

Editors' Introduction: The Third Life of Quantum Logic: Quantum Logic Inspired by Quantum Computing

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Abstract We begin by discussing the history of quantum logic, dividing it into three eras or “lives.” The first life has to do with Birkhoff and von Neumann’s algebraic approach in the 1930’s. The second life has to do with attempt to understand quantum logic as logic that began in the late 1950’s and blossomed in the 1970’s. And the third life has to do with recent developments in quantum logic coming from its connections to quantum computation. We discuss our own work connecting quantum logic to quantum computation (viewing quantum logic as the logic of quantum registers storing qubits), make some speculations about mathematics based on quantum principles, and end by quickly describing the papers by the other authors contributing to this special issue on “Quantum Logic Inspired by Quantum Computing.” The overall aim of this issue is to show how work on this topic is leading to interesting new areas of logic.

Keywords Quantum logic · Quantum computing · Quantum mathematics · Qubit · Hilbert space

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1 History

Section 1 of this paper is an introduction, and it sets the context for this special volume on quantum logic and its relation to quantum computing. We use Section 2 as an occasion to report on some of our own relevant research. Section 3 changes tone a bit, providing some speculative remarks on a different theme: quantum analogues of classical objects like numbers. We close with a brief examination in Section 4 of the papers in the volume, together with comments on how they relate to the themes of this article.

Modern classical logic began with Boole [4], who had two interpretations of the elements in his algebra of logic. The first interpretation was that they were classes; the second was that they were propositions. He connected the two, saying that for purposes of inference a proposition could be regarded as a class.¹

We see in Boole a prescient anticipation of the contemporary mathematization of propositions as sets (of “cases,” “occasions,” “times,” “possible worlds,” “situations,” “set-ups,” “states,” whatever), sometimes called “UCLA propositions.” Conjunction is interpreted as intersection, disjunction as union, and negation as complement (relative to a given underlying set of possible). This way of looking at propositions can be generalized to include other non-classical logics, including quantum logic, though we will see that only conjunction remains in its original form.

1.1 The First Life of Quantum Logic: Birkhoff and von Neumann

Quantum logic began with Birkhoff and von Neumann [3] when they published their pioneering paper titled “The Logic of Quantum Mechanics.” It is adventitious that this volume of the *Journal of Philosophical Logic* comes at the 75th (“diamond”) anniversary of this historic event.²

They point out that in classical dynamics, the state of a single particle can be described as a sextuple $\langle r_1, \dots, r_6 \rangle$ of real numbers, where the first 3 components specify its position and the second 3 components specify its momentum. The “phase space” for n particles can thus be thought of as the product set \mathbb{R}^{6n} . Any subset of \mathbb{R}^{6n} can be thought of as an *event*, or proposition. And conjunction, disjunction, and negation can be understood as Boole’s operations on sets.

But anyone who knows anything about quantum mechanics has heard of the Heisenberg Uncertainty principle, which says that one cannot simultaneously determine both the position and momentum of a particle. Birkhoff and von Neumann accordingly move to the more complicated phase-space on which they build their quantum logic. There is a correspondence with classical dynamics and Boole, except

¹Boole [4] spoke somewhat abstractly of classes of “conceivable cases and conjunctures of circumstances,” whereas Boole [5] took a more concrete temporal interpretation, speaking of classes of “times” (calling these “durations”). See Kneale [19].

²That paper was anticipated by von Neumann’s [29] book on the mathematical foundations of quantum mechanics. There in Section 5, chapter 3, he observed that the projections defined on a Hilbert space could be regarded as representing experimental propositions concerning the properties of a quantum mechanical system. Projections correspond to closed subspaces.

not every set of states determines a proposition – only the closed subspaces. The conjunction \wedge of two subspaces is their set intersection, but their disjunction \vee is the closure of their span.³ And the negation of a subspace is the set of states that are “orthogonal” ($\alpha \perp \beta$) to every state in the subspace, where $\alpha \perp \beta$ means that the “inner product” $\alpha \bullet \beta = 0$.

1.2 Boolean Algebras and Their Generalizations

In this section we quickly review some algebraic structures that have naturally arisen in the study of classical logic and quantum logic.

A *Boolean algebra* is a special kind of bounded distributive lattice where every element x has a complement $\sim x$. Let us go through this a step at a time. A *lattice* can be defined as a partially ordered set (L, \leq) where for any $x, y \in L$, there exists a greatest lower bound $x \wedge y \in L$ and a least upper bound $x \vee y \in L$. (There are several additional laws which we omit.) Think of x, y as propositions, \leq as entailment, \wedge as conjunction, and \vee as disjunction.

The lattice is *bounded* if it has both a least element 0 and a greatest element 1. It is *complemented* if for every element x there is an element x' such that $x \wedge x' = 0$. Complements are unique in a Boolean algebra, so we can introduce a unary operation \sim that takes x to its complement. It can then be shown that \sim is *order inverting* (if $x \leq y$ then $\sim y \leq \sim x$) and of *period two* ($\sim\sim x = x$). In the context of a lattice these are equivalent to the *De Morgan Laws*: $\sim(x \wedge y) = \sim x \vee \sim y$ and $\sim(x \vee y) = \sim x \wedge \sim y$. In a Boolean algebra then we have $x \vee \sim x = 1$. A lattice is *distributive* if $x \wedge (y \vee z) \leq (x \wedge y) \vee (x \wedge z)$. (The converse is true in any lattice.)

A unary operation \sim on an arbitrary lattice is an *orthocomplement* if it is of period two, order inverting, and $\sim x$ is the complement of x . Orthocomplements are not necessarily unique outside of the context of a distributive lattice. An *ortholattice* is a structure (L, \leq, \sim) where (L, \leq) is a lattice and \sim is an orthocomplement.

An ortholattice is a generalization of a Boolean algebra in that it does not need to be distributive.

Let us first consider three critical laws. There are various equivalent ways to state the modular law, and we choose:

$$\text{(Modularity) if } z \leq x, \text{ then } x \wedge (y \vee z) \leq (x \wedge y) \vee z.$$

The modular law holds in classical logic, and in fact it holds without any conditions, since in the context of a lattice the consequent $(x \wedge (y \vee z) \leq (x \wedge y) \vee z)$ comes unconditionally from distribution (and is in fact equivalent). Modularity can also be stated unconditionally as:

$$\text{(Unconditional Modularity) } x \wedge (y \vee [x \wedge z]) \leq (x \wedge y) \vee z.$$

This is not strictly an equation but we can treat all weak inequalities as equations in virtue of the general lattice equivalence $s \leq t$ iff $s \wedge t = s$. This means that the class of modular lattices is equationally definable.

³This enlarges the union in two distinct ways. First by adding all linear combinations (the “span”), and secondly by adding all limit points (the “closure”).

Birkhoff and von Neumann noted that the distributive law fails in their quantum logic, but somewhat surprisingly they also note that the modular law also fails. Instead there is a weaker “orthomodular law”:

$$\text{(Orthomodularity) if } z \leq x, \text{ then } x \wedge (\sim x \vee z) \leq z.^4$$

An *orthomodular lattice* is an ortholattice in which the orthomodular law holds. This leads to a linguistically confusing but important distinction between an orthomodular lattice and a modular ortholattice. Modular ortholattices are special kinds of orthomodular lattices. It is interesting that Birkhoff and von Neumann [3] took the former and not the latter as part of their logic of quantum mechanics. Rédei points out that they prefer the modular law because of its fit with a generalization of classical probability theory.⁵

Our reason for liking the modular law was different and had to do with wanting a generalization of the register of bits in a classical computer, so as to have a quantum register of qubits. We focused on finite registers and it is well-known that finite dimensional orthomodular lattices are modular.

There is an abstraction under which one can fit both probability and dimension. A standard (Kolmogorov) requirement on a probability function

$$p(a \vee b) = p(a) + p(b) - p(a \wedge b)$$

can trivially be restated and generalized (putting a general function f for p) as

$$d(a) + d(b) = d(a \vee b) + d(a \wedge b).$$

Birkhoff [2] calls such a real-valued function a “valuation” and shows that the existence of a strictly monotonic valuation on a lattice implies that the lattice is modular. Birkhoff observes that both probability and dimension are valuations, and both are monotonic. Dimension is obviously also strictly monotonic ($a < b$ implies $d(a) < d(b)$), and so it seems is probability when it is taken in its logical interpretation.⁶

1.3 The Second Life of Quantum Logic: Quantum Logic as Logic

From the late 1950’s, and especially in the 1970’s and 80’s, quantum logic had a second life. As is said by Dalla Chiara and Giuntini after discussing Birkhoff and von Neumann [3]: “Only twenty years later, after the appearance of George Mackey’s book *Mathematical Foundations of Quantum Theory* [20], one has witnessed a ‘renaissance period’ for the logico-algebraic approach to QT. This has been mainly stimulated by the contributions of Jauch, Piron, Varadarajan, Suppes,

⁴This has the philosophically memorable equivalent: $x \wedge (\sim x \vee (x \wedge y)) \leq y$, which has led to regarding $\sim x \vee (x \wedge y)$ as a conditional – the so-called “Sasaki hook,” named after its discoverer.

⁵Rédei [24] contains an interesting discussion of this, as do other publications by Rédei. See particularly Rédei [23] regarding the background correspondence from von Neumann to Birkhoff.

⁶Von Neumann seems to have gone back and forth on how he interpreted probability (frequency or logical), but about this time seemed to favor logical probability. See Rédei [23].

Finkelstein, Foulis, Randall, Geachie, Gudder, Beltrametti, Cassinelli, Mittelstaedt and many others.” The main topic of interest regarding quantum logic regarded the novelty of yet another non-classical logic, and how it compared with intuitionistic logic (the main alternative to non-classical logic at that point). There was also relatively great interest (compared to Birkhoff and von Neumann, and now) about how it should best be conceived (orthomodular lattice, many-valued logic, etc.), and following this in the standard logical issues of semantics, proof-theory, completeness, and decidability. There was a strong preference for the orthomodular approach, and that gave a proof-theory and a semantics (the closed subspaces of a Hilbert space), but connecting the two has proved impossible. It turns out that the lattice of closed subspaces of a Hilbert space satisfies additional laws, even those that can be stated as equations such as the “Orthoarguesian law.”⁷ Another problem is that the axiom system can be given in so-called “Hilbert-style” by translating the axioms for orthomodular lattices into a more standard logical formalism, but to our knowledge no one has yet succeeded in giving an equivalent cut-free Gentzen version (which many people think is the gold-standard approach to proof-theory) of orthomodular logic or modular orthologic.⁸

1.4 The Third Life of Quantum Logic: Quantum Logic Based on Quantum Computation

Dalla Chiara and Giuntini speak of a “Renaissance,” which of course literally means “rebirth,” i.e., a second life. We believe that quantum logic now has a “third life,” inspired by quantum computing. And subtly different algebraic structures arise (with some of the same open questions, but new chances at solving them).

Although Richard Feynman had first suggested the idea of a quantum computer to simulate quantum processes faster than might be done by a classical computer, it was not until 1985 that David Deutsch published a paper describing a general purpose (universal) quantum computer. Deutsch modified the classical Turing machines to make equivalents among other differences put qubits in place of the standard binary digits that appear on the tape of a Turing machine. The equivalent use of quantum gates has become the much more customary way of characterizing quantum computation. This work was once purely theoretical, but after 1994, when Peter Shor gave his famous algorithm for efficiently factoring numbers into prime, the idea of a quantum computer began to take on real practical significance. This is because of the widely used RSA encryption scheme that depends on the difficulty of factoring large numbers into their prime components.

⁷This apparently was an unpublished idea of Alan Day, and the proof was first presented by Greechie [18]. See Dalla et al. [6]

⁸See Nishimura [21] for presentation and history of cut-free Gentzen systems for “minimal quantum logic” (what we are calling orthologic) and its history. See also Egly and Tompits [12]. Chiara and Giuntini [7] in sec. 17 (by G. Battilotti and C Faggian) discuss a Gentzen system for orthologic developed by Sambin, Battilotti, and Faggian that has a cut-free formulation, but they do not address orthomodular logic or modular orthologic.

From its early years logic has been linked to computation. Leibniz's great achievement was to combine the idea of a "lingua universalis" with a "calculus ratiocinator". The two together facilitate "blind thinking," as Leibniz termed it, since reasoning is reduced to arithmetic calculation. The link between classical computing and classical logic is often taken for granted. Many standard classical textbooks contain both, e.g. Kleene's *Introduction to Metamathematics* [17]. But the use of classical logic to describe and design circuits is not even mentioned, whereas this has become almost the standard approach to thinking about quantum logics.

In a classical computer, data is stored as a "string" of bits in a register. Registers come in various sizes, thus a 64-bit register contains strings of length 64. The "register space" can be viewed as the direct product of the 2-element Boolean algebra, i.e., the set of n -element sequences of 0s and 1s. This can be viewed as a Boolean algebra itself by the direct product construction, defining $\wedge, \vee, -$ component wise, e.g., $\neg \langle b_1, \dots, b_n \rangle = \langle \neg b_1, \dots, \neg b_n \rangle$.

What is the logic of the classical n -bit register? Is it the same as the logic of the 1-bit register (classical logic) or not? This is answerable in two steps. We first take classical propositional calculus, and form its "Lindenbaum Algebra" by the "Method of Abstraction." We thus put two provably equivalent formulas φ and φ' into the same equivalence class $[\varphi] = [\varphi']$, and we then define operations on these equivalence classes using the sentential operators, e.g., $\neg[\varphi] = [\neg\varphi]$. For classical propositional calculus this gives a Boolean algebra in which the equivalence class of the theorems $= 1$. The second step is to invoke a form of the Representation Theorem for Boolean algebras [28]: Every Boolean algebra is isomorphic to a subdirect product of the 2-element Boolean algebra (1-bit register). Combining these ideas, classical propositional logic can easily be shown to be the logic of the n -bit register (not just the 2-bit one).

The qubit is a "quantum bit". Unlike the classical bit, 0 and 1 are just two of infinitely many possible states of the qubit. The state of a qubit is the "superposition" (linear combination) $\alpha|0\rangle + \beta|1\rangle$ (where α, β are complex numbers representing "amplitudes"—amplitudes squared give probabilities).⁹ The state of a qubit can be described as a vector (α, β) in the two-dimensional complex vector space \mathbb{C}^2 . The special states 0 and 1 are known as the computational basis states, and form an orthonormal basis for this vector space. According to quantum theory, when we try to measure the qubit in this basis in order to determine its state, we get either 0 with probability $|\alpha|^2$ or 1 with probability $|\beta|^2$. This motivates requiring that $|\alpha|^2 + |\beta|^2 = 1$. (This is a Probability Sum Rule for disjoint events).

Quantum registers contain qubits (quantum bits). The 2-dimensional space of the complex numbers $\mathbb{C}^2 = \mathbb{C} \oplus \mathbb{C}$ can be thought as a quantum register containing a single qubit, and all the pairs of complex numbers in it are then thought of as states of that qubit. The n -qubit register \mathbb{C}^{2^n} can then be defined inductively as $\mathbb{C}^2 \otimes \mathbb{C}^{2^{n-1}}$, i.e., $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \dots \otimes \mathbb{C}^2$ (n -times). It turns out that unlike the analogous case with classical logic, the logic of the n -qubit register is generally different than the logic of the 1-bit quantum register, and indeed the logic of the n -qubit register is always

⁹This is usually written in the Dirac notation as $\alpha|0\rangle + \beta|1\rangle$, but we will not be so fussy in our motivating explanations here.

different from the logic of the m -bit quantum register when $m \neq n$. This was shown in Dunn et al. [11], and the result was improved by Haggé [14] who showed for all $m \neq n$, the logics of \mathbb{C}^m and \mathbb{C}^n differ. (Note that the superscript here is n and not 2^n .) This leads us to wonder whether every subdirectly irreducible modular lattice is isomorphic to the lattice of subspaces of some \mathbb{C}^n . If it were, then by Birkhoff's Subdirect Product Theorem, every modular ortholattice would be isomorphic to a subdirect product of such lattices.

We titled the conference "Quantum Logic Inspired by Quantum Computing" (QLIQC, pronounced "click"), but it turns out it might just as well have been "Quantum Logic Inspired by Quantum Categories" in terms of the talks given (and the subsequent papers published in this volume).

2 Quantum Logic Inspired by Quantum Computing

The BB84 private key protocol [1], Shor's algorithm [26], and Hastings' additivity counterexamples [15] are all pieces of evidence that quantum information theory is strictly richer than classical information theory. The attempt to build a useful quantum computer has begun and rekindled interest in quantum mechanics at all levels: philosophical, mathematical and physical. Unlike the construction of the classical computer, to build a quantum computer might require new physics such as non-abelian topological order [13].

Ever since its appearance, quantum mechanics presents great conceptual difficulty, even for the most brilliant minds. In quantum mechanics, the wave function of a state is a complete description of the physical state, and the Schrodinger equation is a deterministic evolution of the state. When the measuring apparatus is included into the system, the measurement of a quantum system is a deterministic process for the composite system with complete description. Yet our best interpretation for the measurement result is still probabilistic. Probability is usually related to insufficient knowledge. The mismatch of a complete description of a quantum system with the probabilistic interpretation lies at the heart of the debate. Maybe humans are innately not able to apprehend a quantum state. But the emergence of numbers seems to suggest otherwise. Children's counting ability is arguably primitive and dormant, and only through education is the number fully developed into counting with numbers. Historically tally seems to come first, then counting, and finally abstract numbers. An important step in the emergence of numbers is the separation of things to be counted from their associated symbols. Quantum information is taking this step right now. The qubit is the abstraction of 2-level quantum systems, therefore it is not an electron spin; similarly, the number one is not an apple. Hence the qubit likely will play an important role in the evolution of numbers.

2.1 Quantum Logic of Qubits

Qubits are the currency for quantum computing. Their states are represented by non-zero vectors of the Hilbert space $(\mathbb{C}^2)^{\otimes n}$. In this section, we will examine the quantum logic of qubits following the ideas of G. Birkhoff and J. von Neumann [3].

2.1.1 Quantum Logic Determines Dimension

Given a Hilbert space \mathbb{H} , let $L_c(\mathbb{H})$ be the lattice of closed subspaces of \mathbb{H} . Closed subspaces are quantum events, so they are quantum analogues of propositions. We will use $\mathbf{0}, \mathbf{1}$ to denote the 0-subspace and \mathbb{H} , respectively. The meet \wedge of two subspaces is the set intersection, and the join \vee the closure of their span. For any closed subspace p , its negation p^\perp is the orthogonal complement. It is well-known that $L_c(\mathbb{H})$ is an orthomodular lattice and modular if and only if \mathbb{H} is finite dimensional. Propositional formulas consist of alphabet symbols, parenthesis, and connectives \wedge, \vee and $\bar{}$. Let $\{u_i\}$ be a collection of alphabet symbols, and $\{p_i\}$ be a collection of closed subspaces. Given a well-formed formula (wff) $\phi(u_i)$, the evaluation $\phi(p_i)$ is the subspace resulting from substituting each p_i into u_i and performing the corresponding operations. A wff $\phi(u_i)$ is a *tautology* of $L_c(\mathbb{H})$ if for all evaluations $\phi(u_i) = \mathbf{1}$. We will also call any equation of terms $s = t$ in which for all evaluations $s = t$ a *tautology*.

Definition 2.1 Given a Hilbert space \mathbb{H} , the quantum logic $\mathbf{QL}(\mathbb{H})$ is the set of all tautologies of $L_c(\mathbb{H})$.

Theorem 2.2 *Quantum logic $\mathbf{QL}(\mathbb{H})$ determines the dimension of \mathbb{H} .*

Note that the modular law separates infinite dimensional Hilbert spaces from finite dimensional ones. Then the dimension of a finite dimensional Hilbert space is determined by its quantum logic [11, 14].

For notational ease, we will denote $\mathbf{QL}((\mathbb{C}^2)^{\otimes n})$ by $\mathbf{QL}(2^n)$, $n = 0, 1, \dots$. To understand the differences between these logics better, we will exhibit tautologies that distinguish them. For $n = 0$, the quantum logic $\mathbf{QL}(1)$ is just the classical propositional logic. The distributive law holds in $\mathbf{QL}(1)$, but fails in any $\mathbf{QL}(2^n)$, $n \geq 1$. Therefore, the distributive law is a salient feature of classical logic. We will explore the failure of distributive law in $\mathbf{QL}(2^n)$, $n \geq 1$ systematically to arrive at increasingly weakened tautologies. The first such tautology was the m -distributive law:

$$x \wedge (\bigvee_{i=0}^m y_i) = \bigvee_{i=0}^m (x \wedge (\bigvee_{j \neq i} y_j)).$$

It is proven [16] that the m -distributive law holds if and only if $\dim(\mathbb{H}) \leq m$. Dunn et al. [11] and Hagge [14] found another sequence of such tautologies. For simplicity, we will consider only the qubits here. As a bonus of our new tautologies, we will see that $\mathbf{QL}(2^n)$ has no finite universal test sets when $n \geq 1$.

Two closed subspaces a, b are equal if and only if $(a \vee b) \wedge (\bar{a} \vee \bar{b}) = \mathbf{0}$. To see this, if $a = b$, obviously $(a \vee b) \wedge (\bar{a} \vee \bar{b}) = \mathbf{0}$. If $a \neq b$, then either $a \wedge b \neq a$ or $a \wedge b \neq b$. Without loss of generality, we assume $a \wedge b \neq a$. Then the complement of $a \wedge b$ in a , denoted as $\overline{a \wedge b}^a$, is not $\mathbf{0}$. But $\overline{a \wedge b}^a \subset \overline{a \wedge b} = \bar{a} \vee \bar{b}$ and $\overline{a \wedge b}^a \subset a \vee b$. Hence $(a \vee b) \wedge (\bar{a} \vee \bar{b}) \supset \overline{a \wedge b}^a \neq \mathbf{0}$.

Given three subspaces p, q, r , let $a = p \vee (q \wedge r)$ and $b = (p \vee q) \wedge (p \vee r)$, and then define

$$\alpha(p, q, r) = (a \vee b) \wedge (\bar{a} \vee \bar{b}).$$

Note that $a \leq b$, it follows that $\alpha(p, q, r) = b \wedge \bar{a} = [(p \vee q) \wedge (p \vee r)] \wedge [\bar{p} \wedge (\bar{q} \vee \bar{r})] \subseteq \bar{p}$. The distributive law holds if and only if α is always $\mathbf{0}$. Therefore, if α does not vanish for some choice of p, q, r in a Hilbert space \mathbb{H} , then the distributive law is not in $\mathbf{QL}(\mathbb{H})$. Therefore, we will call $\alpha(p, q, r)$ the *distribution test formula*.

From $\alpha(p, q, r) \subseteq \bar{p}$, we deduce $\dim(\alpha(p, q, r)) \leq \dim(\mathbb{H}) - \dim(p)$. In Dunn et al. [11], a direct computation shows $\dim(\alpha(p, q, r)) \leq \dim(p)$. Hence $\dim(\alpha(p, q, r)) \leq \frac{\dim(\mathbb{H})}{2}$.

To define our tautology, we define the restriction of a wff $\phi(u_i)$ to a term β , denoted by $\phi|_\beta$: first using the De Morgan law, we assume that all negations $\bar{}$ are applied to single variables. Next, each variable u_i and its complement \bar{u}_i are replaced by $u_i \wedge \beta$ and $\bar{u}_i \wedge \beta \wedge \bar{\beta}$, respectively. Inductively, we define

$$\alpha^m(p_m, q_m, r_m) = \alpha|_{\alpha^{m-1}}(p_m, q_m, r_m),$$

and $\alpha^1(p_1, q_1, r_1) = \alpha(p_1, q_1, r_1)$, $\alpha^{m-1} = \alpha^{m-1}(p_{m-1}, q_{m-1}, r_{m-1})$. Therefore,

$$\dim(\alpha^m(p_m, q_m, r_m)) \leq \frac{\dim(\alpha^{m-1}(p_{m-1}, q_{m-1}, r_{m-1}))}{2} \leq \dots \leq \frac{\dim(\mathbb{H})}{2^m}.$$

In $\mathbf{QL}(2^n)$, $\dim(\alpha^{n+1}) \leq \frac{2^n}{2^{n+1}} < 1$, so $\alpha^{n+1} = \mathbf{0}$ which gives a tautology in $\mathbf{QL}(2^n)$, which is also true for any $i \leq n$. To show it is not true for $\mathbb{C}^{2^{n+1}}$, we notice that if p, q, r are different subspaces of dimension $\frac{m}{2}$ of \mathbb{C}^m and each pair has trivial intersection in \mathbb{C}^m , then $\dim(\alpha(p, q, r)) = \frac{m}{2}$ if m is even. By choosing subspaces in $\mathbb{C}^{2^{n+1}}$ this way, we have $\dim(\alpha^{n+1}) = \frac{2^{n+1}}{2^{n+1}} = 1$.

Definition 2.3 A set of closed subspaces in \mathbb{C}^m is called a *universal test set* for $\mathbf{QL}(\mathbb{C}^m)$ if the truth of any tautology is determined by the evaluations of the subspaces in this set.

Proposition 2.4 *There are no finite universal test sets for $\mathbf{QL}(\mathbb{C}^m)$, $m \geq 2$.*

To see this, consider the distribution testing formula $\alpha(p, q, r)$. For simplicity, we will only give the details for $m = 2$. In order for the distribution testing formula $\alpha(p, q, r)$ to fail, p, q, r must be three distinct lines. In order for $\alpha(\alpha(\alpha(p, q, r), p, s), q, s), r, s)$ to fail, p, q, r, s must be distinct lines. Continuing in this manner, we can build a complicated formula γ , the failure of which means that the k subspaces p, q, \dots are distinct lines. Since k is arbitrary, no finite set of lines will falsify every invalid formula. This argument works for any \mathbb{C}^m , $m \geq 2$.

For each n -qubit, we have found two tautologies which are not in any qubits m such that $m < n$: the 2^n -distributive law and the iterated distribution test formula. If each law is added to the modular lattice axioms, are the resulting axioms sets equivalent? We leave this as an open problem.

2.1.2 Decidability

Quantum logic for general modular ortholattice is undecidable. Dunn et al. [11] observed that the quantum logic of a finite dimensional Hilbert space is decidable. The decidability $\mathbf{QL}(\mathbb{C}^m)$ is reduced to the decidability of \mathbb{R} . The idea is to associate a matrix M_p to each subspace p so that the kernel of M_p is p . Then new matrix variables are introduced to construct a formula M_ϕ so that a wff ϕ is a tautology if and only if $M_\phi = 0$. This procedure is illustrated for the join in Dunn et al. [11]. The easy cases of meet and negation can be done as follows:

$$r = p \wedge q, \forall u(M_p u = 0 \wedge M_q u = 0 \Leftrightarrow M_r u = 0).$$

$$p = \bar{q}, \forall v(M_p v = 0 \Leftrightarrow (\forall u(M_q u = 0 \Rightarrow \langle v, u \rangle = 0))).$$

Taking all these observations together, we conclude:

Theorem 2.5 *The first-order theories of $\mathbf{QL}(\mathbb{C}^m)$ are uniformly decidable.*

Since decidability of $\mathbf{QL}(\mathbb{C}^m)$ implies its axiomatizability, can $\mathbf{QL}(\mathbb{C}^m)$ be axiomatized with finitely many schemas? Is it sound? Is it complete? Are modular ortholattice axioms plus n -distributivity or the iterated distribution test formula sufficient to axiomatize $\mathbf{QL}(\mathbb{C}^m)$? We believe that these are all interesting open problems.

We might also speculate on a connection between quantum logic and quantum computational complexity. For example, if we choose a finite collection of subspaces of \mathbb{C}^2 including $\mathbf{0}, \mathbf{1}$ that generate a sublattice, then what is the computational complexity for the satisfiability? In particular, if we add one p whose normalized dimension is $\frac{1}{2}$ to $\mathbf{0}, \mathbf{1}$, does the computational complexity depend on the choice of p ? Does quantum computer have any advantage over classical computers for those problems?

2.2 Qubit Continuous Geometry

Birkhoff and von Neumann proposed continuous geometry as quantum propositional logic. In this section, we will focus on a particular continuous geometry—qubit continuous geometry. This turns out to be the famous type Π_1 hyperfinite factor \mathcal{R} in von Neumann algebra theory. Through the study of type Π_1 factors, V. Jones discovered his famous representation of the braid groups and polynomial invariants of knots. Jones’ representation of braid groups are used to describe new particle statistics and are therefore playing a pivotal role in the topological approach to quantum computing.

Let V, W be two Hilbert spaces. Note that neither V nor W is canonically a subspace of $V \otimes W$. But the lattices $L_c(V)$ and $L_c(W)$ are canonical sublattices of $L_c(V \otimes W)$ by including $p \subseteq V$ or W into $V \otimes W$ as $p \otimes W$ or $V \otimes p$, respectively. It follows that $\mathbf{QL}(V)$ and $\mathbf{QL}(W)$ are canonically subsets of $\mathbf{QL}(V \otimes W)$. Therefore, quantum logics of qubits form a compatible decreasing sequence:

$$\mathbf{QL}(1) \supset \mathbf{QL}(2) \supset \mathbf{QL}(4) \supset \dots \supset \mathbf{QL}(2^n) \supset \mathbf{QL}(2^{n+1}) \supset \dots$$

How to describe their intersection $\mathbf{QL}(\infty)$? As remarked in Dunn et al. [11], the intersection $\mathbf{QL}(\infty)$ is not the quantum logic of any infinite dimensional Hilbert space because it contains the modular law.

2.2.1 *Limit of $\mathbf{QL}(2^n)$*

The normalized dimension of a subspace $p \subseteq V$ is $d_V(p) = \frac{\dim(p)}{\dim(V)} : L_c(V) \rightarrow [0, 1]$. The lattice $L_c(2^n)$ with the normalized dimension

$$d_{(\mathbb{C}^2)^{\otimes n}} : L_c((\mathbb{C}^2)^{\otimes n}) \rightarrow [0, 1]$$

is a metric space compatible with the inclusion $L_c(2^n) \subset L_c(2^{n+1})$. Let $L_c(\infty)$ be their direct limit. The ranges of dimensions are all rational numbers with power 2 denominators. Let CG be its metric completion, then CG is a continuous geometry: an irreducible complemented continuous modular lattice. A continuous geometry is a projective geometry whose dimensions cover the unit interval $[0,1]$.

To relate this continuous geometry to the hyperfinite Π_1 factor \mathcal{R} , we consider the sequence of matrix algebras:

$$M_2(\mathbb{C}) \subset M_2(\mathbb{C}) \otimes M_2(\mathbb{C}) \subset \dots$$

with inclusion given by $A \rightarrow A \otimes Id$. The $*$ -algebra limit is the hyperfinite Π_1 factor \mathcal{R} . Let $L_p(\mathcal{R})$ be the set of projectors in $\mathcal{R} : p = p^\dagger, p^2 = p$. Using the identification of a subspace with a projection, we see that $L_p(\mathcal{R}) = CG$. The factor \mathcal{R} can be realized as a subalgebra of the bounded operators of a Hilbert space \mathbb{H} . With this realization, a projector can be identified with the closed subspace $p\mathbb{H} \subset \mathbb{H}$ —invariant vectors of p in \mathbb{H} . We define the partial order, meet and join of two projectors p, q by $p \leq q$ if and only if $p\mathbb{H} \subseteq q\mathbb{H}$, $p \wedge q$ =orthogonal projection onto $p\mathbb{H} \wedge q\mathbb{H}$, $p \vee q$ =orthogonal projection onto $p\mathbb{H} \vee q\mathbb{H}$. The negation of a projector p is $\bar{p} = 1 - p$. Let $\mathbf{QL}(\mathcal{R})$ be the tautologies over $L_p(\mathcal{R})$ or equivalently over the sublattice of $L_c(\mathbb{H})$ consisting of invariant subspaces of a projector in $L_p(\mathcal{R})$.

Theorem 2.6 (1) $\mathbf{QL}(\infty) = \mathbf{QL}(\mathcal{R})$. (2) $\mathbf{QL}(\mathcal{R})$ is decidable.

It is shown by J. Harding in this issue that $\mathbf{QL}(CG) = \mathbf{QL}(\infty)$. Since $L_p(\mathcal{R}) = CG$, therefore $\mathbf{QL}(\mathcal{R}) = \mathbf{QL}(\infty)$. It follows that $L_p(\mathcal{R})$ is a modular lattice. As also proved in this issue by J. Harding, $\mathbf{QL}(\mathcal{R})$ is decidable. Therefore, the intersection $\mathbf{QL}(\infty)$ is decidable, positively answering a question in Dunn et al. [11]. The lattice of projectors $L_p(\mathcal{R})$ is a natural generalization of qubit quantum logic agreeing with Hankel’s principle of the preservation of formal laws. The decidability of quantum logics of general continuous geometries seems to be open.

2.2.2 *Temperley-Lieb Algebra And Jones-Wenzl Projector*

It is clear from the last subsection how to obtain a projector in $L_p(\mathcal{R})$ with its normalized dimension to be any rational number in the interval $[0,1]$. In this section, we construct projectors with irrational algebraic normalized dimensions. It is hard to

imagine projectors with non-computable normalized dimensions. For example, let ω be a Chaitin number in $[0,1]$ which encodes the halting problem for Turing machines. Do the projectors with normalized dimensions ω have any relevance to reality?

To construct such projectors, we introduce the Temperley-Lieb (TL) algebras. The TL algebra $TL_n(A)$ at $A = \pm i e^{\pm \frac{2\pi i}{4r}}$, $r \geq 3$ is a unital algebra with generators $1, e_1, e_2, \dots, e_{n-1}$ and relations:

$$e_i e_j = e_j e_i, \quad \text{if } |i - j| \geq 2, \tag{1}$$

$$e_i e_{i\pm 1} e_i = \frac{1}{d^2} e_i, \tag{2}$$

and

$$e_i^2 = e_i, \tag{3}$$

where $d = -A^2 - A^{-2}$.

$TL_n(A)$ is also defined by the same presentation when A is a variable. In this case, they are matrix algebras over a function field and contain some magic projectors, called Jones-Wenzl projectors: each $TL_n(A)$ contains a unique element p_n characterized by: $p_n^2 = p_n \neq 0$ and $e_i p_n = p_n e_i = 0$ for all $1 \leq i \leq n - 1$. Furthermore p_n can be written as $p_n = 1 + U$ where $U = \sum c_j h_j$, h_j a product of e_i 's, $1 \leq i \leq n - 1$ and $c_j \in \mathbb{C}$.

$TL_n(A)$ can be naturally included into $TL_{n+1}(A)$, hence $p_i, i \leq n$ can be considered as elements of $TL_{n+1}(A)$. When A is chosen as the complex numbers above, the Jones-Wenzl projectors are defined consecutively only for $n = 1, \dots, r - 1$. Moreover, the TL algebra is not a matrix algebra. For a fixed r , their quotients by p_{r-1} considered as an element in each $TL_n(A)$, $n \geq r - 1$ are matrix algebras. We will call those matrix algebras, denoted by $J_n(A)$, the *Jones algebras*. The matrix summands of the decomposition of $J_n(A)$ are indexed by natural numbers $m = n \bmod 2$. Define the n^{th} Chebyshev polynomial $\Delta_n(x)$ inductively by $\Delta_0 = 1$, $\Delta_1 = x$, and $\Delta_{n+1}(x) = x\Delta_n(x) - \Delta_{n-1}(x)$. Then the Markov trace on $J_n(A)$ is the weighted matrix trace $\text{tr}^{Mark}(u) = \sum_m \Delta_m \text{tr}(u)$, where $\text{tr}(u)$ is the usual matrix trace for u . The Jones algebra $J_n(A)$ is included into $J_{n+1}(A)$ naturally. The limit of them is the hyperfinite Π_1 factor \mathcal{R} . The Markov trace is the limit of the normalized dimensions. In the Jones algebras $J_n(A)$, the TL elements e_i 's are Hermitian, i.e. $e_i^\dagger = e_i$, hence $e_i \in L_p(\mathcal{R})$. The same is true for each Jones-Wenzl projector $p_i, i = 0, 1, \dots, p_{r-2}$. Therefore, e_i 's and p_j 's are projectors. The TL relations tell us that the images of e_i and e_j are orthogonal if $|i - j| \geq 2$, and the angle between the i^{th} and $(i + 1)^{th}$ is determined by d . Their trace are given by $\text{tr}(e_i) = \frac{1}{d^2} = \frac{1}{4} \text{sec}^2(\frac{\pi}{r}), i = 1, 2, \dots$, and $\text{tr}(p_j) = \frac{\Delta_j(d)}{d^j}, j = 1, 2, \dots, r - 2$ for any $r \geq 3$. Hence the subspaces $e_i \mathbb{H}$ and $p_j \mathbb{H}$ have normalized dimensions $\{\frac{1}{4} \text{sec}^2(\frac{\pi}{r})\}$ and $\{\frac{\Delta_j(d)}{d^j}\}$ for any $r \geq 3$ and $j = 1, 2, \dots, r - 2$.

Do the projectors $\{p_j, e_i, j = 1, 2, \dots, r - 2, i = 1, 2, \dots\}$ and the subspaces of normalized dimensions with power 2 denominators form a universal test set for $QL(\mathcal{R})$?

2.3 Topological Quantum Computation

Classical physics is the theoretical foundation for the construction of classical computers. The failure of C. Babbage to complete his analytical engine in 1850s was not due to some missing physics, but rather for engineering reasons. The same might occur for some current proposals to build a quantum computer. But one approach is different in this regard: topological quantum computation [13]. The success of topological quantum computation hinges on the discovery of completely new particles: non-abelian anyons. The defining feature of such hypothetical particles are their ground state degeneracy in the plane: suppose several non-abelian anyons are fixed in the plane, well-separated, their lowest energy states are still not unique. There are different internal states of the system which cannot be determined by their positions and other local properties.

2.3.1 Non-Abelian Anyons

The mathematical models of non-abelian anyons are unitary modular tensor categories, or the closely related unitary topological quantum field theories. An anyon is a simple object in the modeling unitary modular category [30].

The Jones algebras can be easily generalized to tensor categories, which are unitary modular tensor categories. The Jones-Wenzl projectors are the simple objects of the resulting unitary modular tensor categories, hence anyons. When $r \geq 4$, the projector p_1 is a non-abelian anyon. Suppose there are m of them in a plane at some fixed locations, well-separated, how do we describe their states? Their Hilbert space decomposes into subspaces of different energies. The lowest energy states are called the ground states. They are not unique and form a Hilbert space of dimension exponential in m . Therefore, we need exponential many states to describe the differences of m non-abelian anyons.

In topological quantum computation, information is encoded into this vast degenerate groundstate manifold of non-abelian anyons and processed by braiding them around each other. Anyons can be brought together to fuse, and the computational answer, encoded in the resulting anyon types, is an approximation of the Jones polynomial at $q = e^{\pm \frac{2\pi i}{r}}$.

Jones theories are predicted to be realized in the fractional quantum Hall liquids. For example, when $r = 4$, the anyon p_1 is predicted to exist in the $\nu = 5/2$ fractional quantum Hall liquid. The ground states will be 2-fold degenerate if a boundary condition is fixed for 4 non-abelian anyons p_1 . Therefore, to “count” the ground states of 4 non-abelian anyons p_1 when $r = 4$, two independent wave functions are required.

2.3.2 Intrinsic Entanglement

The many-anyon state is a state with topological order. Topological order is an internal, dynamical, non-local pattern of many-anyon systems characterized by intrinsic entanglement. In quantum mechanics, entanglement is defined with respect to a tensor decomposition of the relevant Hilbert space, which amounts to a measurement.

In a topological state, the ground state manifold has no natural tensor decompositions. Therefore entanglement of topological states is intrinsic—a salient feature of topological order.

3 Speculative Remarks on Quantum Analogues of Classical Objects

The term *quantum mathematics* is ambiguous. One sense concerns the mathematics needed to explain and work with quantum physics. The other sense is more radical and has to do with an alternative to classical mathematics that is somehow founded on quantum principles. We shall briefly discuss several such approaches.

3.1 Quantum Cantor Set

Von Neumann algebra theory is an axiomatization of quantum mechanics and can be regarded as a non-commutative measure theory. A von Neumann algebra M with a normalized, normal trace ρ is called a noncommutative probability space. An Hermitian operator X , a physical measurement, is a noncommutative random variable. Its eigenvalues $\text{Spec}(X) = \{\lambda_i\} \subset \mathbb{R}$ are in one-one correspondence with projections P_{λ_i} to the eigenspaces. The values $\rho(P_{\lambda_i})$ of the projections $\{P_{\lambda_i}\}$ under ρ define a probability distribution on $\text{Spec}(X)$. Therefore, a type Π_1 factor is a natural noncommutative probability space.

In the classical world, logic, measure theory, and probability fit together via the Stone representation theorem. For bit strings, their limit is the Cantor set. In this analogy, hyperfinite II_1 factor \mathcal{R} with the normalized trace is a probability theory for a quantum Cantor set. What is a quantum Cantor set? What is a quantum Boolean algebra? How to logicize type Π_1 factors?

3.2 Quantum Numbers

Classical computers process bit strings, which can be regarded as numbers denoted by binary notation. Numbers seem to be rooted in a human's need to record the differences between say one sheep and a herd. To understand the physical properties of many quantum particle systems potentially leads us to new numbers. The states of quantum particles cannot be easily described by numbers as they are given by wave functions. Quantum computers are wave function processors. Therefore, we argue that wave functions are quantum numbers.

A real number in base 2 expansion can be considered as an array of bits $\{0, 1\}$ on a bi-infinite Turing tape with a marked square for the separation of integral and fractional parts. The squares are digit holders. If the squares in the Turing tape correspond to basis elements of a Hilbert space, then a wave function can be thought as a generalization of numbers in two aspects: the bases of a Hilbert space is not necessarily an array and the digits $\{0, 1\}$ are replaced by any complex number. There are axiomatizations of both natural numbers—Peano axioms and real numbers—Dedekind cuts. Are there axiomatizations of wave functions—our proposed quantum numbers? For caution, we mention a work of Dunn: if the first order Peano arithmetic is formulated

with orthomodular quantum logic, then it has the same theorems as the first order Peano arithmetic [9]. Quantum mathematics is slippery business as Dunn showed that if one tries to formulate second order orthomodular quantum logic with a certain minimal principle of extensionality, one is doomed to failure in the sense that the resultant system collapses to its classical counterpart [10].¹⁰

In Dedekind's treatise on abstract structure of numbers, he asked "what are numbers and what should they be?", then answered: "numbers are free creations of the human mind. They serve as a means of apprehending more easily and more sharply the difference of things." Wave functions are creations of the human mind. However, as a means of apprehending things, they are neither easier nor more sharply distinguished than numbers. All's fair in love and quantum theory.

4 Papers in this Volume

On May 11-12, 2009, a workshop was held at Indiana University Bloomington entitled "Quantum Logic Inspired by Quantum Computation" (QLIQC). We are grateful to the National Science Foundation for their support of QLIQC. We are also grateful to the authors of the papers in this volume, both for their contributions and for their enthusiasm for our topic. The aim of the workshop was bring together people who are developing new areas of logic coming from quantum computation, and also people who are interested in related projects coming from areas of philosophical logic, mathematics, and theoretical computer science. The papers in this volume represent papers by participants of the workshop, either presented at the workshop, or written afterwards.

We would like to think that the papers in this volume represent a spectrum of approaches that relate logic and quantum computation. We are using both terms, "logic" and "quantum computation" in a broad sense: just as the scope of logic and computation have broadened considerably from their first modern incarnations, so too the topics in this volume represent work on quantum logic and quantum computation which significantly expands – and in some cases differs from – the original versions of both subjects. In many cases, the papers here are as connected to algebra as to logic. (Many of our author seemed to think that QLIQC stood for "Quantum Logic Inspired by Quantum Categories.") This should not be such a surprise, given the closeness of algebraic topics to various flavors of propositional logics.

The paper in our volume which is most directly related to quantum computation as it is emerging today is Dalla Chiara, Ledda, Sergioli, and Guintini's paper, "The Toffoli-Hadamard gate system: an algebraic approach". This paper contains an axiomatic treatment of algebraic systems constructed from Toffoli and Hadamard gates. As with other papers in our volume, it represents a generalization of quantum computation in that it abstracts away from the original Hilbert space formulation.

¹⁰We also note a quite opposing viewpoint, discussed in Dunn [9], which hearsay attributes to a lecture Saul Kripke gave at the University of Pittsburgh in 1974 (see [27]). Kripke apparently argued that given a logicist or set-theoretic understanding of numbers, it can be shown using Putnam's views that $2 + 2 > 4$ since the cartesian product of a 2-membered set with itself has more than 4 ordered pairs.

Stairs and Bub's, "Correlations, contexts and quantum logic" is probably the most conceptually-oriented paper in our volume. It asks "Why the quantum?" whether we can articulate principles that account for the character of quantum probability." It also offers a defense of propositional (lattice-based) models in connection with event structures used in describing quantum information exchanges.

Barnum, Duncan, and Wilce's paper is entitled "Symmetry, compact closure and dagger compactness for categories of convex operational models". In terms of our overall volume, the paper is a kind of bridge: it relates two styles of representing quantum physical systems. An older "convex operational" tradition uses state sets, measurements, and probabilities of outcomes. The more recent categorical approach is more abstract and uses categories with extra structure. The theme of the paper is the interplay between these two representation styles.

"Search for quantum reality" by Stan Gudder represents work on a foundational topic related to quantum physics. He works with a notion of a "coevent," a function from subsets of a universe to truth values. This much is standard from semantics and philosophical logic, where one is often concerned with reconstructing objects from their properties. However, this paper asks questions that are special to the quantum setting having to do with measures and integrals, and it reports general results on a search for "quantum reality."

John Harding's paper, "Daggers, kernels, Baer *-semigroups, and orthomodularity" is concerned with the categorical treatment of quantum mechanics. The leading idea is to associate orthomodular structures from the objects in a given category, and also to study the conditions under which one obtains stronger structures, such as orthomodular lattices or modular ortholattices.

In a broad sense Samson Abramsky's "Coalgebras, Chu spaces, and representations of physical systems" is on the same general topic as Gudder's and Harding's papers: mathematical representations of quantum systems that go beyond the tradition in quantum logic. Abramsky connects the topic with two topics in theoretical computer science: Chu spaces, and also coalgebra.

Our own contribution, the first part of this article, is a historical overview of the subject, together with our own spin (as it were) on the subject and where it could be going. In addition, John Harding establishes a decidability result related to one of ours in his note "Decidability of the equational theory of the continuous geometry $CG(F)$."

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