper.

F. \( n = 6 \) cases

When \( n = 6 \), conditions (43), (45), and (51) have the following four sets of primitive solutions:

\( \Phi_{6/2;Z_6} : (m; S_2, \ldots, S_n) = (2; 0,0,0,0,0), \)

\[ (\frac{m}{n}; h_1^{\infty}, \ldots, h_n^{\infty}) = \left( \frac{2}{6}, \frac{5}{6}, \frac{4}{6}, \frac{3}{6}, \frac{4}{6}, \frac{3}{6}, \frac{2}{6}, 0 \right), \]

\[ (n_0, \ldots, n_{m-1}) = (6, 0). \]

\( \Phi_{2/3;Z_3}, \Phi_{3/2;Z_2} \), and \( \Phi_{1/2} \). Three of the four primitive solutions have been discussed before and only one solution, \( \Phi_{6/2;Z_6} \), is new. The \( \Phi_{6/2;Z_6} \) state is the \( Z_6 \) parafermion state.\(^{19} \)

\( \Phi_{2/3;Z_3} \) and \( \Phi_{3/2;Z_2} \) are the \( Z_2 \) and \( Z_3 \) parafermion states discussed before.

Using \( \Phi_{2/3;Z_3}, \Phi_{3/2;Z_2}, \) and \( \Phi_{6/2;Z_6} \) we can construct some interesting and possibly stable composite states:

\( \Phi_{3/2;Z_3} \Phi_{6/2;Z_6} : (m; S_2, \ldots, S_n) = (6; 0,0,2,4,6), \)

\[ (\frac{m}{n}; h_1^{\infty}, \ldots, h_n^{\infty}) = \left( \frac{6}{3}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, 0 \right), \]

\[ (n_0, \ldots, n_{m-1}) = (3,0,3,0,0,0); \]
\[ \Phi_{2/2:n} : (m; S_2, \ldots, S_n) = \Phi_{2/2:n} \]
\[ \Phi_{3/2:n} : (m; S_2, \ldots, S_n) = \Phi_{3/2:n} \]
\[ \Phi_{4/2:n} : (m; S_2, \ldots, S_n) = \Phi_{4/2:n} \]
\[ \Phi_{5/2:n} : (m; S_2, \ldots, S_n) = \Phi_{5/2:n} \]
\[ \Phi_{2:0} : (m; S_2, \ldots, S_n) = \Phi_{2:0} \]

The filling fractions of those states are given by \( v=n/m \).

**G. \( n=7 \) cases**

When \( n=7 \), conditions (43) and (45) have the following five sets of primitive solutions:

1. \( \Phi_{2/2:n} : (m; S_2, \ldots, S_n) = (2;0,0,0,0,0,0) \)
2. \( \Phi_{3/2:n} : (m; S_2, \ldots, S_n) = (2;0,0,0,0,0,0) \)
3. \( \Phi_{4/2:n} : (m; S_2, \ldots, S_n) = (2;0,0,0,0,0,0) \)
4. \( \Phi_{5/2:n} : (m; S_2, \ldots, S_n) = (2;0,0,0,0,0,0) \)
5. \( \Phi_{2:0} : (m; S_2, \ldots, S_n) = (2;0,0,0,0,0,0) \)

The explicit wave function is given by [see Eq. (25)]

\[ f_{C_n}(z_1, \ldots, z_n) = \frac{1}{z_1 - z_2} \frac{1}{z_2 - z_3} \ldots \frac{1}{z_{n-1} - z_n} \frac{1}{z_n - z_1} \]

To confirm such a result, we note that for \( a=2, \ldots, 7 \), the minimal total powers of \( a \) variables in \( f_{C_n}^a \) are \( s_a \) with \( (s_2, \ldots, s_7) = (-2, -4, -6, -8, -10, -14) \) [see Eq. (29)]. The minimal total powers of \( a \) variables in \( \Pi (z_i - z_j)^2 \) are \( s_a \) with \( (s_2, \ldots, s_7) = (2, 6, 12, 20, 30, 42) \). Thus, the minimal total powers of \( a \) variables in \( \Phi_{7:14:C_n} \) are given by \( s_a = s_a + \tilde{s}_a \). The explicit wave function is given by [see Eq. (25)]

\[ f_{C_n}(z_1, \ldots, z_n) = \frac{1}{z_1 - z_2} \frac{1}{z_2 - z_3} \ldots \frac{1}{z_{n-1} - z_n} \frac{1}{z_n - z_1} \]

In fact, for any \( n \), we have a state \( \Phi_{n:2n:C_n} \) described by \( (h_1^a, \ldots, h_n^a) = (2, \ldots, 2, 0) \). The explicit wave function is given by

\[ \Phi_{n:2n:C_n} = \prod_{i<j} (z_i - z_j)^2 S(f_{C_n}^2(z_1, \ldots, z_n) f_{C_n}^2(z_{n+1}, \ldots, z_{2n})) \]
H. \( n=8 \) cases

When \( n=8 \), conditions (43), (45), and (51) have the following six sets of primitive solutions:

\[
\Phi_{8/2;Z_8} : (m;S_2, \ldots, S_8) = (2;0,0,0,0,0,0,0),
\]

\[
\left( \frac{m}{n} ; h_1^{\text{sc}}, \ldots, h_8^{\text{sc}} \right) = \left( \frac{2 \cdot 7 \cdot 3 \cdot 15}{8 \cdot 8 \cdot 8 \cdot 2}, \frac{15 \cdot 3}{8 \cdot 8 \cdot 2}, \right),
\]

\[
(n_0, \ldots, n_{m-1}) = (8,0);
\]

\[
\Phi_{8/18;Z_8} : (m;S_2, \ldots, S_8) = (18;0,2,8,14,24,36,48),
\]

\[
\left( \frac{m}{n} ; h_1^{\text{sc}}, \ldots, h_8^{\text{sc}} \right) = \left( \frac{18 \cdot 15 \cdot 3 \cdot 7}{8 \cdot 8 \cdot 2}, \frac{7 \cdot 3}{8 \cdot 8 \cdot 2}, \right),
\]

\[
(n_0, \ldots, n_{m-1}) = (2,0,1,0,0,0,2,0,0,0,0,0,0,0,0,0,0);
\]

\[
\Phi_{8/8;C_8/Z_2} : (m;S_2, \ldots, S_8) = (8;0,0,2,4,8,12,16),
\]

\[
\left( \frac{m}{n} ; h_1^{\text{sc}}, \ldots, h_8^{\text{sc}} \right) = \left( \frac{8 \cdot 3}{8 \cdot 2}, \frac{15 \cdot 3 \cdot 2}{8 \cdot 2}, \right),
\]

\[
(n_0, \ldots, n_{m-1}) = (3,0,2,0,3,0,0,0);
\]

\[
\Phi_{4/2;Z_2} ; \Phi_{2;2;Z_2} ; \text{and } \Phi_{1/2};
\]

\[
\Phi_{8/2;Z_8} \text{ is the } Z_8 \text{ parafermion state which can be expressed as a correlation of simple-current operators } \psi_{\text{1}} \text{ in the } Z_8 \text{ parafermion CFT. } \psi_{\text{1}} \text{ has a scaling dimension of } h_{1}^{\text{sc}} = 7/8.
\]

\[
\Phi_{8/18;Z_8} \text{ is a new } Z_8 \text{ parafermion state. } \Phi_{8/18;Z_8} \text{ can be expressed as a correlation of simple-current operators } \psi_{\text{3}} \text{ in the } Z_8 \text{ parafermion CFT. } \psi_{\text{3}} \text{ has a scaling dimension of } h_{1}^{\text{sc}} = 15/8.
\]

Let us discuss the state \( \Phi_{8/8;C_8/Z_2} \) in more details. We note that the \( h \) vector for the \( \Phi_{8/8;C_8/Z_2} \) state is the difference of the \( h \) vectors of the \( \Phi_{8/2;C_2/Z_2} \) state and the \( \Phi_{2;2;Z_2} \) state:

\[
\left( \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2} \right) = (2,2,2,2,2,2,0).
\]

\[
\Phi_{8/8;C_8/Z_2} \text{ has the form}
\]

\[
\Phi_{8/8;C_8/Z_2}(\{z_i\}) = G_{C_8/Z_2}(\{z_i\}) \prod \left( z_i - z_j \right)
\]

where the minimal total power of \( a \) variables in \( G_{C_8/Z_2}(\{z_i\}) \) is given by \( s_a \) with \( (s_2, \ldots, s_8)=(-1, -3, -4, -6, -7, -9, -12) \). We find that

\[
G_{C_8/Z_2}(\{z_i\}) = A[f_{C_8/Z_2}(z_2, \ldots, z_8)f_{C_8/Z_2}(z_9, \ldots, z_{10}) \cdots]
\]

where \( A \) is the antisymmetrization operator and the function \( f_{C_8/Z_2}(z_2, \ldots, z_8) \) is represented by Fig. 4(a).

We would like to mention that a state simpler than \( \Phi_{8/8;C_8/Z_2} \) is \( \Phi_{4/4;C_4/Z_2} \) that has the same pattern of zeros as a composite state of \( Z_4 \) parafermion state \( \Phi_{4/2;Z_4} \):

\[
\Phi_{4/4;C_4/Z_2} = \Phi_{4/2;Z_4} \Phi_{4/2;Z_4}.
\]

Here, \( \sim \) means to have the same pattern of zeros. \( \Phi_{4/4;C_4/Z_2} \) has the form

\[
\Phi_{4/4;C_4/Z_2}(\{z_i\}) = G_{C_4/Z_2}(\{z_i\}) \prod \left( z_i - z_j \right),
\]

where the minimal total power of \( a \) variables in \( G_{C_4/Z_2}(\{z_i\}) \) is given by \( s_a \) with \( (s_2, \ldots, s_8)=(-1, -3, -6) \). We find that

\[
G_{C_4/Z_2}(\{z_i\}) = A[f_{C_4/Z_2}(z_2, \ldots, z_8)f_{C_4/Z_2}(z_9, \ldots, z_{10}) \cdots]
\]

where the function \( f_{C_4/Z_2}(z_2, \ldots, z_8) \) is represented by Fig. 5. In fact, \( f_{C_4/Z_2}(z_2, \ldots, z_8)=\Pi_{i<j}^{-1}z_{ij}^{-4} \) is the only function whose total order of poles for two-, three-, and four-particle clusters are given by 1, 3, and 6, respectively. Such a state is studied recently by Yu.\(^{46}\)
Another interesting state is $\Phi_{6/6;C_6^Z}$ which has the same pattern of zeros as a composite state of $Z_3$ and $Z_6$ parafermion states:

$$\Phi_{6/6;C_6^Z} \approx \Phi_{3/2;Z_3^c} \Phi_{6/2;Z_6^c}.$$ 

$\Phi_{6/6;C_6^Z}$ has the form

$$\Phi_{6/6;C_6^Z} = \mathcal{G}_{C_6^Z}(z_i) \prod (z_i - z_j),$$

where the minimal total power of $a$ variables in $\mathcal{G}_{C_6^Z}(z_i)$ is given by $a_j$ with $(s_2, \ldots, s_6) = (1, 2, 3, 4, 5, -6, -9)$. We find that

$$\mathcal{G}_{C_6^Z}(z_i) = \mathcal{A}[f_{C_6^Z}(z_1, \ldots, z_6)f_{C_6^Z}(z_7, \ldots, z_{12}) \cdots]$$

where the function $f_{C_6^Z}(z_1, \ldots, z_6)$ is represented by Fig. 6(a).

I. $n=9$ cases

When $n=9$, conditions (43), (45), and (51) have the following six sets of primitive solutions:

$$\Phi_{9/2;Z_6^c} : (m; S_2, \ldots, S_n) = (2; 0, 0, 0, 0, 0, 0, 0, 0),$$

$$\left(\frac{m}{n}; h_1^c, \ldots, h_n^c\right) = \left(\frac{2}{9}; \frac{14}{9}, \frac{20}{9}, \frac{20}{9}, \frac{14}{9}, \frac{2}{9}\right),$$

$$(n_0, \ldots, n_{m-1}) = (9, 0);$$

$$\Phi_{9/8;Z_9^c}^{(2)} : (m; S_2, \ldots, S_n) = (8; 0, 0, 0, 2, 6, 10, 14, 18),$$

$$\left(\frac{m}{n}; h_1^c, \ldots, h_n^c\right) = \left(\frac{8}{9}; \frac{14}{9}, \frac{20}{9}, \frac{20}{9}, \frac{14}{9}, \frac{2}{9}\right),$$

$$(n_0, \ldots, n_{m-1}) = (4, 0, 1, 0, 4, 0, 0, 0);$$

$$\Phi_{9/32;Z_9^c}^{(4)} : (m; S_2, \ldots, S_n) = (32; 0, 6, 14, 26, 42, 60, 84, 108),$$

$$\left(\frac{m}{n}; h_1^c, \ldots, h_n^c\right) = \left(\frac{32}{9}; \frac{20}{9}, \frac{14}{9}, \frac{20}{9}, \frac{14}{9}, \frac{2}{9}\right),$$

$$(n_0, \ldots, n_{m-1}) = (2, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 2, 0, 0, 0, 0, 0, 0);$$

$$\Phi_{9/12;C_9^Z} : (m; S_2, \ldots, S_n) = (12; 0, 2, 4, 8, 14, 20, 28, 36),$$

$$\left(\frac{m}{n}; h_1^c, \ldots, h_n^c\right) = \left(\frac{12}{9}; \frac{4}{3}, \frac{4}{3}, \frac{4}{3}, \frac{4}{3}, \frac{4}{3}, \frac{4}{3}, \frac{4}{3}, \frac{4}{3}, \frac{4}{3}\right),$$

$$(n_0, \ldots, n_{m-1}) = (2, 0, 2, 0, 1, 0, 2, 0, 0, 0, 0);$$

$$\Phi_{3/2;Z_3^c}$$ and $\Phi_{1/2}$. 

$\Phi_{9/2;Z_9^c}$ is the old $Z_9$ parafermion state. $\Phi_{9/8;Z_9^c}^{(2)}$ and $\Phi_{9/32;Z_9^c}^{(4)}$ are new $Z_9$ parafermion states, which can be expressed as a correlation of simple-current operators $\theta_2$ and $\theta_4$ in the $Z_9$ parafermion CFT, respectively. We also note that the $h$ vector for the $\Phi_{9/12;C_9^Z}$ state is the difference of the $h$ vectors of the $\Phi_{9/18;C_9^Z}$ state and the $\Phi_{3/2;Z_3^c}$ state:

$$\left(\frac{4}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right) = \left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right) - \left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right).$$

However, we do not know if a symmetric polynomial described by the $h$ vector $(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3})$ really exists or not.

IX. DISCUSSION

In this paper, we use a local condition—the pattern of zeros—to classify symmetric polynomials of infinity variables. We find that symmetric polynomials of $n$-cluster form [see Eqs. (23) and (24)] can be labeled by a set in integers $(m; S_2, \ldots, S_n)$. Those integers must satisfy conditions (43), (45), and (47).

Using the symmetric polynomials labeled by $n$ and $(m; S_2, \ldots, S_n)$, we have constructed a large class of simple FQH states. The constructed FQH states contain both the Laughlin states and non-Abelian states, such as the Read–Rezayi parafermion states, the new generalized parafermion states, and some other new non-Abelian states. Although the constructed FQH states are for bosonic electrons, the bosonic FQH states and the fermionic FQH states have a simple one-to-one correspondence:

$$\Phi_{\text{fermion}} = \Phi_{\text{boson}} \prod_{i<j} (z_i - z_j).$$

We can easily obtain fermionic FQH states from the corresponding bosonic ones.

We have seen that the ground state wave functions of different Abelian and non-Abelian fraction quantum Hall states can be characterized by patterns of zeros $\{S\}$. One may wonder: can we use the data $\{S\}$ to calculate various topological properties of the corresponding fraction quantum Hall state? In Ref. 45, we will show that many topological properties can indeed be calculated from $\{S\}$, such as the number of possible quasiparticle types and their quantum numbers.

However, $\{S\}$ cannot describe all FQH states. More complicated “multicomponent” FQH states, such as the $\nu=2/5$
Abelian FQH state, are not included in our construction. This suggests that certain non-Abelian states, such as the parafermion states, belong to the same class as the simple one-component Laughlin states. Thus, our result can be viewed as a classification of “one-component” FQH states, although the precise meaning of “one-component” remains to be clarified. String-net condensation and the associated tensor category theory provide a fairly complete classification of nonchiral topological orders in two spatial dimensions. We hope the framework introduced in this paper be a step toward a classification of chiral topological orders in two spatial dimensions.

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APPENDIX A: ELECTRONS ON A SPHERE

To test if the FQH Hamiltonian has an energy gap or not, we need to put the FQH state on a compact space to avoid the gapless edge excitations which are always present. In this appendix, we will discuss how to put a FQH state on a sphere. We assume that there is a uniform magnetic field on a sphere with a total $N \Phi_0$ flux quanta. The wave function of one electron in the first Landau level has the form

$$\Psi(\theta, \varphi) = \sum_{m=0}^{N \Phi_0} c_m u^{N \Phi_0} e^{im\theta} \sin^m \theta$$

where $u = \cos(\theta/2) e^{i\varphi/2}$ and $v = \sin(\theta/2) e^{-i\varphi/2}$.

We can also use a complex number $z = u/v$ to parameterize the points on the sphere. In terms of $z$, the wave function becomes

$$\Psi(\theta, \varphi) = e^{-iN \Phi_0 \varphi/2} \sum_{m=0}^{N \Phi_0} c_m e^{im\varphi}$$

$$= e^{-iN \Phi_0 \varphi/2} \frac{1}{(1 + |z|^2)^{N \Phi_0/2}} \sum_{m=0}^{N \Phi_0} c_m e^{im\varphi}$$

We see that we can use a polynomial $\Phi(z)$ to describe the wave function of one electron in the first Landau level:

$$\Phi(z) = \sum_{m=0}^{N \Phi_0} c_m e^{im\varphi}$$

Here, the power of $z$ is equal or less than $N \Phi_0$. Equations (A1) and (A2) allow us to go back and forth between the spinor representation $\Psi(u, v)$ and the polynomial representation $\Phi(z)$ of the states on the sphere.

Since polynomials (A3) represent wave functions on a sphere, hence they form a representation of $SU(2)$ [or $O(3)$] rotation of the sphere. The dimension of the representation is $N \Phi_0 + 1$. Such a representation is said to carry an angular momentum

$$J = \frac{N \Phi_0}{2}$$

The $SU(2)$ Lie algebra is generated by

$$L^z = \hat{z}_1 \hat{J}_z - J_z, \quad L^z = \hat{\partial}_z + 2J_z$$

which satisfy

$$[L^z, L^z] = L^z, \quad [L^z, \hat{L}^z] = 2L^z$$

Those operators act within the space formed by the polynomials of form (A3). The inner product in the space of the polynomials is defined through the inner product of the wave functions

$$\langle \Phi_2 | \Phi_1 \rangle = \int d\theta d\varphi \Psi_2^*(\theta, \varphi) \Psi_1(\theta, \varphi)$$

$$= \int 4 \cos(\theta/2) \sin^3(\theta/2) \frac{d\cos(\theta/2)}{\sin(\theta/2)} d\varphi \Psi_2^* \Psi_1$$

Now let us consider a polynomial of two variables $\Phi(z_1, z_2)$ where the highest power for $z_1$ is $J_1$ and the highest power for $z_2$ is $2J_2$ (here, $2J_1$ and $2J_2$ are integers). $\Phi(z_1, z_2)$ can also be viewed as a representation of $SU(2)$, wherein the generators of the $SU(2)$ Lie algebra are given by

$$L^z = L_1^z + L_2^z, \quad L^z = L_1^z + L_2^z$$

$$L_1 = z_1 \hat{J}_1 - J_1, \quad L_1 = \hat{\partial}_z + 2J_1z_1$$

$$L_2 = z_2 \hat{J}_2 - J_2, \quad L_2 = \hat{\partial}_z + 2J_2z_2$$

$\Phi(z_1, z_2)$ is not an irreducible representation of $SU(2)$. It can be decomposed as $\oplus_{J_1, J_2} H_{J}$ where $H_{J}$ is an angular-momentum $J$ representation of $SU(2)$. We may say that $z_1$ has angular momentum $J_1$ and $z_2$ has an angular momentum $J_2$. Thus, the angular momenta of $\Phi(z_1, z_2)$ are those obtained by combining the angular momentum $J_1$ and the angular momentum $J_2$.

What are the states in the space $H_{J}$? Let $\Phi_{J,m}$, where $m = -J_1, -J_1 + 1, \ldots, J_1$, be the polynomials in the $H_J$ space such that $\Phi_{J,m}$ is the eigenstate of $L^z$ with eigenvalue $m$. Let us also introduce $z = z_1 \pm z_2$. We see that

$$L^z = \hat{\partial}_{z_1} + \hat{\partial}_{z_2} = 2 \hat{\partial}_{z_1}$$

$$L^z = \frac{1}{2} (z_1^2 + z_2^2) \hat{\partial}_{z_1} - z_1 \hat{\partial}_{z_2} + 2J_1 z_1 + 2J_2 z_2$$

$$L^z = z_1 \hat{\partial}_{z_1} + z_2 \hat{\partial}_{z_2} - J_1 - J_2$$

From $L^z \Phi_{J,m} = 0$ and $L^z \Phi_{J,m} = -J \Phi_{J,m}$, we find that

$$\Phi_{J,m} \propto z_1^{J_1} z_2^{J_2} = (z_1 z_2)^{J_1+J_2}$$

$\Phi_{J,m}$’s are generated from $\Phi_{J,m}$ by applying $L^z$’s. Since $L^z$ never reduce the power of $z$, $\Phi_{J,m}$ thus has the form
Thus, decrease the power of zero as $L$ to find the corresponding CFT of a symmetric polynomial together produces a variable with an angular momentum $F$ and contributions $Ca$ of the particle through the spin-statistics theorem. As discussed in Ref. 50, the orbital spin contains two contributions $S_o = S_o^a + h_a$, $S_o^a$ comes from the spin vector and $h_a$ is the intrinsic spin. The intrinsic spin is related to the statistics of the particle through the spin-statistics theorem. The statistics in turn is related to the scaling dimension (for example, bosons always have integral scaling dimensions).

To separate the two contributions, we need to identify the contribution from the spin vector. This can be achieved by noting that the spin vector contribution is proportional to $a$: $S_o^a = Ca$. The key is to find the proportional coefficient $C$.

For this purpose, let us consider $S_{nN_c}$ in Eq. (38). $S_{nN_c}$ is the orbital spin for the bound state of $N_c$ type-$n$ particles. We know that the type-$n$ particles form the Laughlin state (20). For a bound state of $N_c$ type-$n$ particles, its orbital spin $S_{nN_c}$ contains a term linear in $N_c$ which is the contribution from the spin vector and a term quadratic in $N_c$ which is the intrinsic spin. From Eq. (38), we see that the contribution from the spin vector to $S_{nN_c}$ is

$$S_{nN_c}^\text{sv} = N_c \left( S_n - \frac{mn}{2} \right).$$

After replacing $N_c$ by $a/n$, we identify the contribution from the spin vector to $S_a$:

$$S_{a}^{\text{sv}} = a \left( \frac{S_n}{n} - \frac{m}{2} \right). \quad (B2)$$

Thus, the intrinsic spin is

$$h_a = S_a - S_{a}^{\text{sv}} = \left( \frac{S_n}{n} - \frac{m}{2} \right). \quad (B3)$$

$h_a$ is also the scaling dimension of the operator $(V_e)^a$.

2. Symmetric polynomial as a correlation in a conformal field theory

The electron operator $V_e(z)$ in the CFT expression of $\Phi$ [Eq. (B1)] has the form

$$V_e(z) = \phi_1(z) e^{i \phi(z) / \sqrt{2}},$$

where $e^{i \phi / \sqrt{2}}$ is the vertex operator in a Gaussian model. The vertex operator has a scaling dimension $\frac{1}{2} \sqrt{2}$, $\psi_1$ is a simple-current operator; i.e., $\psi_1$ satisfies the following fusion relation:

$$\psi_a \psi_b = \psi_{a+b}, \quad \psi_a = (\psi_1)^a.$$

Such an Abelian fusion rule is closely related to the unique-fusion condition discussed in Sec. III A. If $\Phi(z_1, \ldots, z_N)$ has an $n$-cluster form, $\psi_1$ satisfies

$$\psi_n = (\psi_1)^n \sim 1.$$

$\Phi(z_1, \ldots, z_N)$ can be decomposed according to Eq. (21). The correlation of the Gaussian part $e^{i \phi(z) / \sqrt{2}}$ produces the $\Phi_e$ part of $\Phi$ and the correlation of the simple-current part $\psi_1$ produces the $\Phi_1$ part of $\Phi$.

The intrinsic spin $h_a$ is actually the scaling dimension of the $a$th power of the electron operator, $V_a^n = (V_e)^a$. The scaling dimension of the Gaussian part $e^{i \phi(z) / \sqrt{2}}$ is $\nu^{-1} \frac{a^2}{2} = \frac{am}{2n}$ and

$$h_a^e = h_a - \frac{a^2 m}{2n}. \quad (B4)$$

is the scaling dimension of the simple-current operator $\psi_a$.

We can obtain the scaling dimension $h_a$ of the operator $V_a$ more directly without using the concept of spin vector and orbital spin. First, we note that the derived polynomial $P(\{z_i^{(a)}\})$ can be expressed as a correlation of $V_a(z_i)^a$'s,
Thus, the scaling dimension of
\[ \gamma_1 = h_{a_2} + h_{a_3} + h_{a_4} - h_{a_2} a_{a_3} - h_{a_1} a_{a_4} - h_{a_3} a_{a_2}, \]
where we have used \( h_{a_1} = h_{a} \). The requirement that \( \gamma_1 \geq 0 \) is the third condition in Eq. (45).

3. Generalized vertex algebra

To understand the CFT representation of the symmetric polynomial more deeply, let us consider generalized vertex algebra.\(^5\) The CFT formed by the simple-current operators \( \psi_a \) is a special case of a generalized vertex algebra.

Consider operators \( A(z), B(w), \) etc., which form an operator-product-expansion algebra

\[
A(z)B(w) = \frac{1}{(z-w)^{\alpha_{AB}}} [A]_{\alpha_{AB}}(w) + (z-w)[A]_{\alpha_{AB}-1}(w) + (z-w)^2[A]_{\alpha_{AB}-2}(w) + \cdots,
\]
and

\[
(z-w)^{\alpha_{AB}} A(z)B(w) = \mu_{AB}(w) (z-w)^{\alpha_{AB}} A(z)B(w), \quad \mu_{AB}(w) = \mu_{AB}(w)^{\alpha_{AB}} [A]_{\alpha_{AB}}(w),
\]
where \( \alpha_{AB} \) is a phase factor. Here, \( (z-w)^{\alpha_{AB}} = (z-w)^{\alpha_{AB}} e^{-i\theta_{\alpha_{AB}}} \), where \( \theta_{\alpha_{AB}} = \theta_{\alpha_{AB}} + i \pi < \theta_{\alpha_{AB}} < \pi \).

The operator product in Eq. (B9) is assumed to be radially ordered: \( A(z)B(w) \rightarrow R[A(z)B(w)] \), where

\[
(z-w)^{\alpha_{AB}} R[A(z)B(w)] = \begin{cases} 
(z-w)^{\alpha_{AB}} A(z)B(w) & |z| > |w| \\
\mu_{AB}(w) (z-w)^{\alpha_{AB}} B(w)A(z) & |w| > |z|.
\end{cases}
\]

We see that commutation relation (B10) ensures that the correlation functions of \( A(z) \) and \( B(w) \) are smooth functions. Let \( h_A, h_B \) and \( h_{[AB]} \) be the scaling dimensions of \( A, B, \) and \( [AB] \), then

\[
\alpha_{AB} = h_A + h_B - h_{[AB]}.
\]

The self-consistency of the vertex algebra requires \( \alpha_{AB} \)'s to satisfy\(^5\)

\[
\alpha_{AB} + \alpha_{AC} - \alpha_{AD} = 0 \mod 1,
\]
and \( \alpha_{AB} \)'s to satisfy

\[
\mu_{AB} \mu_{AC} = \mu_{AD} (-)^{\alpha_{AB} \alpha_{AC} - \alpha_{AD}},
\]
where \( D = [BC]_{\theta_{BC}} \).

The fusion rule of the simple-current operators \( \psi_a \) requires that those operators form the following vertex operator algebra:

\[
\psi_a(z) \psi_b(w) = \frac{\epsilon_{ab} \psi_{ab}(w) + O(z-w)}{(z-w)^{h_{a} h_{b} - h_{c} h_{c}}},
\]
and

\[
\psi_a(z) \psi_d(w) = \frac{1 + 2\epsilon_{a} (z-w)^2 T(w) + O[(z-w)^3]}{(z-w)^{2\epsilon_{a}}},
\]
where \( \psi_a = \psi_{a_0} \) and \( \psi_{a_{\text{even}}} = \psi_{a} \).

We see that the above algebra of simple currents \( \psi_a \) is a special case of generalized vertex algebra. We have
\[ \alpha_{ab} = h^a \circ \circ + h^b \circ \circ - h^{ab} \circ \circ. \]

Condition (B11) becomes
\[ \Delta_3(a,b,c) = 0 \mod 1, \quad (B13) \]
where we have used Eqs. (49) and (46). From Eq. (B12), we see that if \( \Delta_3(a,b,c)=\text{odd} \) for certain choices of \( a, b, c \), then \( \mu_{ab} \) cannot be trivial (i.e., \( \mu_{ab}=1 \)). When \( \Delta_3(a,b,c)=\text{even} \) for all \( a, b, c \), then \( \mu_{ab} \) can be a trivial solution \( \mu_{ab}=1 \). Thus condition (51) has a special meaning in CFT.

4. Conditions on the \( h \) vectors

Due to the one-to-one correspondence between the \( S \) vectors and the \( h \) vectors [see Eqs. (34) and (35)], we can translate conditions (43) and (45) on the \( S \) vectors to some conditions on the \( h \) vectors.

Note that an \( h \) vector is specified by \( n, m, \) and \( h_1, \ldots, h_{n-1} \). We extend \( h_1^{\circ \circ}, \ldots, h_{n-1}^{\circ \circ} \) to \( h_n^{\circ \circ} \) for any integer \( a \) by requiring
\[ h_0^{\circ \circ} = 0, \quad h_a^{\circ \circ} = h_{a+n}^{\circ \circ}. \]

Conditions (43) become
\[ S_a = h_a^{\circ \circ} - ah_1^{\circ \circ} + \frac{a(a-1)m}{2n} = \text{non-negative integer}, \]
\[ m > 0, \quad mn = \text{even}, \]
\[ 2nh_1^{\circ \circ} + m = \text{0 mod } n. \quad (B14) \]

From \( 2nh_1^{\circ \circ} + m = 0 \mod n \), we see that \( 2nh_1^{\circ \circ} \) is an integer. From \( 2nh_1^{\circ \circ} - a(2n-1)h_1^{\circ \circ} + a(a-1)m = \text{even integer} \), we see that \( 2nh_1^{\circ \circ} \) are always integers. Also, \( 2nh_2^{\circ \circ} \) are always even integers and \( 2nh_2^{\circ \circ}+m \) are either even or odd all. Since \( h_2^{\circ \circ} = 0 \), thus when \( n=\text{odd} \) \( 2nh_2^{\circ \circ} \) are all even. Only when \( n=\text{even} \) it is \( 2nh_2^{\circ \circ} \) either be all even or all odd.

To generate sets of \( h_n^{\circ \circ} \) that satisfy the above conditions, we will use Eq. (34). Setting \( a=1 \) in Eq. (34), we get \( 2nh_1^{\circ \circ} = m(n-1) - 2S_n \). We see that when \( n=\text{odd} \), \( 2nh_1^{\circ \circ} = \text{even} \). When \( n=\text{even} \), \( 2nh_1^{\circ \circ} = \text{even} \) when \( m=\text{even} \) and \( 2nh_1^{\circ \circ} = \text{odd} \) when \( m=\text{odd} \). We also see that \( S_n = m(n-1)/2 \) implies that \( S_n = m(n-1)/2 \) for \( a=2, 3, \ldots, n \).

Conditions (45) become
\[ nh_2^{\circ \circ} - 2an_1^{\circ \circ} + ma(2a-1) = 0 \mod 2n, \]
\[ h_{2nh_n}^{\circ \circ} - h_n^{\circ \circ} - h_n^{\circ \circ} \geq - \frac{abm}{n}, \]
\[ h_{a+b+c}^{\circ \circ} - h_{a+b}^{\circ \circ} - h_{b+c}^{\circ \circ} - h_{a+c}^{\circ \circ} + h_a^{\circ \circ} + h_b^{\circ \circ} + h_c^{\circ \circ} \geq 0. \quad (B15) \]

Condition (51) becomes
\[ h_{a+b+c}^{\circ \circ} - h_{a+b}^{\circ \circ} - h_{b+c}^{\circ \circ} - h_{a+c}^{\circ \circ} + h_a^{\circ \circ} + h_b^{\circ \circ} + h_c^{\circ \circ} = \text{even}. \quad (B16) \]

The first condition in Eq. (B14) and the second condition in Eq. (B15) imply that
\[ h_{a+b}^{\circ \circ} - h_a^{\circ \circ} - h_b^{\circ \circ} + \frac{abm}{n} \geq 0, \]
\[ h_{a+b}^{\circ \circ} - h_a^{\circ \circ} - h_b^{\circ \circ} + \frac{abm}{n} = 0 \mod 1, \quad (B17) \]
which are part of defining conditions of parafermion CFT if \( m=2 \). Thus, the pattern of zeros of symmetric polynomial may have a natural relation to parafermion CFT.
46 Yue Yu (unpublished).
48 Such a wave function corresponds to a section of a line bundle on a sphere. The Chern number of the line bundle is $N_{\phi}$.